ECE 604, Lecture 4

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Contents

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1 Magnetostatics

The magnetostatic equations where $\partial/\partial t = 0$ are

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{1.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.2}$$

One way to satisfy the second equation is to let

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1.3}$$

because

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{1.4}$$

The above is zero for the same reason that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. In this manner, Gauss's law is automatically satisfied.

From (1.1), we have

$$\nabla \times \left(\frac{\mathbf{B}}{\mu} \right) = \mathbf{J} \tag{1.5}$$

Then using (1.3)

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J} \tag{1.6}$$

In a homogeneous medium, μ is a constant and hence

$$\nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{J} \tag{1.7}$$

We use the vector identity that (see handout on Some Useful Formulas)

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A} \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \end{aligned} \tag{1.8}$$

As a result, we arrive at

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} \tag{1.9}$$

By imposing the Coulomb's gauge that $\nabla \cdot \mathbf{A} = 0$, which will be elaborated in the next section, we arrive at

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \tag{1.10}$$

The above is also known as the vector Poisson's equation. In cartesian coordinates, the above can be viewed as three scalar Poisson's equations. Each of the Poisson's equation can be solved using the Green's function method previously described. Consequently, in free space

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r})}{R} dV' \tag{1.11}$$

where

$$R = |\mathbf{r} - \mathbf{r}'| \tag{1.12}$$

1.1 More on Coulomb's Gauge

However, $\mathbf{A}$ in (1.3) is not unique because one can always define

$$\mathbf{A}' = \mathbf{A} - \nabla\Psi \tag{1.13}$$

Then

$$\nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} - \nabla\Psi) = \nabla \times \mathbf{A} = \mathbf{B} \tag{1.14}$$

where we have made use of that $\nabla \times \nabla\Psi = 0$. Hence, the $\nabla\times$ of both $\mathbf{A}$ and $\mathbf{A}'$ produce the same $\mathbf{B}$.

To find $\mathbf{A}$ uniquely, we have to define or set the divergence of $\mathbf{A}$ or provide a gauge condition. One way is to set the divergence of $\mathbf{A}$ to be zero, namely

$$\nabla \cdot \mathbf{A} = 0 \tag{1.15}$$

Then

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} - \nabla^2\Psi \neq \nabla \cdot \mathbf{A} \tag{1.16}$$

The last non-equal sign follows if $\nabla^2\Psi \neq 0$. However, if we further stipulate that $\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} = 0$, then $-\nabla^2\Psi = 0$. This does not necessary imply that $\Psi = 0$, but if we impose that condition that $\Psi \to 0$ when $\mathbf{r} \to \infty$, then $\Psi = 0$ everywhere.[1] By so doing, $\mathbf{A}$ and $\mathbf{A}'$ are equal to each other, and we obtain (1.10) and (1.11).

2 Boundary Conditions–1D Poisson's Equation

Boundary conditions are embedded in the partial differential equations that the potential or the field satisfy. Two important concepts to keep in mind are:

- Differentiation of a function with discontinuous slope will give rise to step discontinuity.
- Differentiation of a function with step discontinuity will give rise to a Dirac delta function.

Take for example a one dimensional Poisson's equation that

$$\frac{d}{dx}\varepsilon(x)\frac{d}{dx}\Phi(x) = -\varrho(x) \tag{2.1}$$

where $\varepsilon(x)$ represents material property that has the form given in Figure 1. One can actually say a lot about $\Phi(x)$ given $\varrho(x)$ on the right-hand side. If $\varrho(x)$ has a delta function singularity, it implies that $\varepsilon(x)\frac{d}{dx}\Phi(x)$ has a step discontinuity. If $\varrho(x)$ is finite everywhere, then $\varepsilon(x)\frac{d}{dx}\Phi(x)$ must be continuous everywhere.

[1] It is a property of the Laplace boundary value problem that if $\Psi = 0$ on a closed surface S, then $\Psi = 0$ everywhere inside S. Earnshaw's theorem is useful for proving this assertion.

Furthermore, if $\varepsilon(x)\frac{d}{dx}\Phi(x)$ is finite everywhere, it implies that $\Phi(x)$ must be continuous everywhere.

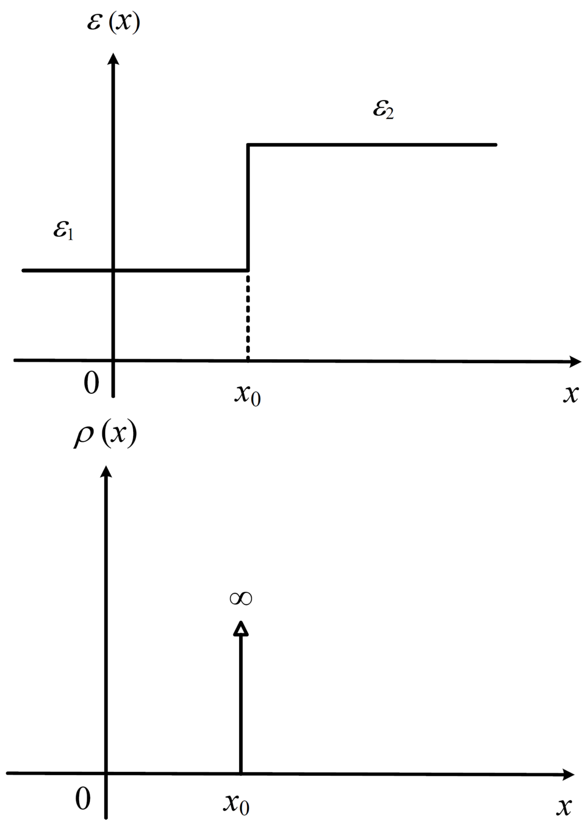

Figure 1:

To see this in greater detail, we illustrate it with the following example. In the above, $\varrho(x)$ represents a charge distribution given by $\varrho(x) = \varrho_s\delta(x - x_0)$. In this case, the charge distribution is everywhere zero except at the location of the surface charge sheet, where the charge density is infinite: it is represented mathematically by a delta function[2] in space.

To find the boundary condition of the potential $\Phi(x)$ at x_0 , we integrate (2.1) over an infinitesimal width around x_0, the location of the charge sheet, namely

$$\int_{x_0-\Delta}^{x_0+\Delta} dx \frac{d}{dx}\varepsilon(x)\frac{d}{dx}\Phi(x) = -\int_{x_0-\Delta}^{x_0+\Delta} dx\varrho(x) \tag{2.2}$$

[2]This function has been attributed to Dirac who used in pervasively, but Cauchy was aware of such a function.

or on the left-hand side, we get

$$\varepsilon(x)\frac{d}{dx}\Phi(x)\bigg|_{x_0-\Delta}^{x_0+\Delta} \cong -\varrho_s \tag{2.3}$$

whereas on the right-hand side, we pick up the contribution from the delta function. Evaluating the left-hand side at their limits, one arrives at

$$\lim_{\Delta\to 0} \varepsilon(x_0^+)\frac{d}{dx}\Phi(x_0^+) - \varepsilon(x_0^-)\frac{d}{dx}\Phi(x_0^-) \cong -\varrho_s, \tag{2.4}$$

In other words, the jump discontinuity is in $\varepsilon(x)\frac{d}{dx}\Phi(x)$ and the amplitude of the jump discontinuity is proportional to the amplitude of the delta function.

Since $\mathbf{E} = \nabla\Phi$, or

$$E_x(x) = -\frac{d}{dx}\Phi(x), \tag{2.5}$$

The above implies that

$$\varepsilon(x_0^+)E_x(x_0^+) - \varepsilon(x_0^-)E_x(x_0^-) = \varrho_s \tag{2.6}$$

or

$$D_x(x_0^+) - D_x(x_0^-) = \varrho_s \tag{2.7}$$

where

$$D_x(x) = \varepsilon(x)E_x(x) \tag{2.8}$$

The lesson learned from above is that boundary condition is obtained by integrating the pertinent differential equation over an infinitesimal small segment. One can also eyeball the differential equation and ascertain the terms that will have the jump discontinuity that will yield the delta function on the right-hand side.

3 Boundary Conditions–Maxwell's Equations

As seen previously, boundary conditions for a field is embedded in the differential equation that the field satisfies. Hence, boundary conditions can be derived from the differential operator forms of Maxwell's equations. To derive these boundary conditions, we will take an unconventional view: namely to see what sources can induce jump conditions on the pertinent fields. Boundary conditions are needed at media interfaces, as well as across current or charge sheets.

3.1 Faraday's Law

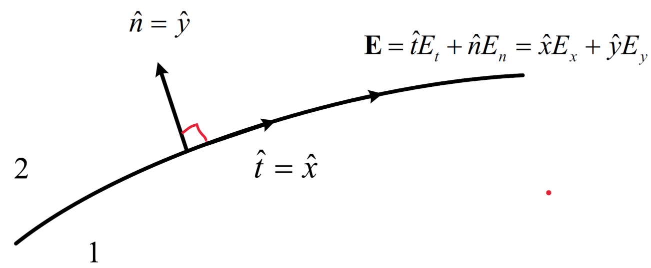

Figure 2:

For this, we start with Faraday's law, which implies that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.1}$$

One quick answer we could have is that if the right-hand side of the above equation is everywhere finite, then there could not be any jump discontinuity on the field $\mathbf{E}$ on the left hand side. To see this quickly, one can project the tangential field component and normal field component to a local coordinate system. In other words, one can think of $\hat{t}$ and $\hat{n}$ as the local $\hat{x}$ and $\hat{y}$ coordinates. Then writing the curl operator in this local coordinates, one gets

$$\nabla \times \mathbf{E} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right) \times (\hat{x}E_x + \hat{y}E_y) \tag{3.2}$$

$$= \hat{z}\frac{\partial}{\partial x}E_y - \hat{z}\frac{\partial}{\partial y}E_x \tag{3.3}$$

In simplifying the above, we have used the distributive property of cross product, and evaluating the cross product in cartesian coordinates. The cross product produces four terms, but only two of the four terms are non-zero as shown above.

Since the right-hand side of (3.1) is finite, the above implies that $\frac{\partial}{\partial x}E_y$ and $\frac{\partial}{\partial y}E_x$ have to be finite. In order words, E_x is continuous in the y direction and E_y is continuous in the x direction. Since in the local coordinate system, $E_x = E_t$, then E_t is continuous across the boundary. The above implies that

$$E_{1t} = E_{2t} \tag{3.4}$$

or

$$\hat{n} \times \mathbf{E}_1 = \hat{n} \times \mathbf{E}_2 \tag{3.5}$$

where $\hat{n}$ is the unit normal at the interface, and $\hat{n} \times \mathbf{E}$ always bring out the tangential component of a vector $\mathbf{E}$ (convince yourself).

3.2 Gauss's Law

From Gauss's law, we have

$$\nabla \cdot \mathbf{D} = \varrho \tag{3.6}$$

where ϱ is the volume charge density.

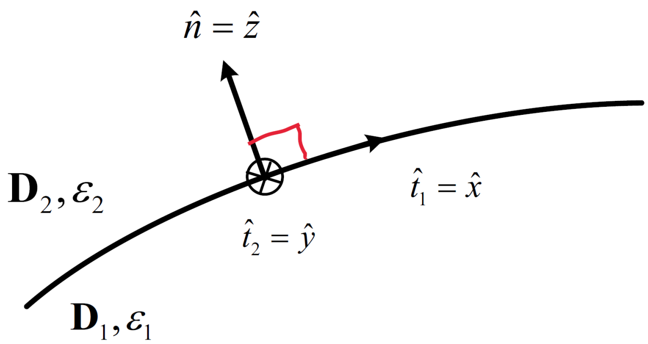

Figure 3:

Expressing the above in local coordinates, then

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x} D_x + \frac{\partial}{\partial y} D_y + \frac{\partial}{\partial z} D_z = \varrho \tag{3.7}$$

If there is a surface layer charge at the interface, then the volume charge density must be infinitely large, and can be expressed in terms of a delta function, or $\varrho = \varrho_s \delta(z)$ in local coordinates. By looking at the above expression, the only term that can produce a $\delta(z)$ is from $\frac{\partial}{\partial z} D_z$. In other words, D_z has a jump discontinuity at $z = 0$; the other terms do not. Then

$$\frac{\partial}{\partial z} D_z = \varrho_s \delta(z) \tag{3.8}$$

Integrating the above from $0 - \Delta$ to $0 + \Delta$, we get

$$D_z(z)\Big|_{0-\Delta}^{0+\Delta} = \varrho_s \tag{3.9}$$

or

$$D_z(0^+) - D_z(0^-) = \varrho_s \tag{3.10}$$

where $0^+ = \lim_{\Delta \to 0} 0 + \Delta$, $0^- = \lim_{\Delta \to 0} 0 - \Delta$. Since $D_z(0^+) = D_{2n}$, $D_z(0^-) = D_{1n}$, the above becomes

$$D_{2n} - D_{1n} = \varrho_s \tag{3.11}$$

or that

$$\hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \varrho_s \tag{3.12}$$

3.3 Ampere's Law

Ampere's law, or the generalized one, stipulates that

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3.13}$$

Again if the right-hand side is everywhere finite, then $\mathbf{H}$ is a continuous field everywhere. However, if the right-hand side has a delta function singularity, then this is not so. For instance, we can project the above equation onto a local coordinates just as we did for Faraday's law.

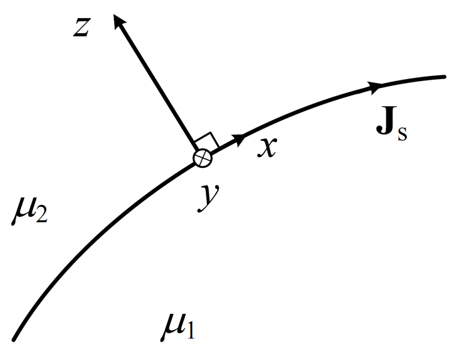

Figure 4:

To be general, we also include the presence of a current sheet at the interface. A current sheet, or a surface current density becomes a delta function singularity when expressed as a volume current density; Thus, rewriting (3.13) in a local coordinate system, assuming that $\mathbf{J} = \hat{x} J_{sx} \delta(z)$, then

$$\nabla \times \mathbf{H} = \hat{x} \left(\frac{\partial}{\partial y} H_z - \frac{\partial}{\partial z} H_y \right) = \hat{x} J_{sx} \delta(z) \tag{3.14}$$

The displacement current term on the right-hand side is ignored since it is regular or finite, and will not induce a jump discontinuity on the field; hence, we have the form of the right-hand side of the above equation. From the above, the only term that can produce a $\delta(z)$ singularity on the left-hand side is the $-\frac{\partial}{\partial z} H_y$ term. Therefore, we conclude that

$$-\frac{\partial}{\partial z} H_y = J_{sx} \delta(z) \tag{3.15}$$

In other words, H_y has to have a jump discontinuity at the interface where the current sheet resides. Or that

$$H_y(z = 0^+) - H_y(z = 0^-) = -J_{sx} \tag{3.16}$$

The above implies that

$$H_{2y} - H_{1y} = -J_{sx} \tag{3.17}$$

But H_y is just the tangential component of the **H** field. Now if we repeat the same exercise with $\mathbf{J} = \hat{y} J_{sy} \delta(z)$, at the interface, we have

$$H_{2x} - H_{1x} = J_{sy} \tag{3.18}$$

Now, (3.17) and (3.18) can be rewritten using a cross product as

$$\hat{z} \times (\hat{y} H_{2y} - \hat{y} H_{1y}) = \hat{x} J_{sx} \tag{3.19}$$

$$\hat{z} \times (\hat{x} H_{2x} - \hat{x} H_{1x}) = \hat{y} J_{sy} \tag{3.20}$$

The above two equations can be combined as one to give

$$\hat{z} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s \tag{3.21}$$

Taking $\hat{z} = \hat{n}$ in general, we have

$$\hat{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s \tag{3.22}$$

3.4 Gauss's Law for Magnetic Flux

Similarly, from Gauss's law for magnetic flux, or that

$$\nabla \cdot \mathbf{B} = 0 \tag{3.23}$$

one deduces that

$$\hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \tag{3.24}$$

or that the normal magnetic fluxes are continuous at an interface.