ECE 604, Lecture 32

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Contents

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1 Image Theory

Image theory can be used to derived closed form solution to boundary value problems when the geometry is simple and has a lot of symmetry. The closed form solutions in turn offer physical insight into the problems.

1.1 A Note on Electrostatic Shielding

For electrostatic problems, a conductive medium suffices to produce surface charges that shield out the electric field from the conductive medium. If the electric field is not zero, then since $\mathbf{J} = \sigma\mathbf{E}$, the electric current inside the conductor will keep flowing until inside the medium $\mathbf{E} = 0$, and no electric current can flow in the conductor. In other words, when the field reaches the quiescent state, the charges redistribute themselves so as to shield out the electric field, and that the total internal electric field, $\mathbf{E} = 0$. And from Faraday's law that tangential $\mathbf{E}$ field is continuous, then $\hat{n} \times \mathbf{E} = 0$ on the conductor surface since $\hat{n} \times \mathbf{E} = 0$ inside the conductor. Figure 1 shows the electric field, in the quiescent state, between two conductors (even though they are not PEC), and the electric field has to be normal to the conductor surfaces.

1.2 Relaxation Time

The time it takes for the charges to move around until they reach their quiescent distribution is called the relaxation time. It is very much similar to the RC time constant of an RC circuit consisting of a resistor in series with a capacitor. It can be proven that this relaxation time is related to ε/σ, but the proof is beyond the scope of this course.

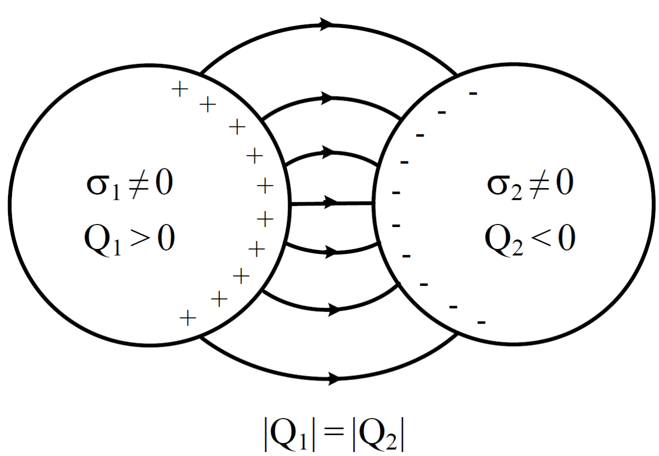

Figure 1:

However, if the conductor charges are induced by an external electric field that is time varying, then the charges have to constantly redistribute/re-orient themselves to try to shield out the incident time-varying electric field. Currents have to constantly flow around the conductor. Then the electric field cannot be zero inside the conductors as shown in Figure 2. In other words, a finite conductor cannot shield out completely a time-varying electric field.

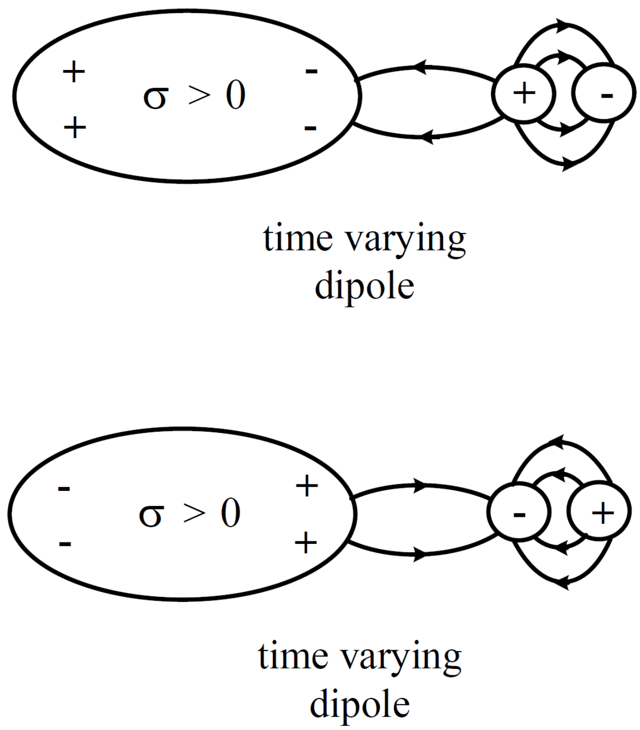

Figure 2:

For a perfect electric conductor (PEC), $\mathbf{E} = 0$ inside with the following argument: Because $\mathbf{J} = \sigma\mathbf{E}$ where $\sigma \to \infty$, let us assume an infinitesimally time-varying electric field in the PEC to begin with. It will yield an infinite electric current, and hence an infinite time-varying magnetic field. A infinite time-varying magnetic field in turn yields an infinite electric field that will drive an electric current, and these fields and current will be infinitely large. This is an unstable sequence of events if it is true. Hence, the only possibility is for the time-varying electromagnetic fields to be zero inside a PEC.

Thus, for the PEC, the charges can re-orient themselves instantaneously on surface when the inducing electric fields from outside are time varying. In other words, the relaxation time is zero. As a consequence, the time-varying electric field **E** is always zero inside PEC, and hence $\hat{n} \times \mathbf{E} = 0$ on the surface of the PEC.

1.3 Electric Charges and Electric Dipoles

Image theory for a flat conductor surface or a half-space is quite easy to derive. To see that, we can start with electro-static theory of putting a positive charge above a flat plane. As mentioned before, for electrostatics, the plane or half-space does not have to be a perfect conductor, but only a conductor (a metal). The tangential static electric field on the surface of the conductor has to be zero.

The tangential static electric field can be canceled by putting an image charge of opposite sign at the mirror location of the original charge. This is shown in Figure 3. Now we can mentally add the total field due to these two charges. When the total static electric field due to the original charge and image charge is sketched, it will look like that in Figure 4. It is seen that the static electric field satisfies the boundary condition at the conductor interface due to symmetry.

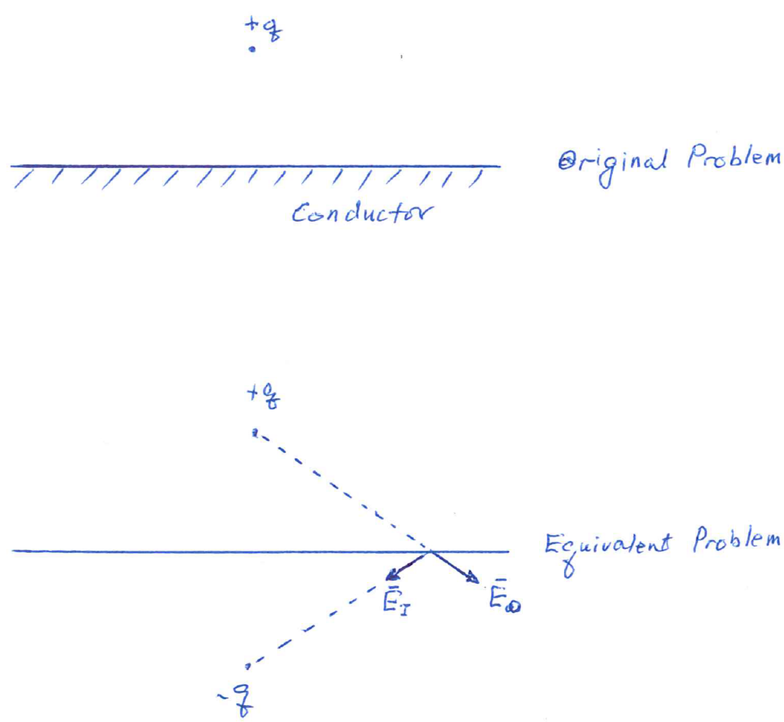

Figure 3:

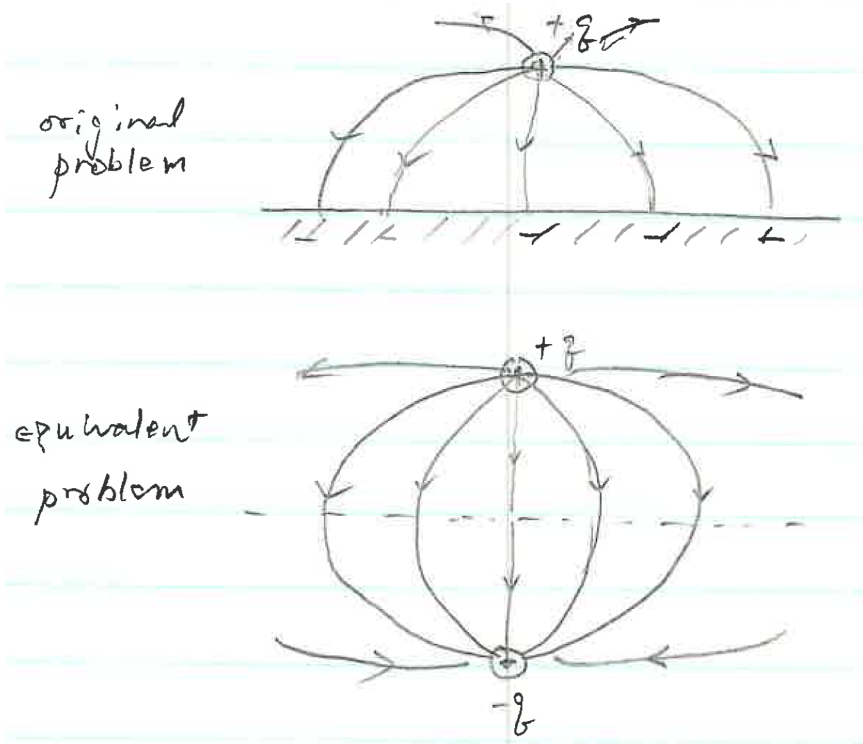

Figure 4:

An electric dipole is made from a positive charge placed in close proximity to a negative charge. Using that a charge reflects to a charge of opposite polarity, from the above, one can easily see that a static horizontal electric dipole reflects to a static horizontal electric dipole of opposite polarity while a static vertical electric dipole reflects to static vertical electric dipole of the same polarity as shown in Figure 5.

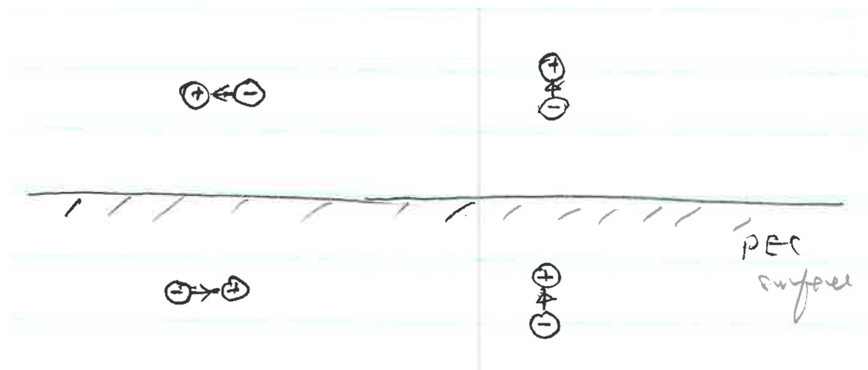

Figure 5:

If this electric dipole is a Hertzian dipole whose field is time-varying, then one needs a PEC half-space to shield out the electric field. Also, the image charges will follow the original dipole charges instantaneously. Then the image theory for static electric dipoles over a half-space still holds true if the dipoles now become Hertzian dipoles.

1.4 Magnetic Charges and Magnetic Dipoles

A static magnetic field can penetrate a conductive medium. This is our experience when we play with a bar magnet over a copper sheet: the magnetic field from the magnet can still be experienced by iron filings put on the other side of the copper sheet.

However, this is not the case for a time-varying magnetic field. Inside a conductive medium, a time-varying magnetic field will produce a time-varying electric field, which in turn produces the conduction current via $\mathbf{J} = \sigma\mathbf{E}$. This is termed eddy current, which by Lenz's law, repels the magnetic field from the conductive medium.[1]

Now, consider a static magnetic field penetrating into a perfect electric conductor, an minute amount of time variation will produce an electric field, which in turn produces an infinitely large eddy current. So the stable state for a static magnetic field inside a PEC is for it to be expelled from the perfect electric conductor. This in fact is what we observe when a magnetic field is brought near a superconductor. Therefore, for the static magnetic field, where $\mathbf{B} = 0$ inside the PEC, then $\hat{n} \cdot \mathbf{B} = 0$ on the PEC surface.

Now, assuming a magnetic monopole exists, it will reflect to itself on a PEC surface so that $\hat{n} \cdot \mathbf{B} = 0$ as shown in Figure 6. Therefore, a magnetic charge reflects to a charge of similar polarity on the PEC surface.

[1] The repulsive force occurs by virtue of energy conservation. Since "work done" is needed to set the eddy current in motion, or to impart kinetic energy to the electrons forming the eddy current, a repulsive force is felt in Lenz's law so that work is done in pushing the magnetic field into the conductive medium.

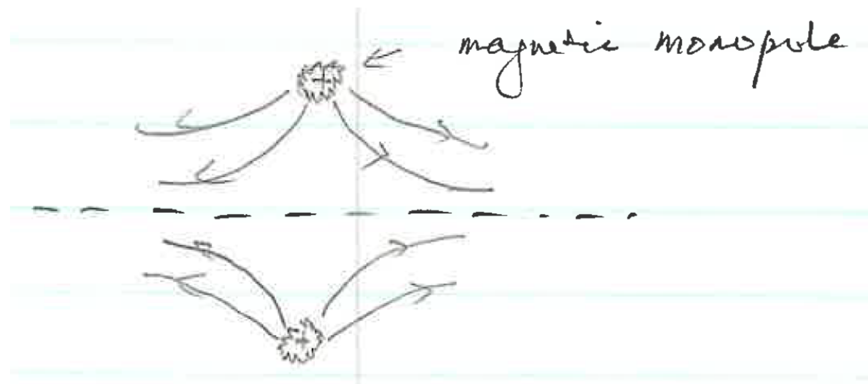

Figure 6:

By extrapolating this to magnetic dipoles, they will reflect themselves to the magnetic dipoles as shown in Figure 7. A horizontal magnetic dipole reflects to a horizontal magnetic dipole of the same polarity, and a vertical magnetic dipole reflects to a vertical magnetic dipole of opposite polarity. Hence, a dipolar bar magnet can be levitated by a superconductor when this magnet is placed closed to it. This is also known as the Meissner effect, which is shown in Figure 8.

A time-varying magnetic dipole can be made from a electric current loop. Over a PEC, a time-varying magnetic dipole will reflect the same way as a static magnetic dipole as shown in Figure 7.

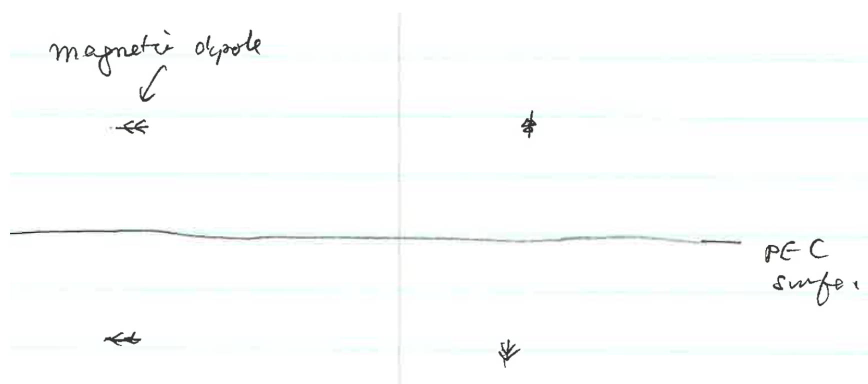

Figure 7:

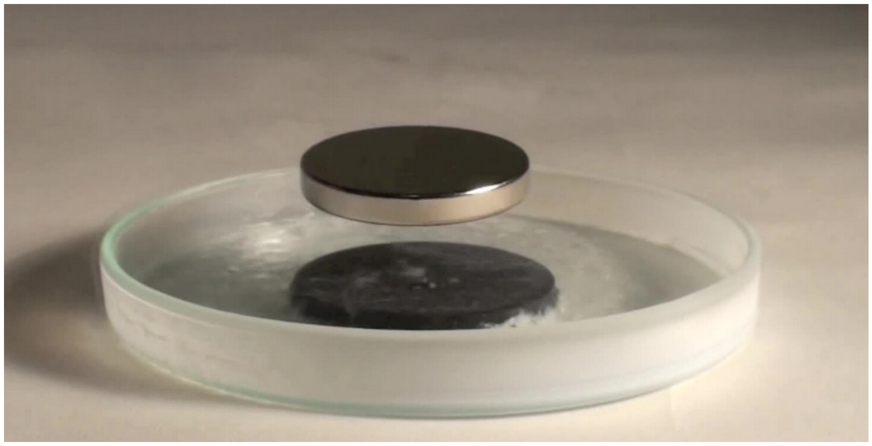

Figure 8: Courtesy of Wikimedia.

1.5 Perfect Magnetic Conductor (PMC) Surfaces

Magnetic conductor does not come naturally in this world since there are no free-moving magnetic charges around. Magnetic monopoles are yet to be discovered. On a PMC surface, by duality, $\hat{n} \times \mathbf{H} = 0$. At low frequency, it can be mimicked by a high μ material. One can see that for magnetostatics, at the interface of a high μ material and air, the magnetic flux is approximately normal to the surface, resembling a PMC surface. High μ materials are hard to find at higher frequencies. Since $\hat{n} \times \mathbf{H} = 0$ on such a surface, no electric current can flow on such a surface. Hence, a PMC can be mimicked by a surface where no surface electric current can flow. This has been achieved in microwave engineering with a mushroom surface as shown in Figure 9. The mushroom structure consisting of a wire and an end-cap, can be thought of as forming an LC tank circuit. Close to the resonance frequency of this tank circuit, the surface of mushroom structures essentially becomes open circuit resembling a PMC.

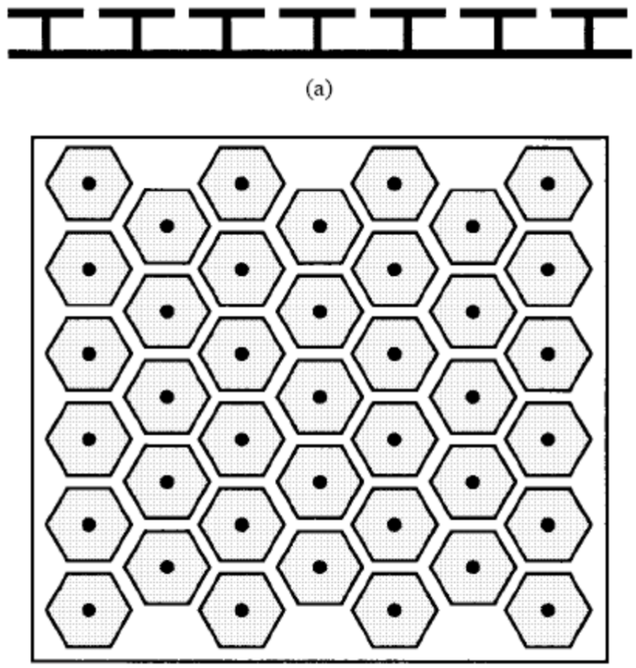

Figure 9: Courtesy of Sievenpiper.

Mathematically, a surface that is dual to the PEC surface is the perfect magnetic conductor (PMC) surface. The magnetic dipole is also dual to the electric dipole. Thus, over a PMC surface, these electric and magnetic dipoles will reflect differently as shown in Figure 10. One can go through Gedanken experiments and verify that the reflection rules are as shown in the figure.

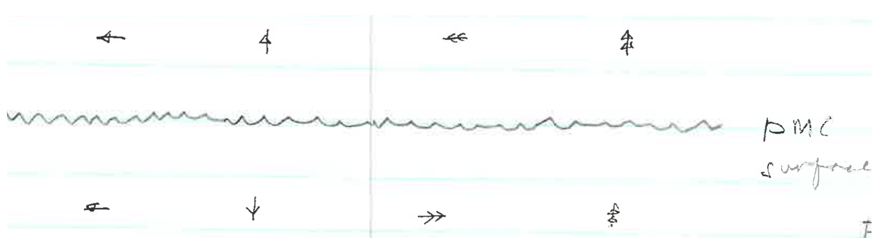

Figure 10:

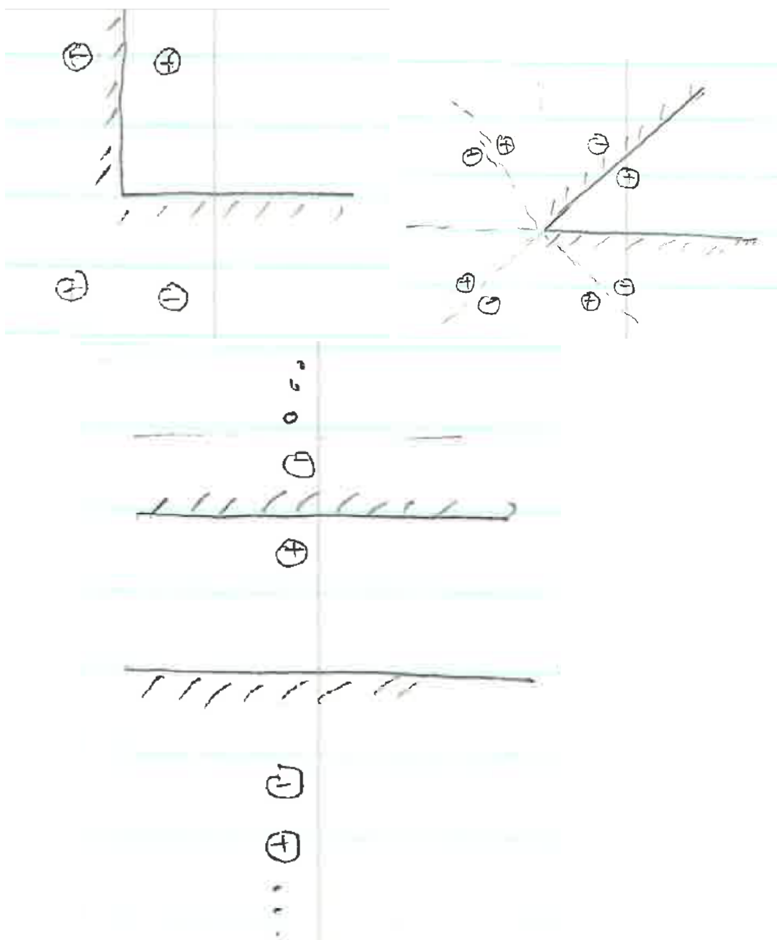

Figure 11:

1.6 Some Special Geometries

For the geometry shown in Figure 11, one can start with electrostatic theory, and convince oneself that $\hat{n} \times \mathbf{E} = 0$ on the metal surface with the placement of charges as shown. For conducting media, they charges will relax to the quiescent distribution after the relaxation time. For PEC surfaces, one can extend these cases to time-varying dipoles because the charges in the PEC medium can re-orient instantaneously (i.e. with zero relaxation time) to shield out or expel the **E** and **H** fields. Again, one can repeat the above exercise for magnetic charges, magnetic dipoles, and PMC surface.

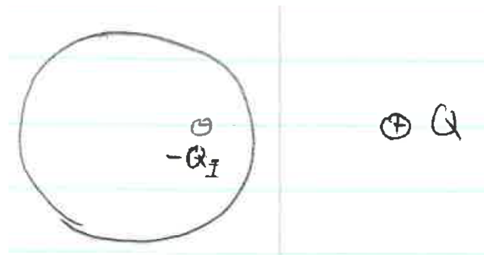

Figure 12:

1.7 Some Special Cases

One curious case is for a static charge placed near a conductive sphere (or cylinder) as shown in Figure 12. This is worked out in p. 48 and p. 49, Ramo et al. A charge of $+Q$ reflects to a charge of $-Q_I$ inside the sphere. For electrostatics, the sphere (or cylinder) need only be a conductor. However, this cannot be generalized to electrodynamics or a time-varying problem, because of the retardation effect: A time-varying dipole or charge will be felt at different points asymmetrically on the surface of the sphere from the original and image charges. Exact cancelation of the tangential electric field cannot occur for time-varying field.

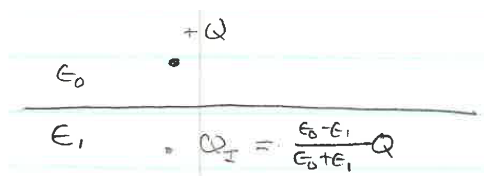

Figure 13:

When a static charge is placed over a dielectric interface, image theory can be used to find the closed form solution. This solution can be derived using Fourier transform technique which we shall learn later. It can also be extended to multiple interfaces. But image theory cannot be used for the electrodynamic case due to the different speed of light in different media, giving rise to different retardation effects.