ECE 604, Lecture 38

Wed, April 24, 2019

Contents

Printed on April 24, 2019 at 15 : 38: W.C. Chew and D. Jiao.

1 Quantum Theory of Light

1.1 Historical Background

Quantum theory is a major intellectual achievement of the twentieth century, even though we are still discovering new knowledge in it. Several major experimental findings led to the revelation of quantum theory or quantum mechanics of nature. In nature, we know that many things are not infinitely divisible. Matter is not infinitely divisible as vindicated by the atomic theory of John Dalton (1766-1844). So fluid is not infinitely divisible: as when water is divided into smaller pieces, we will eventually arrive at water molecule, H_2O, which is the fundamental building block of water.

In turns out that electromagnetic energy is not infinitely divisible either. The electromagnetic radiation out of a heated cavity would obey a very different spectrum if electromagnetic energy is infinitely divisible. In order to fit experimental observation of radiation from a heated electromagnetic cavity, Max Planck (1900s) proposed that electromagnetic energy comes in packets or is quantized. Each packet of energy or a quantum of energy E is associated with the frequency of electromagnetic wave, namely

$$E = \hbar\omega = \hbar 2\pi f = hf \tag{1.1}$$

where $\hbar$ is now known as the Planck constant and $\hbar = h/2\pi = 6.626 \times 10^{-34}$ J·s (Joule-second). Since $\hbar$ is very small, this packet of energy is very small unless ω is large. So it is no surprise that the quantization of electromagnetic field is first associated with light, a very high frequency electromagnetic radiation.

The second experimental evidence that light is quantized is the photo-electric effect. It was found that matter emitted electrons when light shined on it. First, the light frequency has to correspond to the "resonant" frequency of the atom. Second, the number of electrons emitted is proportional to the number of packets of energy $\hbar\omega$ that the light carries. This was a clear indication that light energy traveled in packets or quanta as posited by Einstein in 1905.

That light is a wave has been demonstrated by Newton's ring effect in the eighteenth century (1717) (see Figure 1). In 1801, Thomas Young demonstrated the double slit experiment that further confirmed the wave nature of light (see Figure 2). But by the beginning of the 20-th century, one has to accept that light is both a particle, called a photon, carrying a quantum of energy with momentum, as well as a particle endowed with wave-like behavior. This is called wave-particle duality.

Theory

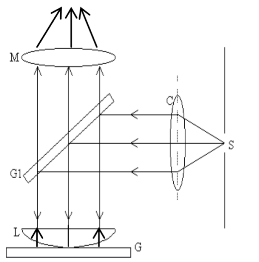

Fig. 1 Experimental set-up to observe Newton's ring

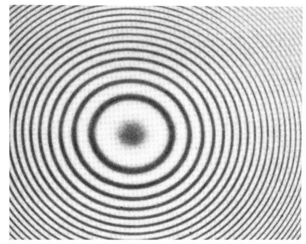

Fig. 2. Newton's rings

Figure 1: Courtesy of http://www.iiserpune.ac.in/∼bhasbapat/phy221_files

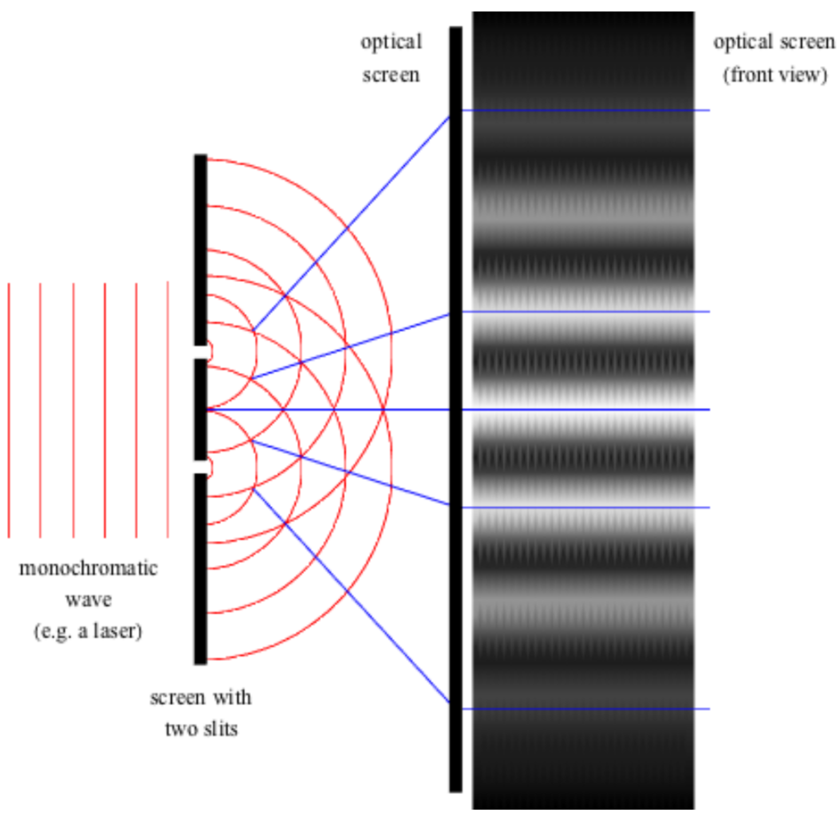

Figure 2: Courtesy of Shmoop.

This concept was not new to quantum theory as electrons were known to behave both like a particle and a wave. The particle nature of an electron was confirmed by the measurement of its charge by Millikan in 1913 in his oil-drop experiment. The double slit experiment for electron was done in 1927 by Davison and Germer, indicating that an electron has a wave nature. In 1924, De Broglie suggested that there is a wave associated with an electron with momentum p such that

$$p = \hbar k \tag{1.2}$$

where $k = 2\pi/\lambda$, the wavenumber. All this knowledge gave hint to the quantum theorists of that era to come up with a new way to describe nature.

Classically, particles like an electron moves through space obeying Newton's law of motion first established in 1687. Old ways of describing particle motion were known as classical mechanics, and the new way of describing particle motion is known as quantum mechanics. Quantum mechanics is very much motivated by a branch of classical mechanics called Hamiltonian mechanics. We will first use Hamiltonian mechanics to study a simple pendulum and connect it with electromagnetic oscillations.

1.2 Connecting Electromagnetic Oscillation to Simple Pendulum

The quantization of electromagnetic field theory was started by Dirac in 1927. In the beginning, it was called quantum electrodynamics important for understanding particle physics phenomena. Later on, it became important in quantum optics where quantum effects in electromagnetic technologies first emerged. Now, microwave photons are measurable and are important in quantum computers. Hence, quantum effects are important in the microwave regime as well.

The cavity modes in electromagnetics are similar to the oscillation of a pendulum. To understand the quantization of electromagnetic field, we start by connecting these cavity modes to a simple pendulum. It is to be noted that fundamentally, electromagnetic oscillation exists because of displacement current. Displacement current exists even in vacuum because vacuum is polarizable, namely that $\mathbf{D} = \varepsilon\mathbf{E}$. Furthermore, displacement current exists because of the $\partial\mathbf{D}/\partial t$ term in the generalized Ampere's law added by Maxwell, namely,

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{1.3}$$

Together with Faraday's law that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.4}$$

(1.3) and (1.4) together allow for the existence of wave. The coupling between the two equations gives rise to the "springiness" of electromagnetic fields.

Wave exists due to the existence of coupled harmonic oscillators, and at a fundamental level, these harmonic oscillators are electron-positron (e-p) pairs. The fact that they are coupled allows waves to propagate through space, and even in vacuum.

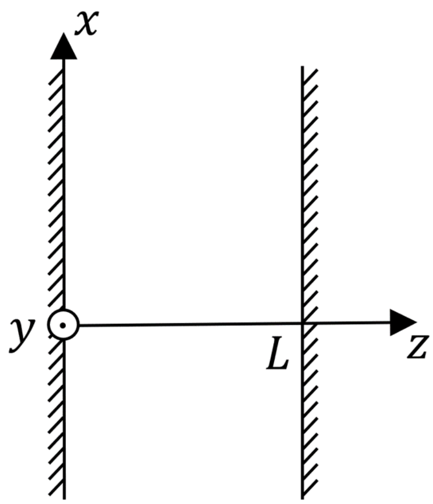

Figure 3:

We can start by looking at a one dimensional cavity formed by two PEC (perfect electric conductor) plates as shown in Figure 3. Assume source-free Maxwell's equations in between the plates and letting $\mathbf{E} = \hat{x}E_x$, $\mathbf{H} = \hat{y}H_y$, then (1.3) and (1.4) become

$$\frac{\partial}{\partial z} H_y = -\varepsilon \frac{\partial}{\partial t} E_x \tag{1.5}$$

$$\frac{\partial}{\partial z} E_x = -\mu \frac{\partial}{\partial t} H_y \tag{1.6}$$

The above are similar to the telegrapher's equations. We can combine them to arrive at

$$\frac{\partial^2}{\partial z^2} E_x = \mu\varepsilon \frac{\partial^2}{\partial t^2} E_x \tag{1.7}$$

We look for a cavity mode solution, which should be a standing wave in this case. We let

$$E_x(z,t) = E_0(t) \sin(k_l z) \tag{1.8}$$

In order to satisfy the boundary conditions at $z = 0$ and $z = L$, one deduces that

$$k_l = \frac{l\pi}{L}, \qquad l = 1, 2, 3, \ldots \tag{1.9}$$

Then,

$$\frac{\partial^2 E_x}{\partial z^2} = -k_l^2 E_x \tag{1.10}$$

Using this back in (1.7), one arrives at

$$-k_l^2 E_x = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} \tag{1.11}$$

Now using (1.8) in the above, and removing the space dependence, one gets

$$-k_l^2 E_0(t) = \mu\varepsilon \frac{\partial^2 E_0(t)}{\partial t^2} \tag{1.12}$$

The general solution for the above equation is that

$$E_0(t) = E_0 \cos(\omega_l t + \psi) \tag{1.13}$$

Since $k_l = \omega_l / c$, the above solution can only exist for discrete frequencies or that

$$\omega_l = \frac{l\pi}{L} c, \qquad l = 1, 2, 3, \ldots \tag{1.14}$$

These are the discrete resonant frequencies ω_l of the 1D cavity.

They can be thought of as the collective oscillations of coupled harmonic oscillators. At the fundamental level, these oscillations are oscillators made by electron-positron pairs. But macroscopically, their collective resonances manifest themselves as giving rise to infinitely many electromagnetic cavity modes.

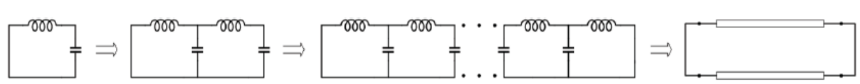

Figure 4:

The resonance between two parallel PEC plates is similar to the resonance of a transmission line of length L shorted at both ends. One can see that the resonance of a shorted transmission line is similar to the coupling of infnitely many LC tank circuits. To see this, as shown in Figure 4, we start with a single LC tank circuit as a simple harmonic oscillator with only one resonant frequency. When two LC tank circuits are coupled to each other, they will have two resonant frequencies. For N of them, they will have N resonant frequencies. For a continuum of them, they will be infinitely many resonant frequencies or modes as indicated by (1.9).

What is more important is that the resonance of each of these modes is similar to the resonance of a simple pendulum or a simple harmonic oscillator. For a fixed point in space, the field due to this oscillation is similar to the oscillation of a simple pendulum.

As we have seen in the Drude-Lorentz-Sommerfeld mode, for a particle of mass m attached to a spring connected to a wall, where the restoring force is like Hooke's law, the equation of motion of a pendulum by Newton's law is

$$m\frac{d^2x}{dt^2} + \kappa x = 0 \tag{1.15}$$

where κ is the spring constant, and we assume that the oscillator is not driven by an external force, but is in natural or free oscillation. By letting[1]

$$x = x_0 e^{-i\omega t} \tag{1.16}$$

the above becomes

$$-m\omega^2 x_0 + \kappa x_0 = 0 \tag{1.17}$$

Again, a non-trivial solution is possible only at the resonant frequency of the oscillator or that when $\omega = \omega_0$ where

$$\omega_0 = \sqrt{\frac{\kappa}{m}} \tag{1.18}$$

2 Hamiltonian Mechanics

Equation (1.15) can be derived by Newton's law but it can also be derived via Hamiltonian mechanics. Since Hamiltonian mechanics motivates quantum mechanics, we will look at the Hamiltonian mechanics view of the equation of motion (EOM) of a simple pendulum given by (1.15).

Hamiltonian mechanics, developed by Hamilton (1805-1865), is motivated by energy conservation. The Hamiltonian H of a system is given by its total energy, namely that

$$H = T + V \tag{2.1}$$

where T is the kinetic energy and V is the potential energy of the system.

For a simple pendulum, the kinetic energy is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2m}m^2v^2 = \frac{p^2}{2m} \tag{2.2}$$

[1] For this part of the lecture, we will switch to using $\exp(-i\omega t)$ time convention as is commonly used in optics and physics literatures.

where $p = mv$ is the momentum of the particle. The potential energy, assuming that the particle is attached to a spring with spring constant κ, is given by

$$V = \frac{1}{2}\kappa x^2 = \frac{1}{2}m\omega_0^2 x^2 \tag{2.3}$$

Hence, the Hamiltonian is given by

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 \tag{2.4}$$

At any instant of time t, we assume that $p(t) = mv(t) = m\frac{d}{dt}x(t)$ is independent of $x(t)$.[2] In other words, they can vary independently of each other. But $p(t)$ and $x(t)$ have to time evolve to conserve energy to keep H, the total energy, constant or independent of time. In other words,

$$\frac{d}{dt}H\left[p(t), x(t)\right] = 0 = \frac{dp}{dt}\frac{\partial H}{\partial p} + \frac{dx}{dt}\frac{\partial H}{\partial x} \tag{2.5}$$

Therefore, the Hamilton equations of motion are derived to be[3]

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}, \qquad \frac{dx}{dt} = \frac{\partial H}{\partial p} \tag{2.6}$$

From (2.4), we gather that

$$\frac{\partial H}{\partial x} = m\omega_0^2 x, \qquad \frac{\partial H}{\partial p} = \frac{p}{m} \tag{2.7}$$

Applying (2.6), we have[4]

$$\frac{dx}{dt} = \frac{p}{m}, \qquad \frac{dp}{dt} = -m\omega_0^2 x \tag{2.8}$$

Combining the two equations in (2.8) above, we have

$$m\frac{d^2x}{dt^2} = -m\omega_0^2 x = -\kappa x \tag{2.9}$$

which is also derivable by Newton's law.

A typical harmonic oscillator solution to (2.9) is

$$x(t) = x_0\cos(\omega_0 t + \psi) \tag{2.10}$$

Hence

$$\begin{aligned} H &= \frac{1}{2}m\omega_0^2 x_0^2 \sin^2(\omega_0 t + \psi) + \frac{1}{2}m\omega_0^2 x_0^2\cos^2(\omega_0 t + \psi) \\ &= \frac{1}{2}m\omega_0^2 x_0^2 = E \end{aligned} \tag{2.11}$$

And the total energy E very much depends on the amplitude x_0 of the oscillation.

[2] $p(t)$ and $x(t)$ are termed conjugate variables in many textbooks.

[3] Note that the Hamilton equations are determined to within a multiplicative constant, because one has not stipulated the connection between space and time, or we have not calibrated our clock.

[4] We can also calibrate our clock here so that it agrees with our definition of momentum in the ensuing equation.

3 Schrodinger Equation (1925)

Having seen the Hamiltonian mechanics for describing a pendulum, we shall next see the quantum mechanics description of the same pendulum: In other words, we will look at a quantum pendulum. To this end, we will invoke Schrodinger equation.

Schrodinger equation cannot be derived just as in the case Maxwell's equations. It is a wonderful result of a postulate and a guessing game based on experimental observations. Hamiltonian mechanics says that

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 = E \tag{3.1}$$

where E is the total energy of the oscillator, or pendulum. In classical mechanics, the position x of the particle associated with the pendulum is known with great certainty. But in the quantum world, this position x of the quantum particle is uncertain and is fuzzy.

To build this uncertainty into a quantum harmonic oscillator, we have to look at it from the quantum world. The position of the particle is described by a wave function,[5] which makes the location of the particle uncertain. To this end, Schrodinger proposed his equation. He was very much motivated by the experimental revelation then. Equation (3.1) can be written more suggestively as

$$\frac{\hbar^2 k^2}{2m} + \frac{1}{2}m\omega_0^2 x^2 = \hbar\omega \tag{3.2}$$

To add more depth to the above equation, one lets the above become an operator equation that operates on a wave function $\psi(x,t)$ so that

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{1}{2}m\omega_0^2 x^2 \psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t) \tag{3.3}$$

If the wave function is of the form

$$\psi(x,t) \sim e^{ikx - i\omega t} \tag{3.4}$$

then upon substituting (3.4) into (3.3), we retrieve (3.2).

Equation (3.3) is Schrodinger equation in one dimension for the quantum version of the simple harmonic oscillator. In Schrodinger equation, we can further posit that the wave function has the general form

$$\psi(x,t) = e^{ikx - i\omega t} A(x,t) \tag{3.5}$$

where $A(x,t)$ is a slowly varying function of x and t, compared to $e^{ikx-i\omega t}$. In other words, this is the expression for a wave packet. With this wave packet,

[5] Since a function is equivalent to a vector, and this wave function describes the state of the quantum system, this is also called a state vector.

the $\partial^2/\partial x^2$ can be again approximated by $-k^2$ as has been done in the paraxial wave approximation. Furthermore, if the signal is assumed to be quasi-monochromatic, then $i\hbar\partial/\partial_t\psi(x,t) \approx \hbar\omega$, we again retrieve the classical equation in (3.2). Hence, the classical equation (3.2) is a short wave, monochromatic approximation of Schrodinger equation. However, as we shall see, the solutions to Schrodinger equation are not limited to just wave packets described by (3.5).

For this course, we need only to study the one-dimensional Schrodinger equation. The above can be converted into eigenvalue problem, just as in waveguide and cavity problems, by letting[6]

$$\psi(x,t) = \psi_n(x)e^{-i\omega_n t} \tag{3.6}$$

By so doing, (3.3) becomes

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2\right]\psi_n(x) = E_n\psi_n(x) \tag{3.7}$$

where $E_n = \hbar\omega_n$ is the eigenvalue of the problem while $\psi_n(x)$ is the eigenfunction. (The parabolic x^2 potential profile is also known as a potential well as it can provide the restoring force to keep the particle bound to the well classically.) The above equation is also similar to the electromagnetic equation for a dielectric slab waveguide, where the second term is a dielectric profile (mind you, varying in the x direction) that can trap a waveguide mode. Therefore, the potential well is a trap for the particle both in classical mechanics or wave physics.

The above equation (3.7) can be solved in closed form in terms of Hermite-Gaussian functions (1864), or that

$$\psi_n(x) = \sqrt{\frac{1}{2^n n!}\sqrt{\frac{m\omega_0}{\pi\hbar}}}e^{-\frac{m\omega_0}{2\hbar}x^2}H_n\left(\sqrt{\frac{m\omega_0}{\hbar}}x\right) \tag{3.8}$$

where the eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0 \tag{3.9}$$

Here, the eigenfunction or eigenstate $\psi_n(x)$ is known as the photon number state of the solution. It corresponds to having n photons in the oscillation. If this is conceived as the collective oscillation of the e-p pairs in a cavity, there are n photons corresponding to energy of $n\hbar\omega_0$ embedded in the collective oscillation. The larger E_n is, the larger the number of photons there is. However, there is a curious mode at $n = 0$. This corresponds to no photon, and yet, there is a wave function $\psi_0(x)$. This is the zero-point energy state. This state is there even if the system is at its lowest energy state.

It is to be noted that in the quantum world, the position x of the pendulum is random. Moreover, this position is mapped to the amplitude of the field. Hence,

[6]Mind you, the following is ω_n, not ω_0.

it is the amplitude of an electromagnetic oscillation that becomes uncertain and fuzzy as shown in Figure 5.)

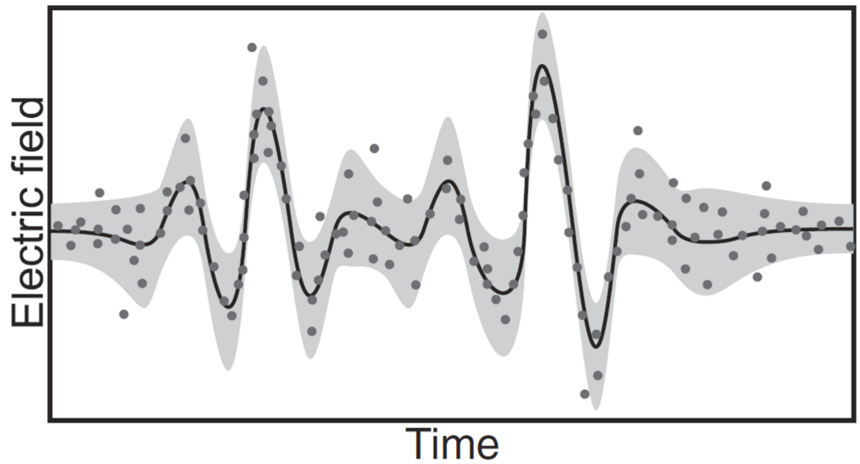

Figure 5: Schematic representation of the randomness of measured electric field. The electric field amplitude maps to the displacement (position) of the quantum harmonic oscillator, which is a random variable (Courtesy of Kira and Koch).

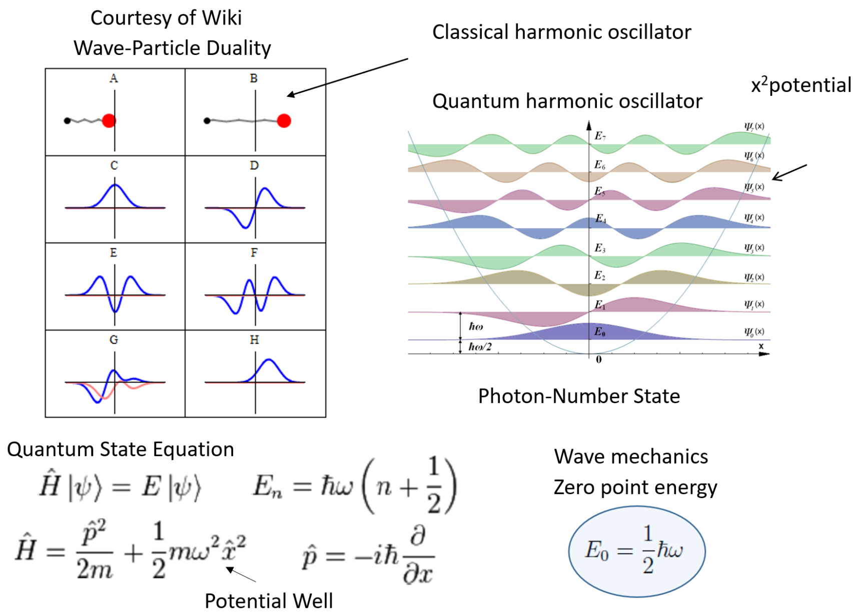

Figure 6: Courtesy of Wiki.

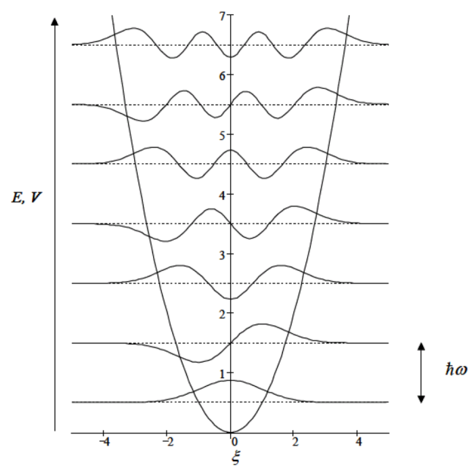

Figure 7: In this figure, $\xi = \sqrt{\frac{m\omega_0}{\hbar}}x$ (Courtesy of D.A.B. Miller).

4 Some Quantum Interpretations–A Preview

Schrodinger used this equation with resounding success. He derived a three-dimensional version of this to study the wave function and eigenvalues of a hydrogen atom. These eigenvalues E_n for a hydrogen atom agreed well with experimental observations that had eluded scientists for decades. Schrodinger did not actually understand what this wave function meant. It was Max Born (1926) who gave a physical interpretation of this wave function.

Given a wave function $\psi(x,t)$, then $|\psi(x,t)|^2\Delta x$ is the probability of finding the particle in the interval $[x, x+\Delta x]$. Therefore, $|\psi(x,t)|^2$ is a probability density function (PDF), and it is necessary that

$$\int_{-\infty}^{\infty} dx|\psi(x,t)|^2 = 1 \tag{4.1}$$

The position x of the particle is uncertain and is now a random variable. The average value or expectation value of x is given by

$$\int_{-\infty}^{\infty} dxx|\psi(x,t)|^2 = \langle x(t)\rangle = \bar{x}(t) \tag{4.2}$$

This is not the most ideal notation, since although x is not a function of time, its expectation value with respect to a time-varying function, $\psi(x,t)$, can be time-varying.

Notice that in going from (3.1) to (3.3), or from a classical picture to a quantum picture, we have let the momentum become p, originally a scalar number in the classical world, become a differential operator, namely that

$$p \to \hat{p} = -i\hbar\frac{\partial}{\partial x} \tag{4.3}$$

The momentum of a particle also becomes uncertain, and its expectation value is given by

$$\int_{\infty}^{\infty} dx\psi^*(x,t)\hat{p}\psi(x,t) = -i\hbar\int_{-\infty}^{\infty} dx\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t) = \langle\hat{p}(t)\rangle = \bar{p}(t) \tag{4.4}$$

The expectation values of position x and the momentum operator $\hat{p}$ are measurable in the laboratory. Hence, they are also called observables.

One more very important aspect of quantum theory is that since $p \to \hat{p} = -i\hbar\frac{\partial}{\partial x}$, $\hat{p}$ and x do not commute. In other words, it can be shown that

$$[\hat{p}, x] = \left[-i\hbar\frac{\partial}{\partial x}, x\right] = -i\hbar \tag{4.5}$$

In the classical world, $[p, x] = 0$, but not in the quantum world. In the equation above, we can elevate x to become an operator by letting $\hat{x} = x\hat{I}$, where $\hat{I}$ is the identity operator. Then both $\hat{p}$ and $\hat{x}$ are now operators, and are on the same footing. In this manner, we can rewrite equation (4.5) above as

$$[\hat{p}, \hat{x}] = -i\hbar\hat{I} \tag{4.6}$$

It can be shown easily that when two operators share the same set of eigenfunctions, they commute. When two operators $\hat{p}$ and $\hat{x}$ do not commute, it means that the expectation values of quantities associated with the operators, $\langle\hat{p}\rangle$ and $\langle\hat{x}\rangle$, cannot be determined to arbitrary precision simultaneously. For instance, $\hat{p}$ and $\hat{x}$ correspond to random variables, then the standard deviation of their measurable values, or their expectation values, obey the uncertainty principle relationship that[7]

$$\Delta p \Delta x \geq \hbar/2 \tag{4.7}$$

where Δp and Δx are the standard deviation of the random variables p and x.

5 Bizarre Nature of the Photon Number States

The photon number states are successful in predicting that the collective e-p oscillations are associated with n photons embedded in the energy of the oscillating modes. However, these number states are bizarre: The expectation values of the position of the quantum pendulum associated these states are always zero. To illustrate further, we form the wave function with a photon-number state

$$\psi(x,t) = \psi_n(x)e^{-i\omega_n t}$$

Previously, since the $\psi_n(x)$ are eigenfunctions, they are mutually orthogonal and they can be orthonormalized meaning that

$$\int_{-\infty}^{\infty} dx \psi_n^*(x)\psi_{n'}(x) = \delta_{nn'} \tag{5.1}$$

Then one can easily show that the expectation value of the position of the quantum pendulum in a photon number state is

$$\langle x(t)\rangle = \bar{x}(t) = \int_{-\infty}^{\infty} dx x|\psi(x,t)|^2 = \int_{-\infty}^{\infty} dx x|\psi_n(x)|^2 = 0 \tag{5.2}$$

because the integrand is always odd symmetric. In other words, the expectation value of the position x of the pendulum is always zero. It can also be shown that the expectation value of the momentum operator $\hat{p}$ is also zero for these photon number states. Hence, there are no classical oscillations that resemble them. Therefore, one has to form new wave functions by linear superposing these photon number states into a coherent state. This will be the discussion next.

[7]The proof of this is quite straightforward but is outside the scope of this course.