ECE 604, Lecture 37

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Contents

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1 Finite-Difference Method

To obtain the transient (time-domain) solution of the wave equation for a more general, inhomogeneous medium, a numerical method has to be used. The finite-difference time-domain (FDTD) method, a numerical method, is particularly suitable for solving transient problems. Moreover, it is quite versatile, and given the present computer technology, it has been used with great success in solving many practical problems.

In the finite-difference method, continuous space-time is replaced with a discrete space-time. Then, in the discrete space-time, partial differential equations are replaced with difference equations. These difference equations are readily implemented on a digital computer. Furthermore, an iterative or time-stepping scheme can be implemented without having to solve large matrices, resulting in a great savings in computer time. More recently, the development of parallel processor architectures in computers has also further enhanced the efficiency of the finite-difference scheme.

1.1 The Finite-Difference Approximation

Consider first a scalar wave equation of the form

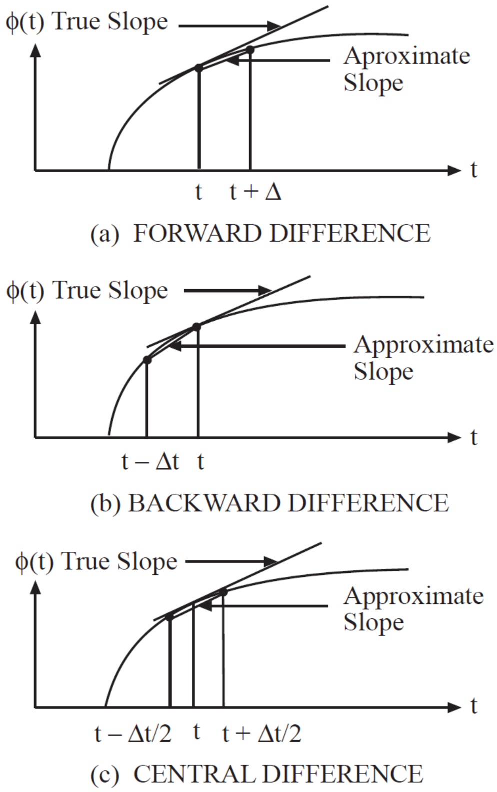
$$\frac{1}{c^2(\mathbf{r})}\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t) = \mu(\mathbf{r})\nabla\cdot\mu^{-1}(\mathbf{r})\nabla\phi(\mathbf{r},t). \tag{1.1}$$

Then, the time derivative can be approximated in many ways. For example,

$$\text{Forward difference:}\quad \frac{\partial\phi(\mathbf{r},t)}{\partial t} \approx \frac{\phi(\mathbf{r},t+\Delta t)-\phi(\mathbf{r},t)}{\Delta t}, \tag{1.2}$$

$$\text{Backward difference:}\quad \frac{\partial\phi(\mathbf{r},t)}{\partial t} \approx \frac{\phi(\mathbf{r},t)-\phi(\mathbf{r},t-\Delta t)}{\Delta t}, \tag{1.3}$$

$$\text{Central difference:}\quad \frac{\partial\phi(\mathbf{r},t)}{\partial t} \approx \frac{\phi(\mathbf{r},t+\frac{\Delta t}{2})-\phi(\mathbf{r},t-\frac{\Delta t}{2})}{\Delta t}, \tag{1.4}$$



(a) FORWARD DIFFERENCE

(b) BACKWARD DIFFERENCE

(c) CENTRAL DIFFERENCE

Figure 1: Different finite-difference approximations.

where Δt is a small number. Of the three methods of approximating the time derivative, the central-difference scheme is the best approximation, as is evident in Figure 1. The errors in the forward and backward differences are $O(\Delta t)$ (first-order error) while the central-difference approximation has an error $O[(\Delta t)^2]$ (second-order error). This can be easily illustrated by Taylor series expanding the right-hand side of (1.2) to (1.4).

Consequently, using the central-difference formula twice, we arrive at

$$\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t) \approx \frac{\partial}{\partial t}\left[\frac{\phi(\mathbf{r},t+\frac{\Delta t}{2}) - \phi(\mathbf{r},t-\frac{\Delta t}{2})}{\Delta t}\right] \tag{1.5}$$

$$\approx \frac{\phi(\mathbf{r},t+\Delta t) - 2\phi(\mathbf{r},t) + \phi(\mathbf{r},t-\Delta t)}{(\Delta t)^2}. \tag{1.6}$$

Next, if the function $\phi(\mathbf{r},t)$ is put on discrete time steps on the t axis, such that $\phi(\mathbf{r},t) = \phi(\mathbf{r},l\Delta t) = \phi^l(\mathbf{r})$, where l is an integer, Equation (1.6) then becomes

$$\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t) \approx \frac{\phi^{l+1}(\mathbf{r}) - 2\phi^l(\mathbf{r}) + \phi^{l-1}(\mathbf{r})}{(\Delta t)^2}. \tag{1.7}$$

1.2 Time Stepping or Time Marching

With this notation and approximations, Equation (1.1) becomes a time-stepping (or time-marching) formula,

$$\phi^{l+1}(\mathbf{r}) = c^2(\mathbf{r})(\Delta t)^2\mu(\mathbf{r})\nabla\cdot\mu^{-1}(\mathbf{r})\nabla\phi^l(\mathbf{r}) + 2\phi^l(\mathbf{r}) - \phi^{l-1}(\mathbf{r}). \tag{1.8}$$

Therefore, given the knowledge of $\phi(\mathbf{r},t)$ at $t = l\Delta t$ and $t = (l-1)\Delta t$ for all $\mathbf{r}$, one can deduce $\phi(\mathbf{r},t)$ at $t = (l+1)\Delta t$. In other words, given the initial values of $\phi(\mathbf{r},t)$ at, for example, $t = 0$ and $t = \Delta t$, $\phi(\mathbf{r},t)$ can be deduced for all subsequent times, provided that the time-stepping formula is stable.

At this point, the right-hand side of (1.8) involves the space derivatives. There exist a plethora of ways to approximate and calculate the right-hand side of (1.8) numerically. Here, we shall illustrate the use of the finite-difference method to calculate the right-hand side of (1.8). Before proceeding further, note that the space derivatives on the right-hand side in Cartesian coordinates are

$$\mu(\mathbf{r})\nabla\cdot\mu^{-1}(\mathbf{r})\nabla\phi(\mathbf{r}) = \mu\frac{\partial}{\partial x}\mu^{-1}\frac{\partial}{\partial x}\phi + \mu\frac{\partial}{\partial y}\mu^{-1}\frac{\partial}{\partial y}\phi + \mu\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}\phi. \tag{1.9}$$

Then, one can approximate, using central differencing.

$$\frac{\partial}{\partial z}\phi(x,y,z) \approx \frac{1}{\Delta z}\left[\phi\left(x,y,z+\frac{\Delta z}{2}\right) - \phi\left(x,y,z-\frac{\Delta z}{2}\right)\right], \tag{1.10}$$

Consequently, using central differencing two times,

$$\begin{aligned}\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}\phi(x,y,z) \approx \frac{1}{(\Delta z)^2}\bigg\{&\mu^{-1}\left(z+\frac{\Delta z}{2}\right)\phi(x,y,z+\Delta z)\\ &-\left[\mu^{-1}\left(z+\frac{\Delta z}{2}\right)+\mu^{-1}\left(z-\frac{\Delta z}{2}\right)\right]\phi(x,y,z)\\ &+\mu^{-1}\left(z-\frac{\Delta z}{2}\right)\phi(x,y,z-\Delta z)\bigg\}.\end{aligned} \tag{1.11}$$

Furthermore, after denoting $\phi(x,y,z) = \phi_{m,n,p}$, $\mu(x,y,z) = \mu_{m,n,p}$, on a discretized grid point at $x = m\Delta x$, $y = n\Delta y$, $z = p\Delta z$, we have

$$\begin{aligned}\frac{\partial}{\partial z}\mu^{-1}\frac{\partial}{\partial z}\phi(x,y,z) \approx \frac{1}{(\Delta z)^2}\Big[&\mu^{-1}_{m,n,p+\frac{1}{2}}\phi_{m,n,p+1}\\ &-\left(\mu^{-1}_{m,n,p+\frac{1}{2}}+\mu^{-1}_{m,n,p-\frac{1}{2}}\right)\phi_{m,n,p}+\mu^{-1}_{m,n,p-\frac{1}{2}}\phi_{m,n,p-1}\Big].\end{aligned} \tag{1.12}$$

With similar approximations to the other terms in (1.9), Equation (1.8) becomes

$$\begin{aligned}\phi^{l+1}_{m,n,p} = (\Delta t)^2 c^2_{m,n,p}\mu_{m,n,p}\Bigg\{ &\frac{1}{(\Delta x)^2}\Big[\mu^{-1}_{m+\frac{1}{2},n,p}\phi^l_{m+1,n,p} \\ &-\left(\mu^{-1}_{m+\frac{1}{2},n,p}+\mu^{-1}_{m-\frac{1}{2},n,p}\right)\phi^l_{m,n,p}+\mu^{-1}_{m-\frac{1}{2},n,p}\phi^l_{m-1,n,p}\Big] \\ &+\frac{1}{(\Delta y)^2}\Big[\mu^{-1}_{m,n+\frac{1}{2},p}\phi^l_{m,n+1,p}-\left(\mu^{-1}_{m,n+\frac{1}{2},p}+\mu^{-1}_{m,n-\frac{1}{2},p}\right)\phi^l_{m,n,p} \\ &\quad+\mu^{-1}_{m,n-\frac{1}{2},p}\phi^l_{m,n-1,p}\Big] \\ &+\frac{1}{(\Delta z)^2}\Big[\mu^{-1}_{m,n,p+\frac{1}{2}}\phi^l_{m,n,p+1}-\left(\mu^{-1}_{m,n,p+\frac{1}{2}}+\mu^{-1}_{m,n,p-\frac{1}{2}}\right)\phi^l_{m,n,p} \\ &\quad+\mu^{-1}_{m,n,p-\frac{1}{2}}\phi^l_{m,n,p-1}\Big]\Bigg\}+2\phi^l_{m,n,p}-\phi^{l-1}_{m,n,p}. \qquad (1.13)\end{aligned}$$

The above can be readily implemented on a computer for time stepping. Notice however, that the use of central differencing results in the evaluation of medium property μ at half grid points. This is inconvenient, as the introduction of half grid points increases computer memory requirements. Hence, it is customary to approximate

$$\mu_{m+\frac{1}{2},n,p} \simeq \frac{1}{2}(\mu_{m+1,n,p}+\mu_{m,n,p}), \qquad (1.14)$$

$$\mu_{m+\frac{1}{2},n,p}+\mu_{m-\frac{1}{2},n,p} \simeq 2\mu_{m,n,p}, \qquad (1.15)$$

and so on. Moreover, if μ is a smooth function of space, it is easy to show that the errors in the above approximations are of second order by Taylor series expansions.

For a homogeneous medium, with $\Delta x = \Delta y = \Delta z = \Delta s$, (1.13) becomes

$$\begin{aligned}\phi^{l+1}_{m,n,p} = \left(\frac{\Delta t}{\Delta s}\right)^2 c^2\Big[&\phi^l_{m+1,n,p}+\phi^l_{m-1,n,p}+\phi^l_{m,n+1,p}+\phi^l_{m,n-1,p}+\phi^l_{m,n,p+1} \\ &+\phi^l_{m,n,p-1}-6\phi^l_{m,n,p}\Big]+2\phi^l_{m,n,p}-\phi^{l-1}_{m,n,p}. \qquad (1.16)\end{aligned}$$

Notice then that with the central-difference approximation, the value of $\phi^{l+1}_{m,n,p}$ is dependent only on $\phi^l_{m,n,p}$, and its nearest neighbors, $\phi^l_{m\pm1,n,p}$, $\phi^l_{m,n\pm1,p}$, $\phi^l_{m,n,p\pm1}$, and $\phi^{l-1}_{m,n,p}$. Moreover, in the finite-difference scheme outlined above, no matrix inversion is required at each time step. Such a scheme is also known as an explicit scheme. The use of an explicit scheme is a major advantage of the finite-difference method compared to the finite-element method. Consequently, in order to update N grid points using (1.13) or (1.16), $O(N)$ multiplications are required for each time step. In comparison, $O(N^3)$ multiplications are required to invert an $N \times N$ full matrix, e.g., using Gaussian elimination. The simplicity and efficiency of these algorithms have made them very popular.

1.3 Stability Analysis

The implementation of the finite-difference scheme does not always lead to a stable scheme. Hence, in order for the solution to converge, the time-stepping scheme must at least be stable. Consequently, it is useful to find the condition under which a numerical finite-difference scheme is stable. To do this, one performs the von Neumann stability analysis (von Neumann 1943) on Equation (1.16). We will assume the medium to be homogeneous to simplify the analysis.

As shown previously, any wave can be expanded in terms of a spectrum of plane waves. So if a scheme is not stable for a plane wave, it would not be stable for any wave. Consequently, to perform the stability analysis, we assume a propagating plane wave as a trial solution

$$\phi(x, y, z, t) = A(t)e^{ik_x x + ik_y y + ik_z z}, \tag{1.17}$$

In discretized form, it is just

$$\phi^l_{m,n,p} = A^l e^{ik_x m\Delta s + ik_y n\Delta s + ik_z p\Delta s}. \tag{1.18}$$

Using (1.18), it is easy to show that for the x space derivative,

$$\begin{aligned}\phi^l_{m+1,n,p} - 2\phi^l_{m,n,p} + \phi^l_{m-1,n,p} &= 2[\cos(k_x\Delta s) - 1]\phi^l_{m,n,p} \\ &= -4\sin^2\left(\frac{k_x\Delta s}{2}\right)\phi^l_{m,n,p}.\end{aligned} \tag{1.19}$$

The space derivatives in y and z directions can be similarly derived.

The time derivative can be treated and it is proportional to

$$\frac{\partial^2}{\partial t^2}\phi(\mathbf{r}, t)(\Delta t)^2 \approx \phi^{l+1}_{m,n,p} - 2\phi^l_{m,n,p} + \phi^{l-1}_{m,n,p}. \tag{1.20}$$

Substituting (1.18) into the above, we have the second time derivative being proportional to

$$\frac{\partial^2}{\partial t^2}\phi(\mathbf{r}, t)(\Delta t)^2 \approx (A^{l+1} - 2A^l + A^{l-1})e^{ik_x m\Delta s + ik_y n\Delta s + ik_z p\Delta s} \tag{1.21}$$

To simplify further, one can assume that

$$A^{l+1} = gA^l. \tag{1.22}$$

This is commensurate with assuming that

$$A(t) = A_0 e^{-i\omega t} \tag{1.23}$$

where ω can be complex. In other words, our trial solution (1.17) is also a quasi-time-harmonic signal. If the finite-difference scheme is unstable for such a signal, it is unstable for all signals.

Consequently, the time derivative is proportional to

$$\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t)(\Delta t)^2 \approx (g-2+g^{-1})\phi^l_{m,n,p} \tag{1.24}$$

For the solution (1.18) to be stable, the requirement is for $|g| \leq 1$. Then, using (1.19) and (1.22) in (1.16), and repeating (1.19) for the n and p variables, one obtains

$$\begin{aligned}(g-2+g^{-1})\phi^l_{m,n,p} &= -4\left(\frac{\Delta t}{\Delta s}\right)^2 c^2 \left[\sin^2\left(\frac{k_x \Delta s}{2}\right) + \sin^2\left(\frac{k_y \Delta s}{2}\right)\right. \\ &\qquad\qquad \left. + \sin^2\left(\frac{k_z \Delta s}{2}\right)\right]\phi^l_{m,n,p} \\ &= -4r^2 s^2 \phi^l_{m,n,p}, \end{aligned} \tag{1.25}$$

where

$$r = \left(\frac{\Delta t}{\Delta s}\right)c, \qquad s^2 = \sin^2\left(\frac{k_x \Delta s}{2}\right) + \sin^2\left(\frac{k_y \Delta s}{2}\right) + \sin^2\left(\frac{k_z \Delta s}{2}\right). \tag{1.26}$$

Equation (16) implies that, for nonzero $\phi^l_{m,n,p}$,

$$g^2 - 2g + 4r^2 s^2 g + 1 = 0, \tag{1.27}$$

or that

$$g = (1-2r^2s^2) \pm 2rs\sqrt{(r^2s^2-1)}\,. \tag{1.28}$$

But if

$$r^2 s^2 < 1, \tag{1.29}$$

the second term in (1.28) is pure imaginary, and

$$|g|^2 = (1-2r^2s^2)^2 + 4r^2s^2(1-r^2s^2) = 1, \tag{1.30}$$

or stability is ensured. Since $s^2 \leq 3$ for all k_x, k_y, and k_z, from (1.29), one concludes that the general condition for stability is

$$r < \frac{1}{\sqrt{3}}, \qquad \text{or} \qquad \Delta t < \frac{\Delta s}{c\sqrt{3}}. \tag{1.31}$$

It is clear from the above analysis that for an n-dimensional problem,

$$\Delta t < \frac{\Delta s}{c\sqrt{n}}. \tag{1.32}$$

One may ponder on the meaning of this inequality further: but it is only natural that the time step Δt has to be bounded from above. Otherwise, one arrives at the ridiculous notion that the time step can be arbitrarily large thus violating causality. Moreover, if the grid points of the finite-difference scheme are regarded as a simple cubic lattice, then the distance $\Delta s/\sqrt{n}$ is also the distance between the closest lattice planes through the simple cubic lattice. Notice that the time for the wave to travel between these two lattice planes is $\Delta s/(c\sqrt{n})$. Consequently, the stability criterion (1.32) implies that the time step Δt has to be less than the shortest travel time for the wave between the lattice planes in order to satisfy causality. In other words, if the wave is time-stepped ahead of the time on the right-hand side of (1.32), instability ensues. The above is also known as the CFL (Courant, Friedrichs, and Lewy 1928) stability criterion. It could be easily modified for $\Delta x \neq \Delta y \neq \Delta z$.

The above analysis implies that we can pick a larger time step if the space steps are larger. A larger time step will allow one to complete generating a time-domain response rapidly. However, one cannot arbitrary make the space step large due to grid-dispersion error, as shall be discussed next.

1.4 Grid-Dispersion Error

When a finite-difference scheme is stable, it still may not produce good results because of the errors in the scheme. Hence, it is useful to ascertain the errors in terms of the size of the grid and the time step. An easy error to analyze is the ***grid-dispersion error***. In a homogeneous, dispersionless medium, all plane waves propagate with the same phase velocity. However, in the finite-difference approximation, all plane waves will not propagate at the same phase velocity. As a consequence, a pulse in the time domain, which is a linear superposition of plane waves, will be distorted if the dispersion introduced by the finite-difference scheme is intolerable. Therefore, to make things simpler, we will analyze the grid-dispersion error in a homogeneous free space medium.

To ascertain the grid-dispersion error, we assume that $A^l = e^{-i\omega l \Delta t}$ in (1.18). In this case, the left-hand side of (1.25) becomes

$$\left(e^{-i\omega\Delta t} - 2 + e^{+i\omega\Delta t}\right)\phi^l_{m,n,p} = -4\sin^2\left(\frac{\omega\Delta t}{2}\right)\phi^l_{m,n,p}. \tag{1.33}$$

Then, from Equation (1.25), it follows that

$$\sin\left(\frac{\omega\Delta t}{2}\right) = rs, \tag{1.34}$$

where r and s are given in (1.26). Now, Equation (1.34) governs the relationship between ω and k_x, k_y, and k_z in the finite-difference scheme, and hence, is a dispersion relation.

But if a medium is homogeneous, it is well known that (1.1) has a plane-wave solution of the type given by (1.17) where

$$\omega = c\sqrt{k_x^2 + k_y^2 + k_z^2} = c|\mathbf{k}| = ck. \tag{1.35}$$

where $\mathbf{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$ is the direction of propagation of the plane wave. Defining the phase velocity to be $\omega/k = c$, this phase velocity is isotropic, or the same in all directions. Moreover, it is independent of frequency. But in (1.34), because of the definition of s as given by (1.26), the dispersion relation between ω and $\mathbf{k}$ is not isotropic. This implies that plane waves propagating in different directions will have different phase velocities.

Equation (1.34) departs from Equation (1.35) as a consequence of the finite-difference approximation. This departure gives rise to errors, which are the consequence of grid dispersion. For example, when c is a constant, (1.35) states that the phase velocities of plane waves of different wavelengths and directions are the same. However, this is not true for (1.34), as shall be shown.

Assuming s small, (1.34) can be written as

$$\frac{\omega \Delta t}{2} = \sin^{-1} rs \cong rs + \frac{r^3 s^3}{6}. \tag{1.36}$$

When Δs is small, using the small argument approximation for the sine function, one obtains from (1.26)

$$s \simeq \frac{\Delta s}{2}(k_x^2 + k_y^2 + k_z^2)^{1/2} \tag{1.37}$$

Equation (1.36), by taking the higher-order Taylor expansion of (1.36), then becomes

$$\frac{\omega \Delta t}{2} \simeq r\frac{\Delta s}{2}(k_x^2 + k_y^2 + k_z^2)^{1/2}\left[1 - \delta\right] \tag{1.38}$$

where (see WFIM)

$$\delta = \frac{\Delta s^2}{24}\frac{k_x^4 + k_y^4 + k_z^4}{k_x^2 + k_y^2 + k_z^2} + \frac{r^2 \Delta s^2}{24}(k_x^2 + k_y^2 + k_z^2) \tag{1.39}$$

Since $\mathbf{k}$ is inversely proportional to wavelength λ, then δ in the correction to the above equation is proportional to $\Delta s^2/\lambda^2$. Hence, to reduce the grid dispersion error, it is necessary to have

$$\left(\frac{\Delta s}{\lambda}\right)^2 \ll 1. \tag{1.40}$$

When this is true, using the fact that $r = c\Delta t/\Delta s$, then (1.38) becomes

$$\frac{\omega}{c} \approx \sqrt{k_x^2 + k_y^2 + k_z^2}\,. \tag{1.41}$$

which is close to the dispersion relation of free space. Consequently, in order for the finite-difference scheme to propagate a certain frequency content accurately, the grid size must be much less than the wavelength of the corresponding frequency. Furthermore, Δt must be chosen so that the CFL stability criterion is met. Hence, the rule of thumb is to choose about 10 to 20 grid points per wavelength.

1.5 The Yee Algorithm

The Yee algorithm (Yee 1966) is specially designed to solve vector electromagnetic field problems on a rectilinear grid. The finite-difference time-domain (FDTD) method (Taflov 1988) when applied to solving electromagnetics problems, usually uses this method. To derive it, Maxwell's equations are first written in Cartesian coordinates:

$$-\frac{\partial B_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}, \tag{1.42}$$

$$-\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \tag{1.43}$$

$$-\frac{\partial B_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}, \tag{1.44}$$

$$\frac{\partial D_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - J_x, \tag{1.45}$$

$$\frac{\partial D_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - J_y, \tag{1.46}$$

$$\frac{\partial D_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - J_z. \tag{1.47}$$

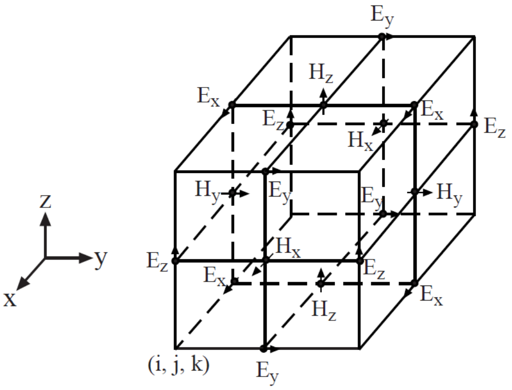


Figure 2: The assignment of fields on a grid in the Yee algorithm.

After denoting $f(n\Delta x, m\Delta y, p\Delta z, l\Delta t) = f^l_{m,n,p}$, and replacing derivatives with central finite-differences in accordance with Figure 2, (1.42) becomes

$$\begin{aligned}\frac{1}{\Delta t}\left[B^{l+\frac{1}{2}}_{x,m,n+\frac{1}{2},p+\frac{1}{2}} - B^{l-\frac{1}{2}}_{x,m,n+\frac{1}{2},p+\frac{1}{2}}\right] &= \frac{1}{\Delta z}\left[E^l_{y,m,n+\frac{1}{2},p+1} - E^l_{y,m,n+\frac{1}{2},p}\right] \\ &\quad - \frac{1}{\Delta y}\left[E^l_{z,m,n+1,p+\frac{1}{2}} - E^l_{z,m,n,p+\frac{1}{2}}\right].\end{aligned} \tag{1.48}$$

Moreover, the above can be repeated for (1.43) and (1.44). Notice that in Figure 2, the electric field is always assigned to the edge center of a cube, whereas the magnetic field is always assigned to the face center of a cube.

In fact, after multiplying (1.48) by $\Delta z \Delta y$, (1.48) is also the approximation of the integral forms of Maxwell's equations when applied at a face of a cube. By doing so, the left-hand side of (1.48) becomes

$$(\Delta y \Delta z / \Delta t) \left[B^{l+\frac{1}{2}}_{x,m,n+\frac{1}{2},p+\frac{1}{2}} - B^{l-\frac{1}{2}}_{x,m,n+\frac{1}{2},p+\frac{1}{2}} \right], \tag{1.49}$$

which is the time variation of the total flux through an elemental area $\Delta y \Delta z$. Moreover, by summing this flux on the six faces of the cube shown in Figure 2, and using the right-hand side of (1.48) and its equivalent, it can be shown that the magnetic flux adds up to zero. Hence, $\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0$ condition is satisfied within the numerical approximations of Yee's algorithm.

Furthermore, a similar approximation of (1.45) leads to

$$\begin{aligned} \frac{1}{\Delta t} \left[D^{l}_{x,m+\frac{1}{2},n,p} - D^{l-1}_{x,m+\frac{1}{2},n,p} \right] &= \frac{1}{\Delta y} \left[H^{l-\frac{1}{2}}_{z,m+\frac{1}{2},n+\frac{1}{2},p} - H^{l-\frac{1}{2}}_{z,m+\frac{1}{2},n-\frac{1}{2},p} \right] \\ &\quad - \frac{1}{\Delta z} \left[H^{l-\frac{1}{2}}_{y,m+\frac{1}{2},n,p+\frac{1}{2}} - H^{l-\frac{1}{2}}_{y,m+\frac{1}{2},n,p-\frac{1}{2}} \right] - J^{l-\frac{1}{2}}_{x,m+\frac{1}{2},n,p}. \end{aligned} \tag{1.50}$$

Also, similar approximations apply for (1.46) and (1.47). In addition, the above has an interpretation similar to (1.48) if one thinks in terms of a cube that is shifted by half a grid point in each direction. Hence, the approximations of (1.45) to (1.47) are consistent with the approximation of $\frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = -\nabla \cdot \mathbf{J}$. This way of alternatively solving for the **B** and **D** fields in tandem while the fields are placed on a staggered grid is also called the leap-frog scheme.

In the above, $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$. Since the magnetic field and the electric field are assigned on staggered grids, μ and ϵ may have to be assigned on staggered grids. This does not usually lead to serious problems if the grid size is small. Alternatively, (1.14) and (1.15) can be used to remove this problem.

By eliminating the **E** or the **H** field from the Yee algorithm, it can be shown that the Yee algorithm is equivalent to finite differencing the vector wave equation directly. Hence, the Yee algorithm is also constrained by the CFL stability criterion.

The following figures show some results of FDTD simulations. Because the answers are in the time-domain, beautiful animations of the fields are also available online:

https://www.remcom.com/xfdtd-3d-em-simulation-software

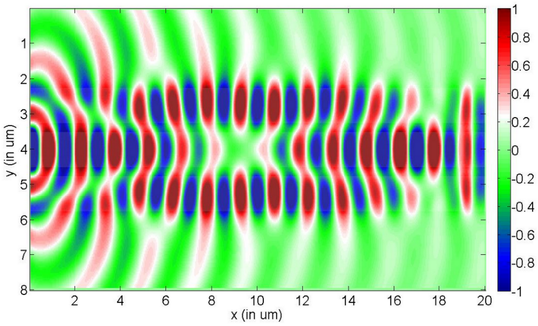


Figure 3: The 2D FDTD simulation of complicated optical waveguides (courtesy of Mathworks).

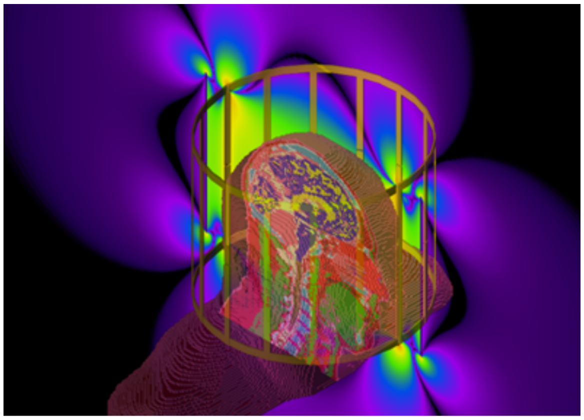


Figure 4: FDTD simulation of human head in a squirrel cage of an MRI (magnetic resonance imaging) system (courtesy of REMCOM).

1.6 Finite-Difference Frequency Domain Method

Unlike electrical engineering, in many fields, nonlinear problems are prevalent. But when we have a linear time-invariant problem, it is simpler to solve the problem in the frequency domain. This is analogous to perform a time Fourier transform of the pertinent linear equations.

Consequently, one can write (1.42) to (1.47) in the frequency domain to remove the time derivatives. Then one can apply the finite difference approximation to the space derivatives using the Yee grid. As a result, one arrives at a matrix equation

$$\overline{\mathbf{A}} \cdot \mathbf{x} = \mathbf{b} \tag{1.51}$$

where $\mathbf{x}$ is an unknown vector containing $\mathbf{E}$ and $\mathbf{H}$ fields, and $\mathbf{b}$ is a source vector that drives the system containing $\mathbf{J}$. Due to the near-neighbor interactions of the fields on the Yee grid, the matrix $\overline{\mathbf{A}}$ is highly sparse and contains $O(N)$ non-zero elements. When an iterative method is used to solve the above equation, the major cost is in performing a matrix-vector product $\overline{\mathbf{A}} \cdot \mathbf{x}$. However, in practice, the matrix $\overline{\mathbf{A}}$ is never generated nor stored. Because of the simplicity of the Yee algorithm, a code can be written to produce the action of $\overline{\mathbf{A}}$ on $\mathbf{x}$. This can greatly result in memory savings: such methods are called matrix-free methods.