ECE 604, Lecture 7

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Contents

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1 More on Constitutive Relations

1.1 Isotropic Frequency Dispersive Media

First let us look at the simple constitutive relation for the where

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{1.1}$$

We have a simple model where

$$\mathbf{P} = \varepsilon_0 \chi_0 \mathbf{E} \tag{1.2}$$

where χ_0 is the electric susceptibility. When used in the generalized Ampere's law, $\mathbf{P}$, the polarization density, plays an important role for the flow of the displacement current through space. The polarization density can be due to the presence of polar atoms or molecules that become little dipoles in the presence of an electric field. For instance, water, which is H_2O, is a polar molecule that becomes a small dipole when an electric field is applied.

We can think of displacement current flow as capacitive coupling of polarization current flow through space. Namely, for a source-free medium,

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} \tag{1.3}$$

Figure 1:

For example, for a sinusoidal oscillating field, the dipoles will flip back and forth giving rise to flow of displacement current just as how time-harmonic electric current can flow through a capacitor as shown in Figure 1.

This linear relationship can be generalized to that of a linear time-invariant system, or that at any given $\mathbf{r}$,

$$\mathbf{P}(\mathbf{r}, t) = \varepsilon_0 \chi_e(\mathbf{r}, t) \circledast \mathbf{E}(\mathbf{r}, t) \tag{1.4}$$

where $\circledast$ here implies a convolution. In the frequency domain or the Fourier space, the above relationship becomes

$$\mathbf{P}(\mathbf{r}, \omega) = \varepsilon_0 \chi_0(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega), \tag{1.5}$$

$$\mathbf{D}(\mathbf{r}, \omega) = \varepsilon_0 (1 + \chi_0(\mathbf{r}, \omega)) \mathbf{E}(\mathbf{r}, \omega) = \varepsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega) \tag{1.6}$$

where $\varepsilon(\mathbf{r}, \omega) = \varepsilon_0(1 + \chi_0(\mathbf{r}, \omega))$ at any point $\mathbf{r}$ in space. There is a rich variety of ways at which $\chi_e(\omega)$ can manifest itself.

1.2 Anisotropic Media

For anisotropic media,

$$\begin{aligned}\mathbf{D} &= \varepsilon_0\mathbf{E} + \varepsilon_0\overline{\boldsymbol{\chi}}_0(\omega)\cdot\mathbf{E}\\ &= \varepsilon_0(\overline{\mathbf{I}} + \overline{\boldsymbol{\chi}}_0(\omega))\cdot\mathbf{E} = \overline{\varepsilon}(\omega)\cdot\mathbf{E}\end{aligned} \tag{1.7}$$

In the above, $\overline{\varepsilon}$ is a 3×3 matrix also known as a tensor in electromagnetics. The above implies that $\mathbf{D}$ and $\mathbf{E}$ do not necessary point in the same direction, the meaning of anisotropy. A tensor is often associated with a physical notion, whereas a matrix is not.

Previously, we have assume that χ_0 to be frequency independent. This is not usually the case as all materials have χ_0's that are frequency dependent. This will become clear later. Also, since $\overline{\varepsilon}(\omega)$ is frequency dependent, we should view it as a transfer function where the input is $\mathbf{E}$, and the output $\mathbf{D}$. This implies that in the time-domain, the above relation becomes a time-convolution relation.

Similarly for conductive media,

$$\mathbf{J} = \sigma\mathbf{E}, \tag{1.8}$$

This can be used in Maxwell's equations in the frequency domain to yield the definition of complex permittivity. Using the above in Ampere's law in the frequency domain, we have

$$\nabla\times\mathbf{H}(\mathbf{r}) = j\omega\varepsilon\mathbf{E}(\mathbf{r}) + \sigma\mathbf{E}(\mathbf{r}) = j\omega\underset{\sim}{\varepsilon}(\omega)\mathbf{E}(\mathbf{r}) \tag{1.9}$$

where the complex permittivity $\underset{\sim}{\varepsilon}(\omega) = \varepsilon - j\sigma/\omega$.

For anisotropic conductive media, one can have

$$\mathbf{J} = \overline{\boldsymbol{\sigma}}(\omega)\cdot\mathbf{E}, \tag{1.10}$$

Here, again, due to the tensorial nature of the conductivity $\overline{\boldsymbol{\sigma}}$, the electric current $\mathbf{J}$ and electric field $\mathbf{E}$ do not necessary point in the same direction.

The above assumes a local or point-wise relationship between the input and the output of a linear system. This need not be so. In fact, the most general linear relationship between $\mathbf{P}(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$ is

$$\mathbf{P}(\mathbf{r},t) = \int_{-\infty}^{\infty}\overline{\boldsymbol{\chi}}(\mathbf{r}-\mathbf{r}', t-t')\cdot\mathbf{E}(\mathbf{r}',t')d\mathbf{r}'dt' \tag{1.11}$$

In the Fourier transform space, the above becomes

$$\mathbf{P}(\mathbf{k},\omega) = \overline{\boldsymbol{\chi}}(\mathbf{k},\omega)\cdot\mathbf{E}(\mathbf{k},\omega) \tag{1.12}$$

where

$$\overline{\boldsymbol{\chi}}(\mathbf{k},\omega) = \int_{-\infty}^{\infty}\overline{\boldsymbol{\chi}}(\mathbf{r},t)\exp(j\mathbf{k}\cdot\mathbf{r} - j\omega t)d\mathbf{r}dt \tag{1.13}$$

Such a medium is termed spatially dispersive as well as frequency dispersive.

In general

$$\overline{\boldsymbol{\varepsilon}}(\mathbf{k},\omega) = 1 + \overline{\boldsymbol{\chi}}(\mathbf{k},\omega) \tag{1.14}$$

where

$$\mathbf{D}(\mathbf{k},\omega) = \overline{\boldsymbol{\varepsilon}}(\mathbf{k},\omega) \cdot \mathbf{E}(\mathbf{k},\omega) \tag{1.15}$$

The above can be extended to magnetic field and magnetic flux yielding

$$\mathbf{B}(\mathbf{k},\omega) = \overline{\boldsymbol{\mu}}(\mathbf{k},\omega) \cdot \mathbf{H}(\mathbf{k},\omega) \tag{1.16}$$

for a general spatial and frequency dispersive magnetic material. In optics, most materials are non-magnetic, and hence, $\mu = \mu_0$, whereas it is quite easy to make anisotropic magnetic materials in radio and microwave frequencies, such as ferrites.

1.3 Bi-anisotropic Media

In the previous section, the electric flux $\mathbf{D}$ depends on the electric field $\mathbf{E}$ and the magnetic flux $\mathbf{B}$ depends on the magnetic field $\mathbf{H}$. The concept of constitutive relation can be extended to where $\mathbf{D}$ and $\mathbf{B}$ depend on both $\mathbf{E}$ and $\mathbf{H}$. In general, one can write

$$\mathbf{D} = \overline{\boldsymbol{\varepsilon}}(\omega) \cdot \mathbf{E} + \overline{\boldsymbol{\xi}}(\omega) \cdot \mathbf{H} \tag{1.17}$$

$$\mathbf{B} = \overline{\boldsymbol{\zeta}}(\omega) \cdot \mathbf{E} + \overline{\boldsymbol{\mu}}(\omega) \cdot \mathbf{H} \tag{1.18}$$

A medium where the electric flux or the magnetic flux is dependent on both $\mathbf{E}$ and $\mathbf{H}$ is known as a bi-anisotropic medium.

1.4 Inhomogeneous Media

If any of the $\overline{\boldsymbol{\varepsilon}}$, $\overline{\boldsymbol{\xi}}$, $\overline{\boldsymbol{\zeta}}$, or $\overline{\boldsymbol{\mu}}$ is a function of position $\mathbf{r}$, the medium is known as an inhomogeneous medium or a heterogeneous medium. There are usually no simple solution to problems associated with such media.

1.5 Uniaxial and Biaxial Media

Anisotropic optical materials are quite often encountered in optics. Examples of them are the biaxial and uniaxial media. They are optical materials where the permittivity tensor can be written as

$$\overline{\boldsymbol{\varepsilon}} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \tag{1.19}$$

When $\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3$, the medium is known as a biaxial medium. But when $\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3$, then the medium is a uniaxial medium.

In the biaxial medium, all the three components of the electric field feel different permittivity constants. But in the uniaxial medium, the electric field in the xy plane feels the same permittivity constant, but the electric field in the z direction feels a different permittivity constant. As shall be shown, different light polarization will propagate with different behavior through such a medium.

1.6 Nonlinear Media

In the previous cases, we have assumed that χ_0 is independent of the field $\mathbf{E}$. The relationships between $\mathbf{P}$ and $\mathbf{E}$ can be written more generally as

$$\mathbf{P} = \varepsilon_0 \overline{\boldsymbol{\chi}}_0(\mathbf{E}) \tag{1.20}$$

where the relationship can appear in many different forms. For nonlinear media, the relationship can be non-linear as indicated in the above.

2 Wave Phenomenon in the Frequency Domain

Given that we have seen the emergence of wave phenomenon in the time domain, it will be interesting to ask how this phenomenon presents itself for time-harmonic field or in the frequency domain. In the frequency domain, the source-free Maxwell's equations are

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu_0 \mathbf{H}(\mathbf{r}) \tag{2.1}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\varepsilon_0 \mathbf{E}(\mathbf{r}) \tag{2.2}$$

Taking the curl of (2.1) and then substituting (2.2) into its right-hand side, one obtains

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu_0 \nabla \times \mathbf{H}(\mathbf{r}) = \omega^2\mu_0\varepsilon_0 \mathbf{E}(\mathbf{r}) \tag{2.3}$$

Again, using the identity that

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \cdot \nabla \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \tag{2.4}$$

and that $\nabla \cdot \mathbf{E} = 0$ in a source-free medium, (2.3) becomes

$$(\nabla^2 + \omega^2\mu_0\varepsilon_0)\mathbf{E}(\mathbf{r}) = 0 \tag{2.5}$$

This is known as the Helmholtz wave equation or just the Helmholtz equation.

For simplicity of seeing the wave phenomenon, we let $\mathbf{E} = \hat{x}E_x(z)$, a field pointing in the x direction, but varies only in the z direction. Evidently, $\nabla \cdot \mathbf{E}(\mathbf{r}) = \partial E_x(z)/\partial x = 0$. Then (2.5) simplifies to

$$\left(\frac{d^2}{dz^2} + k_0^2\right) E_x(z) = 0 \tag{2.6}$$

where $k_0^2 = \omega^2\mu_0\varepsilon_0 = \omega^2/c_0^2$. The general solution to (2.6) is of the form

$$E_x(z) = E_{0+}e^{-jk_0z} + E_{0-}e^{jk_0z} \tag{2.7}$$

One can convert the above back to the time domain using phasor technique, or by using that $E_x(z,t) = \Re e[E_x(z,\omega)e^{j\omega t}]$, yielding

$$E_x(z,t) = |E_{0+}|\cos(\omega t - k_0z + \alpha_+) + |E_{0-}|\cos(\omega t + k_0z + \alpha_-) \tag{2.8}$$

where we have assumed that

$$E_{0\pm} = |E_{0\pm}|e^{j\alpha_\pm} \tag{2.9}$$

The physical picture of the above expressions can be appreciated by rewriting

$$\cos(\omega t \mp k_0z + \alpha_\pm) = \cos\left[\frac{\omega}{c_0}(c_0t \mp z) + \alpha_\pm\right] \tag{2.10}$$

where we have used the fact that $k_0 = \frac{\omega}{c_0}$. One can see that the first term on the right-hand side of (2.8) is a sinusoidal plane wave traveling to the right, while the second term is a sinusoidal plane wave traveling to the left, with velocity c_0. The above plane wave is uniform and a constant in the xy plane and propagating in the z direction. Hence, it is also called a uniform plane wave in 1D.

Moreover, for a fixed t or at $t = 0$, the sinusoidal functions are proportional to $\cos(\mp k_0z + \alpha_\pm)$. From this, we can see that whenever $k_0z = 2n\pi$, $n \in Q$ where Q is the set of integers, the functions repeat themselves. Calling this repetition length the wavelength λ_0, we deduce that $\lambda_0 = \frac{2\pi}{k_0}$, or that

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{\omega}{c_0} = \frac{2\pi f}{c_0} \tag{2.11}$$

One can see that because c_0 is a humongous number, λ_0 can be very large. You can plug in the frequency of your local AM station to see how big λ_0 is.

3 Uniform Plane Waves in 3D

By repeating the previous derivation for a homogeneous lossless medium, the vector Helmholtz equation for a source-free medium is given by

$$\nabla \times \nabla \times \mathbf{E} - \omega^2\mu\varepsilon\mathbf{E} = 0 \tag{3.1}$$

By the same derivation as before for the free-space case, one has

$$\nabla^2\mathbf{E} + \omega^2\mu\varepsilon\mathbf{E} = 0 \tag{3.2}$$

if $\nabla \cdot \mathbf{E} = 0$.

The general solution to (3.2) is hence

$$\mathbf{E} = \mathbf{E}_0e^{-jk_xx-jk_yy-jk_zz} = \mathbf{E}_0e^{-j\mathbf{k}\cdot\mathbf{r}} \tag{3.3}$$

where $\mathbf{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$, $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$ and $\mathbf{E}_0$ is a constant vector. And upon substituting (3.3) into (3.2), it is seen that

$$k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon \tag{3.4}$$

This is called the dispersion relation for a plane wave.

In general, k_x, k_y, and k_z can be arbitrary and even complex numbers as long as this relation is satisfied. To simplify the discussion, we will focus on the case where k_x, k_y, and k_z are all real. When this is the case, the vector function represents a uniform plane wave propagating in the $\mathbf{k}$ direction. As can be seen, when $\mathbf{k}\cdot\mathbf{r}$ = constant, it is represented by all points of $\mathbf{r}$ that represents a flat plane. This flat plane represents the constant phase wave front. By increasing the constant, we obtain different planes for progressively changing phase fronts.[1]

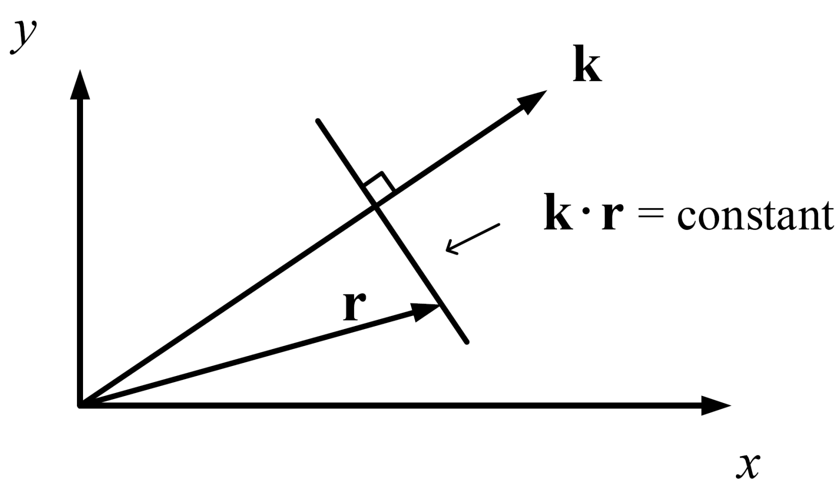

Figure 2:

Further, since $\nabla\cdot\mathbf{E} = 0$, we have

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \nabla\cdot\mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z} = \nabla\cdot\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}\\ &= (-\hat{x}jk_x - \hat{y}jk_y - \hat{z}jk_z)\cdot\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}\\ &= -j(\hat{x}k_x + \hat{y}k_y + \hat{z}k_z)\cdot\mathbf{E} = 0 \end{aligned}\tag{3.5}$$

or that

$$\mathbf{k}\cdot\mathbf{E}_0 = \mathbf{k}\cdot\mathbf{E} = 0 \tag{3.6}$$

[1] In the $\exp(j\omega t)$ time convention, this phase front is decreasing, whereas in the $\exp(-i\omega t)$ time convention, this phase front is increasing.

Thus, $\mathbf{E}$ is orthogonal to $\mathbf{k}$ for a uniform plane wave.

The above exercise shows that whenever $\mathbf{E}$ is a plane wave, and when the ∇ operator operates on such a vector function, one can do the substitution that $\nabla \to -j\mathbf{k}$.

Hence, in a source-free homogenous medium,

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{3.7}$$

the above equation simplifies to

$$-j\mathbf{k} \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{3.8}$$

or that

$$\mathbf{H} = \frac{\mathbf{k} \times \mathbf{E}}{\omega\mu} \tag{3.9}$$

In other words, $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{k}$ form a right-handed system, or that $\mathbf{E} \times \mathbf{H}$ point in the direction of $\mathbf{k}$. Such a wave, where the electric field and magnetic field are transverse to the direction of propagation, is called a transverse electromagnetic (TEM) wave.

Also, from

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} \tag{3.10}$$

we get that

$$\mathbf{E} = -\frac{\mathbf{k} \times \mathbf{H}}{\omega\varepsilon} = -\frac{\mathbf{k} \times (\mathbf{k} \times \mathbf{E})}{\omega^2\mu\varepsilon} \tag{3.11}$$

Again, using the vector identity, the above simplifies to

$$\mathbf{E} = -\frac{\mathbf{k}(\mathbf{k} \cdot \mathbf{E}) - (\mathbf{k} \cdot \mathbf{k})\mathbf{E}}{\omega^2\mu\varepsilon} = \frac{\mathbf{k} \cdot \mathbf{k}}{\omega^2\mu\varepsilon}\mathbf{E} \tag{3.12}$$

where $\mathbf{k} \cdot \mathbf{E} = 0$ has been used. For the above equation to be consistent, it is necessary that

$$\mathbf{k} \cdot \mathbf{k} = k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon \tag{3.13}$$

or that $\mathbf{k}$ has to satisfy the dispersion relation previously derived in (3.4).

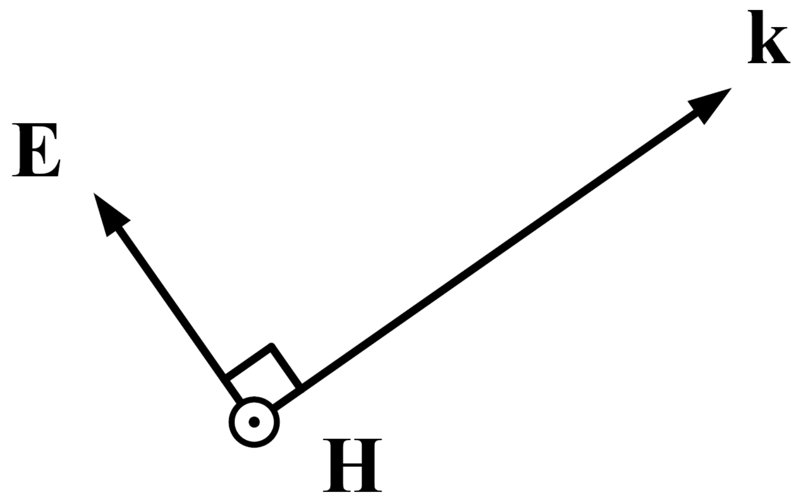

Figure 3:

Figure 3 shows that $\mathbf{k} \cdot \mathbf{E} = 0$, and that $\mathbf{k} \times \mathbf{E}$ points in the direction of $\mathbf{H}$ as shown in (3.9). Figure 3 also shows, as $\mathbf{k}$, $\mathbf{E}$, and $\mathbf{H}$ are orthogonal to each other. Hence, taking the magnitude of (3.9), and using that $\mathbf{k}$ and $\mathbf{E}$ are orthogonal to each other, then

$$|\mathbf{H}| = \frac{|\mathbf{k}||\mathbf{E}|}{\omega\mu} = \sqrt{\frac{\varepsilon}{\mu}}|\mathbf{E}| = \frac{1}{\eta}|\mathbf{E}| \tag{3.14}$$

where the quantity

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{3.15}$$

is call the intrinsic impedance. For vacuum or free-space, it is about 377Ω. It is also noted that $\mathbf{E} \times \mathbf{H}^*$ points in the direction of the vector $\mathbf{k}$. This is also required by the Poynting's theorem.

In the above, when k_x, k_y, and k_z are not all real, the wave is known as an inhomogeneous wave.[2]

[2]The term inhomogeneous plane wave is used sometimes, but it is a misnomer since there is no more a planar wave front in this case.