ECE 604, Lecture 31

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Contents

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1 Equivalence Theorem or Equivalence Principle

Another theorem that is closely related to uniqueness theorem is the equivalence theorem or equivalence principle. We can consider two cases: (1) The inside out case. (2) The outside in case.

1.1 Inside-Out Case

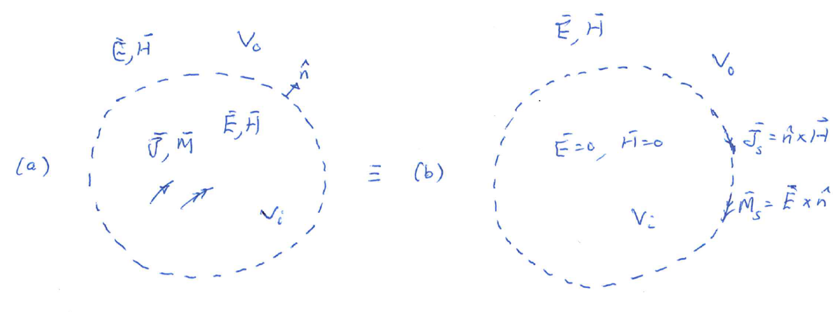

Figure 1:

In this case, we let **J** and **M** be the radiating sources inside a surface S radiating into a region V. They produce **E** and **H** everywhere. We can construct an equivalence problem by first constructing an imaginary surface S. Then impressed surface current sources are placed on this surface. They are

$$\mathbf{J}_s = \hat{n} \times \mathbf{H}, \qquad \mathbf{M}_s = \mathbf{E} \times \hat{n} \tag{1.1}$$

Furthermore, the equivalence theorem says that these sources will produce the same **E** and **H** fields in region V or outside S. In order to ensure that $\hat{n} \times \mathbf{H}$ on S in (a) in Figure 2 is the same as $\hat{n} \times \mathbf{H}$ on S in (b) in the same figure, it is necessary that $\mathbf{H} = 0$ inside S. If $\mathbf{H} = 0$, then $\mathbf{E} = 0$ for consistency with Maxwell's equations. Similarly, in this case, $\mathbf{E} \times \hat{n}$ on S in (a) is the same as $\mathbf{E} \times \hat{n}$ on S in (b). As a consequence, $\hat{n} \times \mathbf{H}$ and $\mathbf{E} \times \hat{n}$ on S in both cases are the same.

By the uniqueness theorem, only the equality of one of them $\mathbf{E} \times \hat{n}$, or $\hat{n} \times \mathbf{H}$ ons S, will guarantee that **E** and **H** outside S are the same in both cases. Also, both the fields inside and outside the surface S are Maxwellian, implying that they are solutions to Maxwell's equations. The fact that these equivalent currents generate zero fields inside S is known as the extinction theorem.

1.2 Outside-in Case

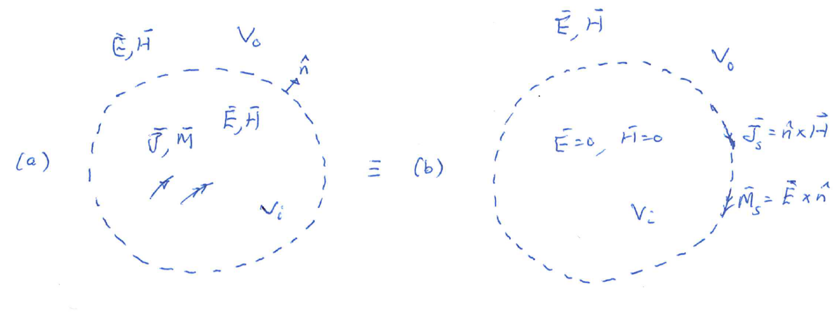

Figure 2:

Similar to before, in order to impress equivalence current $\mathbf{E}_i \times \hat{n}$ and $\hat{n} \times \mathbf{H}_i$ on S that generates the same $\mathbf{E}_i$, $\mathbf{H}_i$ inside S, these equivalence currents have to produce zero fields outside. Then by the uniqueness theorem, the fields $\mathbf{E}_i$, $\mathbf{H}_i$ inside V in both cases are the same. Again, by the extinction theorem, the fields produced by $\mathbf{E}_i \times \hat{n}$ and $\hat{n} \times \mathbf{H}_i$ are zero outside S.

From these two cases, we can create a rich variety of equivalence problems. By linear superposition of the inside-out problem, and the outside-in problem, then a third equivalence problem is shown in Figure 3:

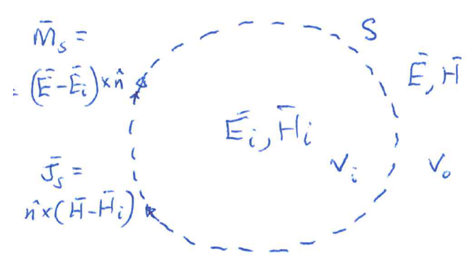

Figure 3:

1.3 Electric Current on a PEC

From reciprocity theorem, it is quite easy to proof that an impressed current on the PEC cannot radiate. Using a Gedanken experiment, since the fields inside S is zero, one can insert an PEC object inside S without disturbing the fields $\mathbf{E}$ and $\mathbf{H}$ outside. As the PEC object grows to snugly fit the surface S, then the electric current $\mathbf{J}_s = \hat{n} \times \mathbf{H}$ does not radiate by reciprocity. Only one of the two currents is radiating, namely, the magnetic current $\mathbf{M}_s = \mathbf{E} \times \hat{n}$ is radiating. This is commensurate with the uniqueness theorem that only the knowledge of $\mathbf{E} \times \hat{n}$ is needed to uniquely determine the fields outside S.

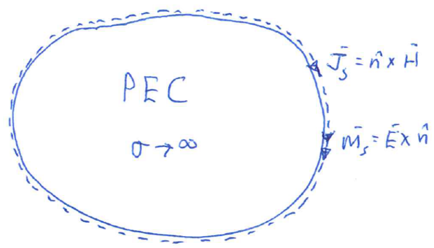

Figure 4:

1.4 Magnetic Current on a PMC

Again, from reciprocity, an impressed magnetic current on a PMC cannot radiate. By the same token, we can perform the Gedanken experiment by inserting a PMC object inside S. It will not alter the fields outside S, as the fields inside S is zero. As the PMC object grows to snugly fit the surface S, only the electric current $\mathbf{J}_s = \hat{n} \times \mathbf{H}$ radiates, and the magnetic current $\mathbf{M}_s = \mathbf{E} \times \hat{n}$ does not radiate. This is again commensurate with the uniqueness theorem that only the knowledge of the $\hat{n} \times \mathbf{H}$ is needed to uniquely determine the fields outside S.

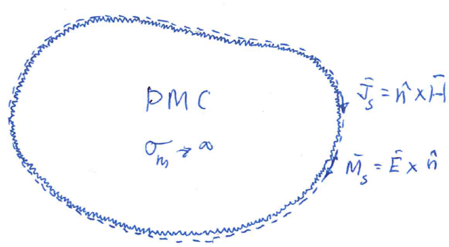

Figure 5:

2 Huygens' Principle and Green's Theorem

Huygens' principle shows how a wave field on a surface S determines the wave field outside the surface S. This concept was expressed by Huygens in the 1600s.

But the mathematical expression of this idea was due to George Green[1] in the 1800s. This concept can be expressed mathematically for both scalar and vector waves. The derivation for the vector wave case is homomorphic to the scalar wave case. But the algebra in the scalar wave case is much simpler. Therefore, we shall first discuss the scalar wave case first, followed by the electromagnetic vector wave case.

2.1 Scalar Waves Case

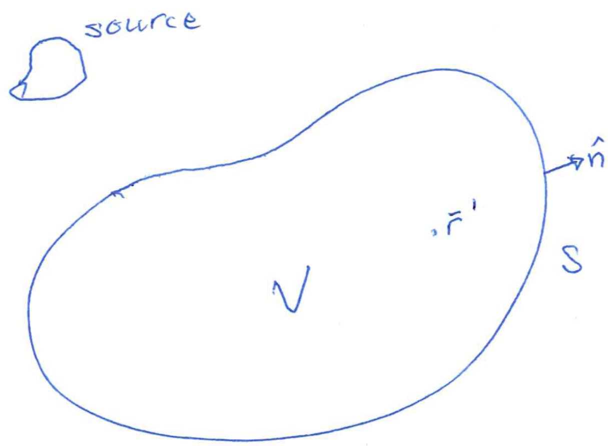

Figure 6:

For a $\psi(\mathbf{r})$ that satisfies the scalar wave equation

$$(\nabla^2 + k^2)\,\psi(\mathbf{r}) = 0, \tag{2.1}$$

the corresponding scalar Green's function $g(\mathbf{r}, \mathbf{r}')$ satisfies

$$(\nabla^2 + k^2)\,g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'). \tag{2.2}$$

Next, we multiply (2.1) by $g(\mathbf{r}, \mathbf{r}')$ and (2.2) by $\psi(\mathbf{r})$. And then, we subtract the resultant equations and integrating over a volume V as shown in Figure 6. There are two cases to consider: when $\mathbf{r}'$ is in V, or when $\mathbf{r}'$ is outside V. Thus, we have

$$\int_V d\mathbf{r}\,[g(\mathbf{r}, \mathbf{r}')\nabla^2\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla^2 g(\mathbf{r}, \mathbf{r}')] = \begin{cases} \psi(\mathbf{r}'), & \text{if } \mathbf{r}' \in V \\ 0, & \text{if } \mathbf{r}' \notin V \end{cases} \tag{2.3}$$

[1]George Green (1793-1841) was self educated and the son of a miller in Nottingham, England.

The right-hand side evaluates to different value depending on where $\mathbf{r}'$ is due to the sifting property of the delta function $\delta(\mathbf{r}-\mathbf{r}')$. Since $g\nabla^2\psi - \psi\nabla^2 g = \nabla\cdot(g\nabla\psi - \psi\nabla g)$, the left-hand side of (2.3) can be rewritten using Gauss' divergence theorem, giving[2]

$$\left.\begin{array}{ll}\text{if } \mathbf{r}'\in V, & \psi(\mathbf{r}')\\ \text{if } \mathbf{r}'\notin V, & 0\end{array}\right\} = \oint_S dS\,\hat{n}\cdot[g(\mathbf{r},\mathbf{r}')\nabla\psi(\mathbf{r}) - \psi(\mathbf{r})\nabla g(\mathbf{r},\mathbf{r}')], \tag{2.4}$$

where S is the surface bounding V. The above is the mathematical expression that once $\psi(\mathbf{r})$ and $\hat{n}\cdot\nabla\psi(\mathbf{r})$ are known on S, then $\psi(\mathbf{r}')$ away from S could be found. This is similar to the expression of equivalence principle where $\hat{n}\cdot\nabla\psi(\mathbf{r})$ and $\psi(\mathbf{r})$ are equivalence sources on the surface S. In acoustics, these are known as monopole layer and double layer sources, respectively. The above is also the mathematical expression of the extinction theorem that says if $\mathbf{r}'$ is outside V, the left-hand side evaluates to 0.

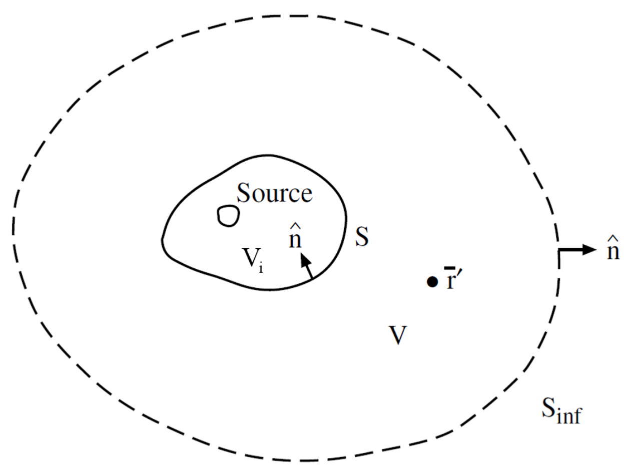

Figure 7:

If the volume V is bounded by S and S_{inf} as shown in Figure 7, then the surface integral in (2.4) should include an integral over S_{inf}. But when $S_{inf}\to\infty$, all fields look like plane wave, and $\nabla\to -\hat{r}jk$ on S_{inf}. Furthermore, $g(\mathbf{r}-\mathbf{r}')\sim O(1/r)$,[3] when $r\to\infty$, and $\psi(\mathbf{r})\sim O(1/r)$, when $r\to\infty$, if $\psi(\mathbf{r})$ is due to a source of finite extent. Then, the integral over S_{inf} in (2.4) vanishes, and (2.4) is valid for the case shown in Figure 7 as well. Here, the field outside

[2]The equivalence of the volume integral in (2.3) to the surface integral in (2.4) is also known as Green's theorem.

[3]The symbol "O" means "of the order."

S at $\mathbf{r}'$ is expressible in terms of the field on S. This is similar to the inside-out equivalence principle we have discussed previously.

Notice that in deriving (2.4), $g(\mathbf{r}, \mathbf{r}')$ has only to satisfy (2.2) for both $\mathbf{r}$ and $\mathbf{r}'$ in V but no boundary condition has yet been imposed on $g(\mathbf{r}, \mathbf{r}')$. Therefore, if we further require that $g(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r} \in S$, then (2.4) becomes

$$\psi(\mathbf{r}') = -\oint_S dS\, \psi(\mathbf{r})\, \hat{n} \cdot \nabla g(\mathbf{r}, \mathbf{r}'), \qquad \mathbf{r}' \in V. \tag{2.5}$$

On the other hand, if require additionally that $g(\mathbf{r}, \mathbf{r}')$ satisfies (2.2) with the boundary condition $\hat{n} \cdot \nabla g(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r} \in S$, then (2.4) becomes

$$\psi(\mathbf{r}') = \oint_S dS\, g(\mathbf{r}, \mathbf{r}')\, \hat{n} \cdot \nabla \psi(\mathbf{r}), \qquad \mathbf{r}' \in V. \tag{2.6}$$

Equations (2.4), (2.5), and (2.6) are various forms of Huygens' principle, or equivalence principle for scalar waves (acoustic waves) depending on the definition of $g(\mathbf{r}, \mathbf{r}')$. Equations (2.5) and (2.6) stipulate that only $\psi(\mathbf{r})$ or $\hat{n} \cdot \nabla \psi(\mathbf{r})$ need be known on the surface S in order to determine $\psi(\mathbf{r}')$. The above are analogous to the PEC and PMC equivalence principle considered previously. (Note that in the above derivation, k^2 could be a function of position as well.)

2.2 Electromagnetic Waves Case

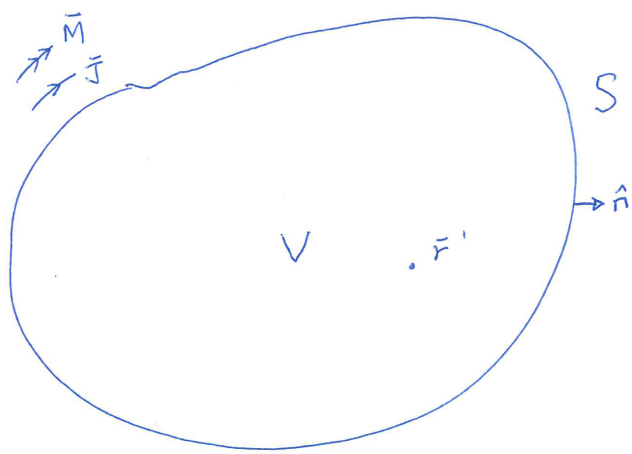

Figure 8:

In a source-free region, an electromagnetic wave satisfies the vector wave equation

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\, \mathbf{E}(\mathbf{r}) = 0. \tag{2.7}$$

The analogue of the scalar Green's function for the scalar wave equation is the dyadic Green's function for the electromagnetic wave case. Moreover, the dyadic Green's function satisfies the equation[4]

$$\nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') - k^2\, \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \overline{\mathbf{I}}\, \delta(\mathbf{r}-\mathbf{r}'). \tag{2.8}$$

It can be shown by direct back substitution that the dyadic Green's function in free space is

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) g(\mathbf{r}-\mathbf{r}') \tag{2.9}$$

The above allows us to derive the vector Green's theorem.

Then, after post-multiplying (2.7) by $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$, pre-multiplying (2.8) by $\mathbf{E}(\mathbf{r})$, subtracting the resultant equations and integrating the difference over volume V, considering two cases as we did for the scalar wave case, we have

$$\left.\begin{array}{ll} \text{if } \mathbf{r}' \in V, & \mathbf{E}(\mathbf{r}') \\ \text{if } \mathbf{r}' \notin V, & 0 \end{array}\right\} = \int_V dV\, \Big[\mathbf{E}(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \\ - \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\Big]. \tag{2.10}$$

Next, using the vector identity that[5]

$$-\nabla \cdot \left[\mathbf{E}(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') + \nabla \times \mathbf{E}(\mathbf{r}) \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right] \\ = \mathbf{E}(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') - \nabla \times \nabla \times \mathbf{E}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}'), \tag{2.11}$$

Equation (2.10), with the help of Gauss' divergence theorem, can be written as

$$\left.\begin{array}{ll} \text{if } \mathbf{r}' \in V, & \mathbf{E}(\mathbf{r}') \\ \text{if } \mathbf{r}' \notin V, & 0 \end{array}\right\} = -\oint_S dS\, \hat{n} \cdot \left[\mathbf{E}(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') + \nabla \times \mathbf{E}(\mathbf{r}) \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right] \\ = -\oint_S dS\, \left[\hat{n} \times \mathbf{E}(\mathbf{r}) \cdot \nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') + i\omega\mu\, \hat{n} \times \mathbf{H}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right]. \tag{2.12}$$

The above is just the vector analogue of (2.4). Since $\mathbf{E} \times \hat{n}$ and $\hat{n} \times \mathbf{H}$ are associated with surface magnetic current and surface electric current, respectively,

[4]A dyad is an outer product between two vectors, and it behaves like a tensor, except that a tensor is more general than a dyad. A purist will call the above a tensor Green's function, as the above is not a dyad in its strictest definition.

[5]This identity can be established by using the identity $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}$. We will have to let (2.11) act on a constant vector to convert the dyad into a vector before we can apply this identity. The equality of the volume integral in (2.10) to the surface integral in (2.12) is also known as vector Green's theorem.

the above can be thought of having these equivalent surface currents radiating via the dyadic Green's function. Again, notice that (2.12) is derived via the use of (2.8), but no boundary condition has yet been imposed on $\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ on S even though we have given a closed form solution for the free-space case.

Now, if we require the addition boundary condition that $\hat{n} \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r} \in S$. This corresponds to a point source radiating in the presence of a PEC surface. Then (2.12) becomes

$$\mathbf{E}(\mathbf{r}') = -\oint_S dS\, \hat{n} \times \mathbf{E}(\mathbf{r}) \cdot \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}'), \qquad \mathbf{r}' \in V \tag{2.13}$$

for it could be shown that $\hat{n} \times \mathbf{H} \cdot \overline{\mathbf{G}} = \mathbf{H} \cdot \hat{n} \times \overline{\mathbf{G}}$ implying that the second term in (2.12) is zero. On the other hand, if we require that $\hat{n} \times \nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = 0$ for $\mathbf{r} \in S$, then (2.12) becomes

$$\mathbf{E}(\mathbf{r}') = -i\omega\mu \oint_S dS\, \hat{n} \times \mathbf{H}(\mathbf{r}) \cdot \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}'), \qquad \mathbf{r}' \in V \tag{2.14}$$

Equations (2.13) and (2.14) state that $\mathbf{E}(\mathbf{r}')$ is determined if either $\hat{n} \times \mathbf{E}(\mathbf{r})$ or $\hat{n} \times \mathbf{H}(\mathbf{r})$ is specified on S. This is in agreement with the uniqueness theorem. These are the mathematical expressions of the PEC and PMC equivalence problems we have considered in the previous sections.

The dyadic Green's functions in (2.13) and (2.14) are for a closed cavity since boundary conditions are imposed on S for them. But the dyadic Green's function for an unbounded, homogeneous medium, given in (2.10) can be written as

$$\overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \frac{1}{k^2}[\nabla \times \nabla \times \bar{\mathbf{I}}\, g(\mathbf{r} - \mathbf{r}') - \bar{\mathbf{I}}\, \delta(\mathbf{r} - \mathbf{r}')], \tag{2.15}$$

$$\nabla \times \overline{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \nabla \times \bar{\mathbf{I}}\, g(\mathbf{r} - \mathbf{r}'). \tag{2.16}$$

Then, (2.12), for $\mathbf{r}' \in V$ and $\mathbf{r}' \neq \mathbf{r}$, becomes

$$\mathbf{E}(\mathbf{r}') = -\nabla' \times \oint_S dS\, g(\mathbf{r} - \mathbf{r}')\, \hat{n} \times \mathbf{E}(\mathbf{r}) + \frac{1}{i\omega\epsilon} \nabla' \times \nabla' \times \oint_S dS\, g(\mathbf{r} - \mathbf{r}')\, \hat{n} \times \mathbf{H}(\mathbf{r}). \tag{2.17}$$

The above can be applied to the geometry in Figure 7 where $\mathbf{r}'$ is enclosed in S and S_{inf}. However, the integral over S_{inf} vanishes by virtue of the radiation condition as for (2.4). Then, (2.17) relates the field outside S at $\mathbf{r}'$ in terms of only the field on S. This is similar to the inside-out problem in the equivalence theorem. It is also related to the fact that if the radiation condition is satisfied, then the field outside of the source region is uniquely satisfied. Hence, this is also related to the uniqueness theorem.