ECE 604, Lecture 35

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Contents

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1 Spectral Representations of Sources

A plane wave is an idealization that does not exist in the real world. In practice, waves are nonplanar in nature as they are generated by finite sources, such as antennas and scatterers. For example, a point source generates a spherical wave which is nonplanar. Fortunately, these waves can be expanded in terms of plane waves. Once this is done, then the study of non-plane-wave reflections from a layered medium becomes routine. In the following, we shall show how waves resulting from a point source can be expanded in terms of plane waves.

1.1 A Point Source

From this point onward, we will adopt the $\exp(-i\omega t)$ time convention to be commensurate with the optics and physics literatures.

The spectral decomposition or the plane-wave expansion of the field due to a point source could be derived using Fourier transform technique. First, notice that the scalar wave equation with a point source is

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2\right] \phi(x, y, z) = -\delta(x)\,\delta(y)\,\delta(z). \tag{1.1}$$

The above equation could then be solved in the spherical coordinates, yielding the solution

$$\phi(r) = \frac{e^{ik_0 r}}{4\pi r}. \tag{1.2}$$

Next, assuming that the Fourier transform of $\phi(x, y, z)$ exists, we can write

$$\phi(x, y, z) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} dk_x dk_y dk_z\, \tilde{\phi}(k_x, k_y, k_z) e^{ik_x x + ik_y y + ik_z z}. \tag{1.3}$$

Then we substitute the above into (1.1), after exchanging the order of differentiation and integration, one can convert

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = -k_x^2 - k_y^2 - k_z^2$$

Then, together with the Fourier representation of the delta function, which is

$$\delta(x)\,\delta(y)\,\delta(z) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} dk_x dk_y dk_z\, e^{ik_x x + ik_y y + ik_z z} \tag{1.4}$$

we convert (1.1) into

$$\iiint_{-\infty}^{\infty} dk_x dk_y dk_z \, [k_0^2 - k_x^2 - k_y^2 - k_z^2]\tilde{\phi}(k_x, k_y, k_z) e^{ik_x x + ik_y y + ik_z z} \tag{1.5}$$

$$= - \iiint_{-\infty}^{\infty} dk_x dk_y dk_z \, e^{ik_x x + ik_y y + ik_z z}. \tag{1.6}$$

Since the above is equal for all x, y, and z, we can Fourier inverse transform the above to get

$$\tilde{\phi}(k_x, k_y, k_z) = \frac{-1}{k_0^2 - k_x^2 - k_y^2 - k_z^2}. \tag{1.7}$$

Consequently, we have

$$\phi(x, y, z) = \frac{-1}{(2\pi)^3} \iiint_{-\infty}^{\infty} d\mathbf{k} \, \frac{e^{ik_x x + ik_y y + ik_z z}}{k_0^2 - k_x^2 - k_y^2 - k_z^2}. \tag{1.8}$$

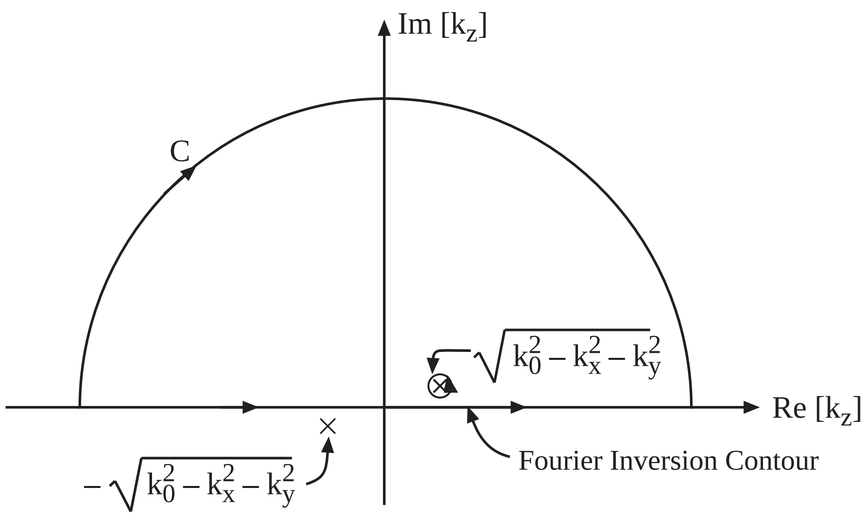

Figure 1: The integration along the real axis is equal to the integration along C plus the residue of the pole at $(k_0^2 - k_x^2 - k_y^2)^{1/2}$, by invoking Jordan's lemma.

In the above, if we examine the k_z integral first, then the integrand has poles at $k_z = \pm(k_0^2 - k_x^2 - k_y^2)^{1/2}$. Moreover, for real k_0, and real values of k_x and k_y, these two poles lie on the real axis, rendering the integral in (1.8) undefined. However, if a small loss is assumed in k_0 such that $k_0 = k_0' + ik_0''$, then the poles are off the real axis (see Figure 1), and the integrals in (1.8) are well-defined.

The reason is that $\phi(x, y, z)$ is not strictly absolutely integrable for a lossless medium, and hence, its Fourier transform may not exist. But the introduction of a small loss also guarantees the radiation condition and the uniqueness of the solution to (1.1), and therefore, the equality of (1.2) and (1.8).

Observe that in (1.8), when $z > 0$, the integrand is exponentially small when $\Im m[k_z] \to \infty$. Therefore, by Jordan's lemma, the integration for k_z over the contour C as shown in Figure 1 vanishes. Then, by Cauchy's theorem, the integration over the Fourier inversion contour on the real axis is the same as integrating over the pole singularity located at $(k_0^2 - k_x^2 - k_y^2)^{1/2}$, yielding the residue of the pole (see Figure 1). Consequently, after doing the residue evaluation, we have

$$\phi(x, y, z) = \frac{i}{2(2\pi)^2} \iint_{-\infty}^{\infty} dk_x dk_y \, \frac{e^{ik_x x + ik_y y + ik'_z z}}{k'_z}, \quad z > 0, \tag{1.9}$$

where $k'_z = (k_0^2 - k_x^2 - k_y^2)^{1/2}$.

Similarly, for $z < 0$, we can add a contour C in the lower-half plane that contributes to zero to the integral, one can deform the contour to pick up the pole contribution. Hence, the integral is equal to the pole contribution at $k'_z = -(k_0^2 - k_x^2 - k_y^2)^{1/2}$ (see Figure 1). As such, the result for all z can be written as

$$\phi(x, y, z) = \frac{i}{2(2\pi)^2} \iint_{-\infty}^{\infty} dk_x dk_y \, \frac{e^{ik_x x + ik_y y + ik'_z |z|}}{k'_z}, \quad \text{all } z. \tag{1.10}$$

By the uniqueness of the solution to the partial differential equation (1.1) satisfying radiation condition at infinity, we can equate (1.2) and (1.10), yielding the identity

$$\frac{e^{ik_0 r}}{r} = \frac{i}{2\pi} \iint_{-\infty}^{\infty} dk_x dk_y \, \frac{e^{ik_x x + ik_y y + ik_z |z|}}{k_z}, \tag{1.11}$$

where $k_x^2 + k_y^2 + k_z^2 = k_0^2$, or $k_z = (k_0^2 - k_x^2 - k_y^2)^{1/2}$. The above is known as the ***Weyl identity*** (Weyl 1919). To ensure the radiation condition, we require that $\Im m[k_z] > 0$ and $\Re e[k_z] > 0$ over all values of k_x and k_y in the integration. Furthermore, Equation (1.11) could be interpreted as an integral summation of plane waves propagating in all directions, including evanescent waves. It is the plane-wave expansion of a spherical wave.

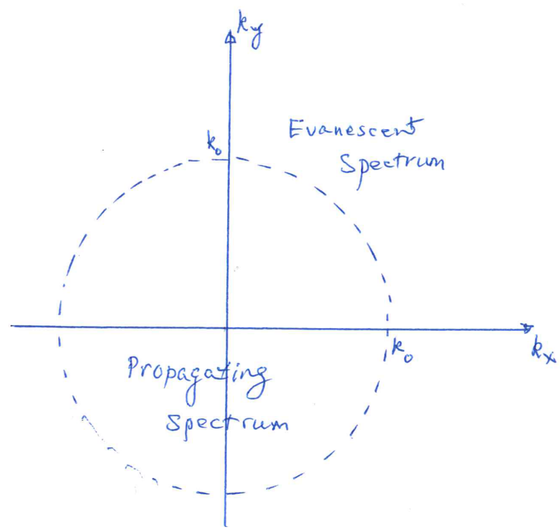

Figure 2: The the wave is propagating for $\mathbf{k}_\rho$ vectors inside the disk, while the wave is evanescent for $\mathbf{k}_\rho$ outside the disk.

One can also interpret the above as a 2D surface integral in the Fourier space over the k_x and k_y variables. When $k_x^2+k_y^2 < k_0^2$, or inside a disk of radius k_0, the waves are propagating waves. But for contributions outside this disk, the waves are evanescent (see Figure 2). And the high Fourier (or spectral) components of the Fourier spectrum correspond to evanescent waves. Since high spectral components are needed to reconstruct the singularity of the Green's function, the evanescent waves are important for reconstructing the singularity of the Green's function.

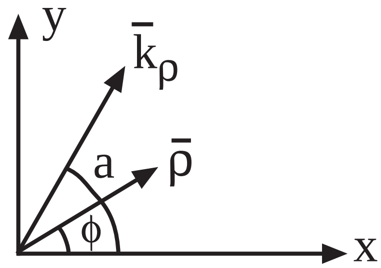

Figure 3: The $\mathbf{k}_\rho$ and the $\boldsymbol{\rho}$ vector on the xy plane.

In (1.11), we can write $\mathbf{k}_\rho = \hat{x}k_\rho \cos\alpha + \hat{y}k_\rho \sin\alpha$, $\boldsymbol{\rho} = \hat{x}\rho\cos\phi + \hat{y}\rho\sin\phi$ (see Figure 3), and $dk_x dk_y = k_\rho dk_\rho\, d\alpha$. Then, $k_x x + k_y y = \mathbf{k}_\rho \cdot \boldsymbol{\rho} = k_\rho \cos(\alpha - \phi)$, and we have

$$\frac{e^{ik_0 r}}{r} = \frac{i}{2\pi} \int_0^\infty k_\rho dk_\rho \int_0^{2\pi} d\alpha \frac{e^{ik_\rho \rho \cos(\alpha-\phi)+ik_z|z|}}{k_z}, \tag{1.12}$$

where $k_z = (k_0^2 - k_x^2 - k_y^2)^{1/2} = (k_0^2 - k_\rho^2)^{1/2}$, where in cylindrical coordinates, in the $\mathbf{k}_\rho$-space, or the Fourier space, $k_\rho^2 = k_x^2 + k_y^2$. Then, using the integral identity for Bessel functions given by[1]

$$J_0(k_\rho \rho) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha\, e^{ik_\rho \rho \cos(\alpha-\phi)}, \tag{1.13}$$

(1.12) becomes

$$\frac{e^{ik_0 r}}{r} = i \int_0^\infty dk_\rho \frac{k_\rho}{k_z} J_0(k_\rho \rho) e^{ik_z|z|}. \tag{1.14}$$

The above is also known as the ***Sommerfeld identity*** (Sommerfeld 1909; 1949, p. 242). Its physical interpretation is that a spherical wave can be expanded as an integral summation of conical waves or cylindrical waves in the ρ direction, times a plane wave in the z direction over all wave numbers k_ρ. This wave is evanescent in the $\pm z$ direction when $k_\rho > k_0$.

By using the fact that $J_0(k_\rho\rho) = 1/2[H_0^{(1)}(k_\rho\rho) + H_0^{(2)}(k_\rho\rho)]$, and the reflection formula that $H_0^{(1)}(e^{i\pi}x) = -H_0^{(2)}(x)$, a variation of the above identity can be derived as

$$\frac{e^{ik_0 r}}{r} = \frac{i}{2} \int_{-\infty}^\infty dk_\rho \frac{k_\rho}{k_z} H_0^{(1)}(k_\rho \rho) e^{ik_z|z|}. \tag{1.15}$$

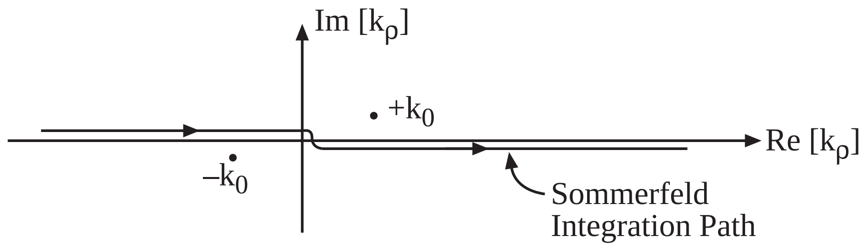

Figure 4: Sommerfeld integration path.

Since $H_0^{(1)}(x)$ has a logarithmic branch-point singularity at $x = 0$,[2] and $k_z = (k_0^2 - k_\rho^2)^{1/2}$ has algebraic branch-point singularities at $k_\rho = \pm k_0$, the integral

[1] See Chew, WFIM, or Whitaker and Watson(1927).

[2] $H_0^{(1)}(x) \sim \frac{2i}{\pi} \ln(x)$, see Chew, WFIM, p. 14, or Abromawitz or Stegun.

in Equation (1.15) is undefined unless we stipulate also the path of integration. Hence, a path of integration adopted by Sommerfeld, which is even good for a lossless medium, is shown in Figure 4. Because of the manner in which we have selected the reflection formula for Hankel functions, i.e., $H_0^{(1)}(e^{i\pi}x) = -H_0^{(2)}(x)$, the path of integration should be above the logarithmic branch-point singularity at the origin.

1.2 Riemann Sheets and Branch Cuts

The function $k_z = (k_0^2 - k_\rho^2)^{1/2}$ in (1.14) and (1.15) are double-value functions because, in taking the square root of a number, two values are possible. In particular, k_z is a double-value function of k_ρ. Consequently, for every point on a complex k_ρ plane in Figure 4, there are two possible values of k_z. Therefore, the integral (1.10) is undefined unless we stipulate which of the two values of k_z is adopted in performing the integration.

A multivalue function is denoted on a complex plane with the help of ***Riemann sheets***. For instance, a double-value function such as k_z is assigned two Riemann sheets to a single complex plane. On one of these Riemann sheets, k_z assumes a value just opposite in sign to the value on the other Riemann sheet. The correct sign for k_z is to pick the square root solution so that $\Im m(k_z) > 0$. This will ensure a decaying wave from the source.

2 A Source on Top of a Layered Medium

It can be shown that plane waves reflecting from a layered medium can be decomposed into TE-type plane waves, where $E_z = 0$, $H_z \neq 0$, and TM-type plane waves, where $H_z = 0$, $E_z \neq 0$.[3] One also sees how the field due to a point source can be expanded into plane waves in Section 1.

In view of the above observations, when a point source is on top of a layered medium, it is then best to decompose its field in terms of waves of TE-type and TM-type. Then, the nonzero component of E_z characterizes TM waves, while the nonzero component of H_z characterizes TE waves. Hence, given a field, its TM and TE components can be extracted readily. Furthermore, if these TM and TE components are expanded in terms of plane waves, their propagations in a layered medium can be studied easily.

The problem of a vertical electric dipole on top of a half space was first solved by Sommerfeld (1909) using Hertzian potentials, which are related to the z components of the electromagnetic field. The work is later generalized to layered media, as discussed in the literature. Later, Kong (1972) suggested the use of the z components of the electromagnetic field instead of the Hertzian potentials.

[3] *Chew, Waves and Fields in Inhomogeneous Media*; *Kong, Electromagnetic Wave Theory.*

2.1 Electric Dipole Fields

The **E** field in a homogeneous medium due to a point current source or a Hertzian dipole directed in the $\hat{\alpha}$ direction, $\mathbf{J} = \hat{\alpha} I\ell\, \delta(\mathbf{r})$, is derivable via the vector potential method or the dyadic Green's function approach. Then, using the dyadic Green's function approach, or the vector/scalar potential approach, the field due to a Hertzian dipole is given by

$$\mathbf{E}(\mathbf{r}) = i\omega\mu \left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k^2} \right) \cdot \hat{\alpha} I\ell\, \frac{e^{ikr}}{4\pi r}, \tag{2.1}$$

where $I\ell$ is the current moment and $k = \omega\sqrt{\mu\epsilon}$, the wave number of the homogeneous medium. Furthermore, from $\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}$, the magnetic field due to a Hertzian dipole is given by

$$\mathbf{H}(\mathbf{r}) = \nabla \times \hat{\alpha} I\ell\, \frac{e^{ikr}}{4\pi r}. \tag{2.2}$$

With the above fields, their TM and TE components can be extracted easily.

2.1.1 (a) Vertical Electric Dipole (VED)

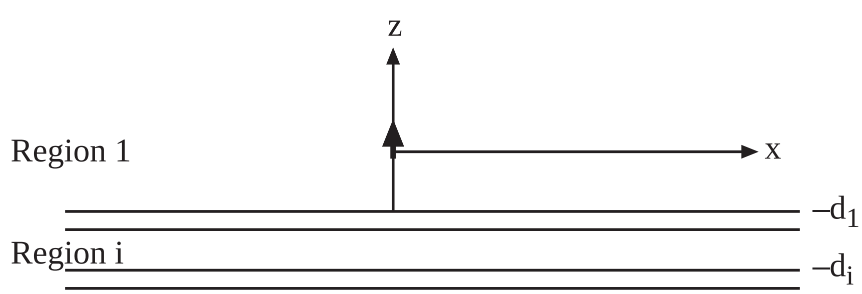

Figure 5: A vertical electric dipole over a layered medium.

A vertical electric dipole shown in Figure 5 has $\hat{\alpha} = \hat{z}$; hence, the TM component of the field is characterized by

$$E_z = \frac{i\omega\mu I\ell}{4\pi k^2} \left(k^2 + \frac{\partial^2}{\partial z^2} \right) \frac{e^{ikr}}{r}, \tag{2.3}$$

and the TE component of the field is characterized by

$$H_z = 0, \tag{2.4}$$

implying the absence of the TE field.

Next, using the Sommerfeld identity (1.15) in the above, and after exchanging the order of integration and differentiation, we have[4]

$$E_z = \frac{-I\ell}{8\pi\omega\epsilon} \int_{-\infty}^{\infty} dk_\rho \, \frac{k_\rho^3}{k_z} H_0^{(1)}(k_\rho \rho) e^{ik_z|z|}, \tag{2.5}$$

after noting that $k_\rho^2 + k_z^2 = k^2$. Notice that now Equation (2.5) expands the z component of the electric field in terms of cylindrical waves in the ρ direction and a plane wave in the z direction. Since cylindrical waves actually are linear superpositions of plane waves, because we can work backward from (1.15) to (1.11) to see this. As such, the integrand in (2.5) in fact consists of a linear superposition of TM-type plane waves. The above is also the ***primary field*** generated by the source.

Consequently, for a VED on top of a stratified medium as shown, the down-going plane wave from the point source will be reflected like TM waves with the generalized reflection coefficient $\tilde{R}_{12}^{TM}$. Hence, over a stratified medium, the field in region 1 can be written as

$$E_{1z} = \frac{-I\ell}{8\pi\omega\epsilon_1} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_{1z}} H_0^{(1)}(k_\rho \rho) \left[e^{ik_{1z}|z|} + \tilde{R}_{12}^{TM} e^{ik_{1z}z + 2ik_{1z}d_1} \right], \tag{2.6}$$

where $k_{1z} = (k_1^2 - k_\rho^2)^{\frac{1}{2}}$, and $k_1^2 = \omega^2 \mu_1 \epsilon_1$, the wave number in region 1.

The phase-matching condition dictates that the transverse variation of the field in all the regions must be the same. Consequently, in the i-th region, the solution becomes

$$\epsilon_i E_{iz} = \frac{-I\ell}{8\pi\omega} \int_{-\infty}^{\infty} dk_\rho \frac{k_\rho^3}{k_{1z}} H_0^{(1)}(k_\rho \rho) A_i \left[e^{-ik_{iz}z} + \tilde{R}_{i,i+1}^{TM} e^{ik_{iz}z + 2ik_{iz}d_i} \right]. \tag{2.7}$$

Notice that Equation (2.7) is now expressed in terms of $\epsilon_i E_{iz}$ because $\epsilon_i E_{iz}$ reflects and transmits like H_{iy}, the transverse component of the magnetic field or TM waves.[5] Therefore, $\tilde{R}_{i,i+1}^{TM}$ and A_i could be obtained using the methods discussed in *Chew, Waves and Fields in Inhomogeneous Media.*

This completes the derivation of the integral representation of the electric field everywhere in the stratified medium. These integrals are known as ***Sommerfeld integrals***. The case when the source is embedded in a layered medium can be derived similarly

[4]By using (1.15) in (2.3), the $\partial^2/\partial z^2$ operating on $e^{ik_z|z|}$ produces a Dirac delta function singularity. Detail discussion on this can be found in the chapter on dyadic Green's function in *Chew, Waves and Fields in Inhomogeneous Media.*

[5]See *Chew, Waves and Fields in Inhomogeneous Media*, p. 46, (2.1.6) and (2.1.7)

2.1.2 (b) Horizontal Electric Dipole (HED)

For a horizontal electric dipole pointing in the x direction, $\hat{\alpha} = \hat{x}$; hence, (2.1) and (2.2) give the TM and the TE components as

$$E_z = \frac{iI\ell}{4\pi\omega\epsilon} \frac{\partial^2}{\partial z \partial x} \frac{e^{ikr}}{r}, \tag{2.8}$$

$$H_z = -\frac{I\ell}{4\pi} \frac{\partial}{\partial y} \frac{e^{ikr}}{r}. \tag{2.9}$$

Then, with the Sommerfeld identity (1.15), we can expand the above as

$$E_z = \pm \frac{iI\ell}{8\pi\omega\epsilon} \cos\phi \int_{-\infty}^{\infty} dk_\rho \, k_\rho^2 H_1^{(1)}(k_\rho \rho) e^{ik_z|z|} \tag{2.10}$$

$$H_z = i\frac{I\ell}{8\pi} \sin\phi \int_{-\infty}^{\infty} dk_\rho \, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) e^{ik_z|z|}. \tag{2.11}$$

Now, Equation (2.10) represents the wave expansion of the TM field, while (2.11) represents the wave expansion of the TE field. Observe that because E_z is odd about $z = 0$ in (2.10), the downgoing wave has an opposite sign from the upgoing wave. At this point, the above are just the primary field generated by the source.

On top of a stratified medium, the downgoing wave is reflected accordingly, depending on its wave type. Consequently, we have

$$E_{1z} = \frac{iI\ell}{8\pi\omega\epsilon_1} \cos\phi \int_{-\infty}^{\infty} dk_\rho \, k_\rho^2 H_1^{(1)}(k_\rho \rho) \left[\pm e^{ik_{1z}|z|} - \tilde{R}_{12}^{TM} e^{ik_{1z}(z+2d_1)} \right], \tag{2.12}$$

$$H_{1z} = \frac{iI\ell}{8\pi} \sin\phi \int_{-\infty}^{\infty} dk_\rho \, \frac{k_\rho^2}{k_{1z}} H_1^{(1)}(k_\rho \rho) \left[e^{ik_{1z}|z|} + \tilde{R}_{12}^{TE} e^{ik_{1z}(z+2d_1)} \right]. \tag{2.13}$$

Notice that the negative sign in front of $\tilde{R}_{12}^{TM}$ in (2.12) follows because the downgoing wave in the primary field has a negative sign.

2.2 Some Remarks

Even though we have arrived at the solutions of a point source on top of a layered medium by heuristic arguments of plane waves propagating through layered media, they can also be derived more rigorously. For example, Equation (2.6) can be arrived at by matching boundary conditions at every interface. The reason why a more heuristic argument is still valid is due to the completeness of Fourier transforms. It is best explained by putting a source over a half space and a scalar problem.

We can expand the scalar field in the upper region as

$$\Phi_1(x,y,z) = \iint_{-\infty}^{\infty} dk_x dk_y \tilde{\Phi}_1(k_x,k_y,z) e^{ik_x x + ik_y y} \tag{2.14}$$

and the scalar field in the lower region as

$$\Phi_2(x,y,z) = \iint_{-\infty}^{\infty} dk_x dk_y \tilde{\Phi}_2(k_x,k_y,z) e^{ik_x x + ik_y y} \tag{2.15}$$

If we require that the two fields be equal to each other at $z = 0$, then we have

$$\iint_{-\infty}^{\infty} dk_x dk_y \tilde{\Phi}_1(k_x,k_y,z=0) e^{ik_x x + ik_y y} = \iint_{-\infty}^{\infty} dk_x dk_y \tilde{\Phi}_2(k_x,k_y,z=0) e^{ik_x x + ik_y y} \tag{2.16}$$

In order to remove the integral, and replace it with a simple scalar problem, one has to impose the above equation for all x and y. Then the completeness of Fourier transform implies that[6]

$$\tilde{\Phi}_1(k_x,k_y,z=0) = \tilde{\Phi}_2(k_x,k_y,z=0) \tag{2.17}$$

The above equation is much simpler than that in (2.16). In other words, due to the completeness of Fourier transform, one can match a boundary condition spectral-component by spectral-component. If the boundary condition is matched for all spectral components, than (2.16) is also true.

[6]Or that we can perform a Fourier inversion on the above integrals.