ECE 604, Lecture 30

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Contents

[1]The author is indebted to inspiration from E. Kudeki of UIUC for this part of the lecture notes.

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1 Reciprocity Theorem

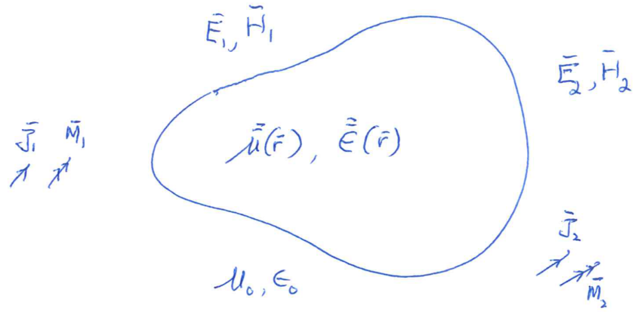

Figure 1:

Reciprocity theorem is like "tit-for-tat" relationship in humans: good-will is reciprocated with good will while ill-will is reciprocated with ill-will. Not exactly as in electromagnetics, this relationship can be expressed precisely and succinctly using mathematics. We shall see how this is done.

Consider a general anisotropic inhomogeneous medium where both $\overline{\boldsymbol{\mu}}(\mathbf{r})$ and $\overline{\varepsilon}(\mathbf{r})$ are described by permeability tensor and permittivity tensor over a finite part of space as shown in Figure 1. This representation of the medium is quite general, and it can include conductive media as well. It can represent complex terrain as well as complicated electronic circuit structures in circuit boards or microchips, and complicated antenna structures.

When only $\mathbf{J}_1$ and $\mathbf{M}_1$ are turned on, they generate fields $\mathbf{E}_1$ and $\mathbf{H}_1$ in this medium. On the other hand, when only $\mathbf{J}_2$ and $\mathbf{M}_2$ are turned on, they generate $\mathbf{E}_2$ and $\mathbf{H}_2$ in this medium. Therefore, the pertinent equations for these two cases are[2]

$$\nabla \times \mathbf{E}_1 = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}_1 - \mathbf{M}_1 \tag{1.1}$$
$$\nabla \times \mathbf{H}_1 = j\omega\overline{\varepsilon} \cdot \mathbf{E}_1 + \mathbf{J}_1 \tag{1.2}$$
$$\nabla \times \mathbf{E}_2 = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}_2 - \mathbf{M}_2 \tag{1.3}$$
$$\nabla \times \mathbf{H}_2 = j\omega\overline{\varepsilon} \cdot \mathbf{E}_2 + \mathbf{J}_2 \tag{1.4}$$

From the above, we can show that

$$\mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 = -j\omega\mathbf{H}_2 \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}_1 - \mathbf{H}_2 \cdot \mathbf{M}_1 \tag{1.5}$$
$$\mathbf{E}_1 \cdot \nabla \times \mathbf{H}_2 = j\omega\mathbf{E}_1 \cdot \overline{\varepsilon} \cdot \mathbf{E}_2 + \mathbf{E}_1 \cdot \mathbf{J}_2 \tag{1.6}$$

[2]The current sources are impressed currents so that they are immutable, and not changed by the environment they are immersed in.

Then,

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2) = \mathbf{H}_2 \cdot \nabla \times \mathbf{E}_1 - \mathbf{E}_1 \cdot \nabla \cdot \mathbf{H}_2$$
$$= -j\omega \mathbf{H}_2 \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}_1 - j\omega \mathbf{E}_1 \cdot \overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}_2 - \mathbf{H}_2 \cdot \mathbf{M}_1 - \mathbf{E}_1 \cdot \mathbf{J}_2 \tag{1.7}$$

By the same token,

$$\nabla \cdot (\mathbf{E}_2 \times \mathbf{H}_1) = -j\omega \mathbf{H}_1 \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}_2 - j\omega \mathbf{E}_2 \cdot \overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}_1 - \mathbf{H}_1 \cdot \mathbf{M}_2 - \mathbf{E}_2 \cdot \mathbf{J}_1 \tag{1.8}$$

If one assumes that

$$\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^t, \qquad \overline{\boldsymbol{\varepsilon}} = \overline{\boldsymbol{\varepsilon}}^t \tag{1.9}$$

or when the tensors are symmetric, then $\mathbf{H}_1 \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}_2 = \mathbf{H}_2 \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}_1$ and $\mathbf{E}_1 \cdot \overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}_2 = \mathbf{E}_2 \cdot \overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}_1$.[3]

Upon subtracting (1.7) and (1.8), one gets

$$\nabla \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = -\mathbf{H}_2 \cdot \mathbf{M}_1 - \mathbf{E}_1 \cdot \mathbf{J}_2 + \mathbf{H}_1 \cdot \mathbf{M}_2 + \mathbf{E}_2 \cdot \mathbf{J}_1 \tag{1.10}$$

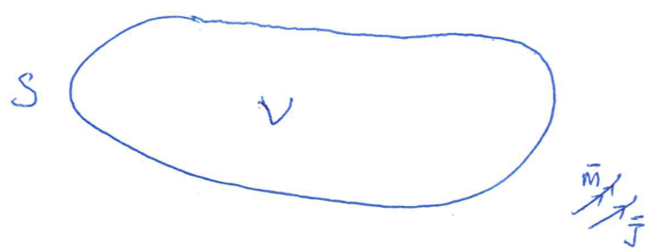

Figure 2:

[3]It is to be noted that in matrix algebra, the dot product between two vectors are often written as $\mathbf{a}^t \cdot \mathbf{b}$, but in the physics literature, the transpose on $\mathbf{a}$ is implied. Therefore, the dot product between two vectors is just written as $\mathbf{a} \cdot \mathbf{b}$.

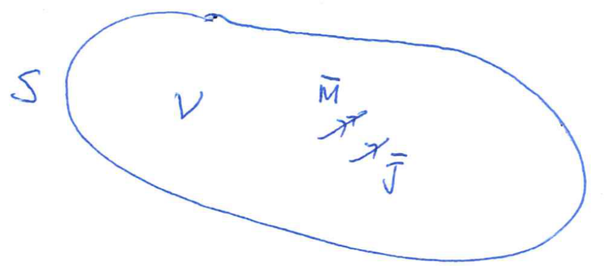

Figure 3:

Now, integrating (1.10) over a volume V bounded by a surface S, and invoking Gauss' divergence theorem, we have the reciprocity theorem that

$$\oiint_S d\mathbf{S} \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = -\iiint_V dV[\mathbf{H}_2 \cdot \mathbf{M}_1 + \mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_1 \cdot \mathbf{M}_2 - \mathbf{E}_2 \cdot \mathbf{J}_1] \quad (1.11)$$

When the volume V contains no sources (see Figure 2), the reciprocity theorem reduces to

$$\oiint_S d\mathbf{S} \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) = 0 \quad (1.12)$$

The above is also called Lorentz reciprocity theorem by some authors.[4]

Next, when the surface S contains all the sources (see Figure 3), then the right-hand side of (1.11) will not be zero. On the other hand, when the surface $S \to \infty$, $\mathbf{E}_1$ and $\mathbf{H}_2$ becomes spherical waves sharing the same $\boldsymbol{\beta}$ vector. Moreover $\omega\mu_0\mathbf{H}_2 = \boldsymbol{\beta} \times \mathbf{E}_2$, $\omega\mu_0\mathbf{H}_1 = \boldsymbol{\beta} \times \mathbf{E}_1$, then

$$\mathbf{E}_1 \times \mathbf{H}_2 \sim \mathbf{E}_1 \times (\boldsymbol{\beta} \times \mathbf{E}_2) = \mathbf{E}_1(\boldsymbol{\beta} \cdot \mathbf{E}_2) - \boldsymbol{\beta}(\mathbf{E}_1 \cdot \mathbf{E}_2) \quad (1.13)$$

$$\mathbf{E}_2 \times \mathbf{H}_1 \sim \mathbf{E}_2 \times (\boldsymbol{\beta} \times \mathbf{E}_1) = \mathbf{E}_2(\boldsymbol{\beta} \cdot \mathbf{E}_1) - \boldsymbol{\beta}(\mathbf{E}_2 \cdot \mathbf{E}_1) \quad (1.14)$$

But $\boldsymbol{\beta} \cdot \mathbf{E}_2 = \boldsymbol{\beta} \cdot \mathbf{E}_1 = 0$ in the far field because the spherical waves emanated by the sources resemble a plane wave, and the $\boldsymbol{\beta}$ vectors are parallel to each other. Therefore, the two terms on the left-hand side of (1.11) cancel each other, and it vanishes when $S \to \infty$. Furthermore, they cancel each other so that the remnant field vanishes faster than $1/r^2$. This is necessary as the surface area S is growing larger and proportional to r^2.

As a result, (1.11) can be rewritten simply as

$$\int_V dV[\mathbf{E}_2 \cdot \mathbf{J}_1 - \mathbf{H}_2 \cdot \mathbf{M}_1] = \int_V dV[\mathbf{E}_1 \cdot \mathbf{J}_2 - \mathbf{H}_1 \cdot \mathbf{M}_2] \quad (1.15)$$

[4] Harrington, Time-Harmonic Electric Field.

The inner product symbol is often used to rewrite the above as

$$\langle \mathbf{E}_2, \mathbf{J}_1 \rangle - \langle \mathbf{H}_2, \mathbf{M}_1 \rangle = \langle \mathbf{E}_1, \mathbf{J}_2 \rangle - \langle \mathbf{H}_1, \mathbf{M}_2 \rangle \tag{1.16}$$

where the inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \int_V dV \mathbf{A}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r})$.

The above inner product is also called reaction, a concept introduced by Rumsey. The above is rewritten as

$$\langle 2, 1 \rangle = \langle 1, 2 \rangle \tag{1.17}$$

where

$$\langle 2, 1 \rangle = \langle \mathbf{E}_2, \mathbf{J}_1 \rangle - \langle \mathbf{H}_2, \mathbf{M}_1 \rangle \tag{1.18}$$

The concept of inner product or reaction can be thought of as a kind of "measurement". The reciprocity theorem can be stated as that the fields generated by sources 2 as "measured" by sources 1 is equal to fields generated by sources 1 as "measured" by sources 2. This measurement concept is more lucid if we think of these sources as Hertzian dipoles.

1.1 Conditions for Reciprocity

It is seen that the above proof hinges on (1.9). In other words, the anisotropic medium has to be described by symmetric tensors. This include conductive media, but not gyrotropic media. A ferrite biased by a magnetic field is often used in electronic circuits, and it corresponds to a gyrotropic medium. Also, our starting equations (1.1) to (1.4) assume that the medium and the equations are linear time invariant so that Maxwell's equations can be written down in the frequency domain easily.

1.2 Application to a Two-Port Network

Figure 4:

The reciprocity theorem can be used to distill and condense the interaction between two antennas over a complex terrain as long as the terrain comprises reciprocal media. In Figure 4, we assume that antenna 1 is driven by current $\mathbf{J}_1$ while antenna 2 is driven by current $\mathbf{J}_2$. Since the system is linear time invariant, it can be written as the interaction between two ports as in circuit theory as shown in Figure 5. Assuming that these two ports are small compared to wavelengths, then we can apply circuit concepts like potential theory at the ports.

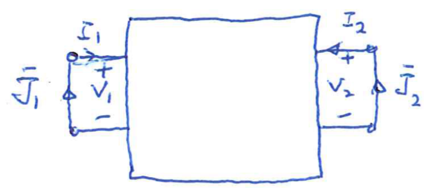

Figure 5:

Focusing on a two-port network as shown in Figure 5, we have

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{1.19}$$

Then assuming that the port 2 is turned on with $\mathbf{J}_2 \neq 0$, and port 1 is turned off with $\mathbf{J}_1 = 0$. In other words, port 1 is open circuit, and the source $\mathbf{J}_2$ will produce an electric field $\mathbf{E}_2$ at port 1. Consequently,

$$\langle \mathbf{E}_2, \mathbf{J}_1 \rangle = \int_V dV(\mathbf{E}_2 \cdot \mathbf{J}_1) = I_1 \int_{\text{Port 1}} \mathbf{E}_2 \cdot d\mathbf{l} = -I_1 V_1^{oc} \tag{1.20}$$

Even though port 1 is assumed to be off, the $\mathbf{J}_1$ to be used above is the $\mathbf{J}_1$ when port 1 is turned on. Given that the port is in the circuit physics regime, then the current $\mathbf{J}_1$ is a constant current at the port when it is turned on. The current $\mathbf{J}_1 = \hat{l}I_1/A$ where A is the cross-sectional area of the wire, and $\hat{l}$ is a unit vector aligned with the axis of the wire. The volume integral $dV = Adl$, and hence the second equality follows above, where $d\mathbf{l} = \hat{l}dl$. Since $\int_{\text{Port 1}} \mathbf{E}_2 \cdot d\mathbf{l} = -V_1^{oc}$, we have the last equality above.

We can repeat the derivation with port 2 to arrive at

$$\langle \mathbf{E}_1, \mathbf{J}_2 \rangle = I_2 \int_{\text{Port 2}} \mathbf{E}_1 \cdot d\mathbf{l} = -I_2 V_2^{oc} \tag{1.21}$$

But from (1.19), we can set the pertinent currents to zero to find these open circuit voltages. Therefore, $V_1^{oc} = Z_{12}I_2$, $V_2^{oc} = Z_{21}I_1$. Since $I_1V_1^{oc} = I_2V_2^{oc}$ by the reaction concept or by reciprocity, then $Z_{12} = Z_{21}$. The above analysis can be easily generalized to an N-port network.

The simplicity of the above belies its importance. The above shows that the reciprocity concept in circuit theory is a special case of reciprocity theorem. The terrain can also be replaced by complex circuits as in a circuit board, as long as the materials are reciprocal, linear and time invariant. The complex terrain can also be replaced by complex antenna structures.

1.3 Voltage Sources in Electromagnetics

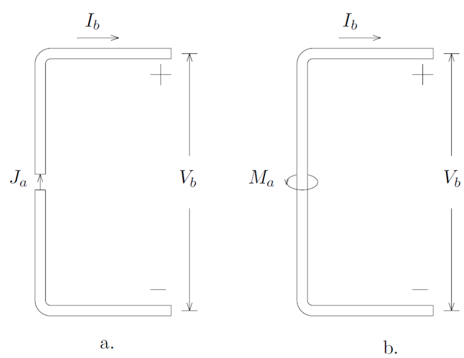

Figure 6: Courtesy of Kong, ELectromagnetic Wave Theory.

In the above discussions, we have used current sources in reciprocity theorem to derive certain circuit concepts. Before we end this section, it is prudent to mention how voltage sources are modeled in electromagnetic theory. The use of the impressed currents so that circuit concepts can be applied is shown in Figure 6. The antenna in (a) is driven by a current source. But a magnetic current can be used as a voltage source in circuit theory as shown by Figure 6b. By using duality concept, an electric field has to curl around a magnetic current just in Ampere's law where magnetic field curls around an electric current. This electric field will cause a voltage drop between the metal above and below the magnetic current loop making it behave like a voltage source.[5]

1.4 Hind Sight

The proof of reciprocity theorem for Maxwell's equations is very deeply to the symmetry of the operator involved. We can see this from linear algebra. Given a matrix equation driven by two different sources, they can be written as

$$\overline{\mathbf{A}} \cdot \mathbf{x}_1 = \mathbf{b}_1 \tag{1.22}$$

$$\overline{\mathbf{A}} \cdot \mathbf{x}_2 = \mathbf{b}_2 \tag{1.23}$$

We can left dot multiply the first equation with $\mathbf{x}_2$ and do the same with the second equation with $\mathbf{x}_1$ to arrive at

$$\mathbf{x}_2^t \cdot \overline{\mathbf{A}} \cdot \mathbf{x}_1 = \mathbf{x}_2^t \cdot \mathbf{b}_1 \tag{1.24}$$

$$\mathbf{x}_1^t \cdot \overline{\mathbf{A}} \cdot \mathbf{x}_2 = \mathbf{x}_1^t \cdot \mathbf{b}_2 \tag{1.25}$$

[5]More can be found in Jordain and Balmain, Electromagnetic Waves and Radiation Systems.

If $\overline{\mathbf{A}}$ is symmetric, the left-hand side of both equations are equal to each other. Subtracting the two equations, we arrive at

$$\mathbf{x}_2^t \cdot \mathbf{b}_1 = \mathbf{x}_1^t \cdot \mathbf{b}_2 \tag{1.26}$$

The above is analogous to the statement of the reciprocity theorem. The above inner product is that of dot product in matrix theory, but the inner product for reciprocity is that for infinite dimensional space. So if the operators in Maxwell's equations are symmetrical, then reciprocity theorem applies.

1.5 Transmit and Receive Patterns of an Antennna[6]

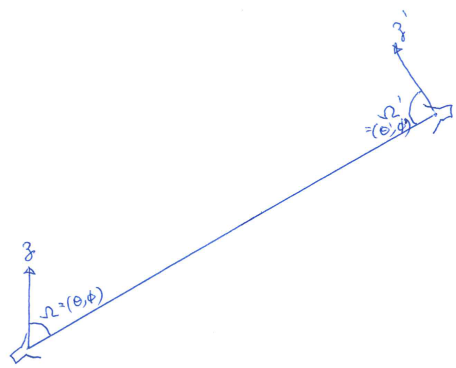

Figure 7:

Reciprocity also implies that the transmit and receive properties of an antenna is similar to each other. Consider an antenna in the transmit mode. Then the radiation power density that it will yield around the antenna is

$$S_{\text{rad}} = \frac{P_t}{4\pi r^2} G(\theta, \phi) \tag{1.27}$$

where P_t is the total power radiated by the transmit antenna, and $G(\theta, \phi)$ is its directive gain function.

1.5.1 Effective Gain versus Directive Gain

At this juncture, it is important to introduce the concept of effective gain versus directive gain. The effective gain, also called the power gain, is

$$G_e(\theta, \phi) = f_e G(\theta, \phi) \tag{1.28}$$

[6] The author is indebted to inspiration from E. Kudeki of UIUC for this part of the lecture notes.

where f_e is the efficiency of the antenna, a factor less than 1. It accounts for the fact that not all power pumped into the antenna is delivered as radiated power. For instance, power can be loss in the circuits and mismatch of the antenna. Therefore, the correct formula the radiated power density is

$$S_{\text{rad}} = \frac{P_t}{4\pi r^2} G_e(\theta, \phi) \tag{1.29}$$

If this power density is intercepted by a receive antenna, then the receive antenna will see an incident power density as

$$S_{\text{inc}} = S_{\text{rad}} = \frac{P_t}{4\pi r^2} G_e(\theta, \phi) \tag{1.30}$$

The effective area or aperture of a receive antenna is used to characterize its receive property. The power received by such an antenna is then

$$P_r = S_{\text{inc}} A_e(\theta', \phi') \tag{1.31}$$

where (θ', ϕ') are the angles at which the plane wave is incident upon the receiving antenna (see Figure 7). Combining the above formulas, we have

$$P_r = \frac{P_t}{4\pi r^2} G_e(\theta, \phi) A_e(\theta', \phi') \tag{1.32}$$

Now assuming that the transmit and receive antennas are identical. We swap their roles of transmit and receive, and also the circuitries involved in driving the transmit and receive antennas. Then,

$$P_r = \frac{P_t}{4\pi r^2} G_e(\theta', \phi') A_e(\theta, \phi) \tag{1.33}$$

We also assume that the receive antenna, that now acts as the transmit antenna is transmitting in the (θ', ϕ') direction. Moreover, the transmit antenna, that now acts as the receive antenna is receiving in the (θ, ϕ) direction (see Figure 7).

By reciprocity, these two powers are the same, because $Z_{12} = Z_{21}$. Furthermore, since these two antennas are identical, $Z_{11} = Z_{22}$. So by swapping the transmit and receive electronics, the power transmitted and received will not change.

Consequently, we conclude that

$$G_e(\theta, \phi) A_e(\theta', \phi') = G_e(\theta', \phi') A_e(\theta, \phi) \tag{1.34}$$

The above implies that

$$\frac{A_e(\theta, \phi)}{G_e(\theta, \phi)} = \frac{A_e(\theta', \phi')}{G_e(\theta', \phi')} = \text{constant} \tag{1.35}$$

The above Gedanken experiment is carried out for arbitrary angles. Therefore, the constant is independent of angles. Moreover, this constant is independent of the size, shape, and efficiency of the antenna, as we have not used their shape, size, and efficienty in the above discussion. One can repeat the above for a Hertzian dipole, wherein the mathematics of calculating P_r and P_t is a lot simpler. This constant is found to be $\lambda^2/(4\pi)$.[7] Therefore, an interesting relationship between the effective aperture (or area) and the directive gain function is that

$$A_e(\theta, \phi) = \frac{\lambda^2}{4\pi} G_e(\theta, \phi) \tag{1.36}$$

One amusing point about the above formula is that the effective aperture, say of a Hertzian dipole, becomes very large when the frequency is low, or the wavelength is very long. Of course, this cannot be physically true, and I will let you meditate on this paradox and muse over this point.

[7]See Kong, p. 700. The derivation is for 100% efficient antenna. A thermal equilibrium argument is used in Wikipedia as well.