ECE 604, Lecture 25

Wed, Mar 8, 2019

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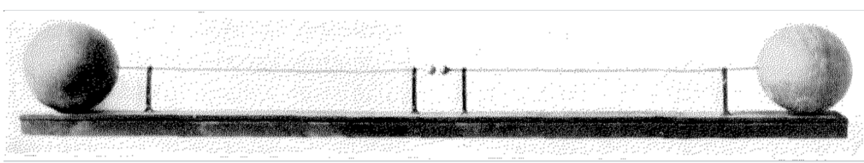
1 Radiation by a Hertzian Dipole

Radiation by arbitrary sources is an important problem for antennas and wireless communications. We will start with studying the Hertzian dipole which is the simplest of a radiation source we can think of.

1.1 History

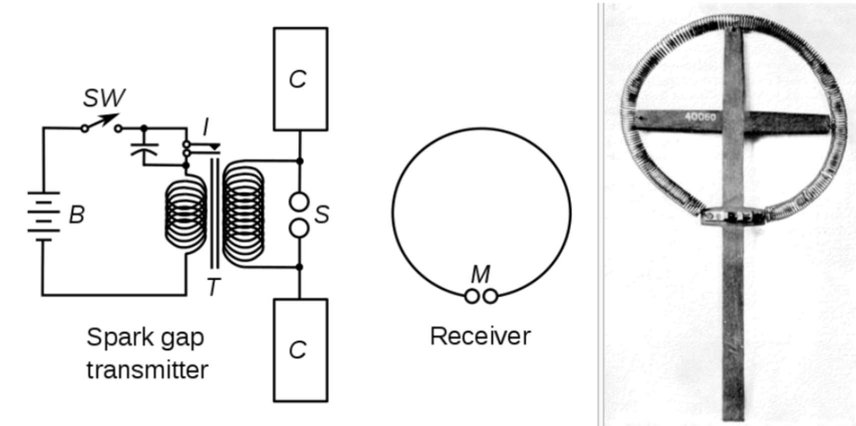
The original historic Hertzian dipole experiment is shown in Figure 1. It was done in 1887 by Heinrich Hertz. The schematics for the original experiment is also shown in Figure 2.

A metallic sphere has a capacitance in closed form with respect to infinity or a ground plane. Hertz could use those knowledge to estimate the capacitance of the sphere, and also, he could estimate the inductance of the leads that are attached to the dipole, and hence, the resonance frequency of his antenna. The large sphere is needed to have a large capacitance, so that current can be driven through the wires. As we shall see, the radiation strength of the dipole is proportional to $p = ql$ the dipole moment. To get a large dipole moment, the current flowing in the lead should be large.



Hertz's first radio transmitter: a dipole resonator consisting of a pair of one meter copper wires with a 7.5 mm spark gap between them, ending in 30 cm zinc spheres.[12] When an induction coil applied a high voltage between the two sides, sparks across the spark gap created standing waves of radio frequency current in the wires, which radiated radio waves. The frequency of the waves was roughly 50 MHz, about that used in modern television transmitters.

Figure 1:



Hertz's 1887 apparatus for generating and detecting radio waves: a spark transmitter *(left)* consisting of a dipole antenna with a spark gap *(S)* powered by high voltage pulses from a Ruhmkorff coil *(T)*, and a receiver *(right)* consisting of a loop antenna and spark gap.

One of Hertz's radio wave receivers: a loop antenna with an adjustable micrometer spark gap *(bottom)*.[12]

Figure 2:

1.2 Approximation by a Point Source

A Hertzian dipole is a dipole which is much smaller than the wavelength under consideration so that we can approximate it by a point current distribution, mathematically given by

$$\mathbf{J}(\mathbf{r}) = \hat{z}Il\delta(\mathbf{r}) \tag{1.1}$$

The dipole may look like the following schematically;

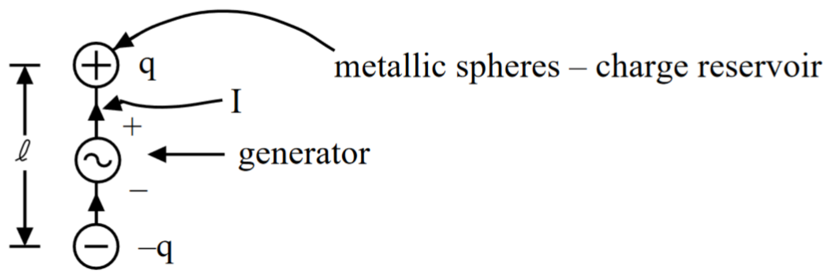


Figure 3:

In (1.1), l is the effective length of the dipole so that the dipole moment $p = ql$. The charge q is varying time harmonically because it is driven by the generator. Since

$$\frac{dq}{dt} = I,$$

we have

$$Il = \frac{dq}{dt} l = j\omega q l = j\omega p \tag{1.2}$$

for a Hertzian dipole. We have learnt previously that the vector potential is related to the current as follows:

$$\mathbf{A}(\mathbf{r}) = \mu \iiint d\mathbf{r}' \mathbf{J}(\mathbf{r}') \frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{1.3}$$

Therefore, the corresponding vector potential is given by

$$\mathbf{A}(\mathbf{r}) = \hat{z} \frac{\mu I l}{4\pi r} e^{-j\beta r} \tag{1.4}$$

The magnetic field is obtained, using cylindrical coordinates, as

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} = \frac{1}{\mu} \left(\hat{\rho} \frac{1}{\rho} \frac{\partial}{\partial \phi} A_z - \hat{\phi} \frac{\partial}{\partial \rho} A_z \right) \tag{1.5}$$

where $\frac{\partial}{\partial \phi} = 0$, $r = \sqrt{\rho^2 + z^2}$. In the above,

$$\frac{\partial}{\partial \rho} = \frac{\partial r}{\partial \rho} \frac{\partial}{\partial r} = \frac{\rho}{\sqrt{\rho^2 + z^2}} \frac{\partial}{\partial r} = \frac{\rho}{r} \frac{\partial}{\partial r}.$$

Hence,

$$\mathbf{H} = -\hat{\phi} \frac{\rho}{r} \frac{Il}{4\pi} \left(-\frac{1}{r^2} - j\beta \frac{1}{r} \right) e^{-j\beta r} \tag{1.6}$$

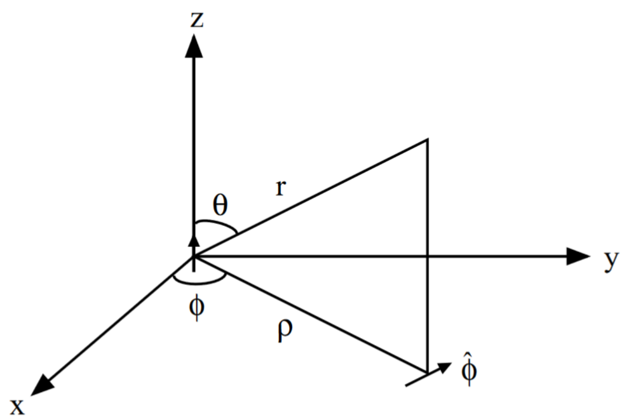


Figure 4:

In spherical coordinates, $\frac{\rho}{r} = \sin\theta$, and (1.6) becomes

$$\mathbf{H} = \hat{\phi}\frac{Il}{4\pi r^2}(1 + j\beta r)e^{-j\beta r}\sin\theta \tag{1.7}$$

The electric field can be derived using Maxwell's equations.

$$\mathbf{E} = \frac{1}{j\omega\epsilon}\nabla\times\mathbf{H} = \frac{1}{j\omega\epsilon}\left(\hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H_\phi - \hat{\phi}\frac{1}{r}\frac{\partial}{\partial r}rH_\phi\right) \tag{1.8}$$

$$= \frac{Ile^{-j\beta r}}{j\omega\epsilon 4\pi r^3}\left[\hat{r}2\cos\theta(1 + j\beta r) + \hat{\theta}\sin\theta(1 + j\beta r - \beta^2 r^2)\right] \tag{1.9}$$

1.3 Case I. Near Field, $\beta r \ll 1$

$$\mathbf{E} \cong \frac{p}{4\pi\epsilon r^3}(\hat{r}2\cos\theta + \hat{\theta}\sin\theta), \qquad \beta r \ll 1 \tag{1.10}$$

$$\mathbf{H} \ll \mathbf{E}, \qquad \text{when } \beta r \ll 1 \tag{1.11}$$

where $p = ql$ is the dipole moment, and βr could be made very small by making $\frac{r}{\lambda}$ small or by making $\omega \to 0$. The above is like the static field of a dipole. The reason being that in the near field, the field varies rapidly, and space derivatives are much larger than the time derivative. For instance,

$$\frac{\partial}{\partial x} \gg \frac{\partial}{c\partial t}$$

1.4 Case II. Far Field (Radiation Field), $\beta r \gg 1$

In this case,

$$\mathbf{E} \cong \hat{\theta} j\omega\mu \frac{Il}{4\pi r} e^{-j\beta r} \sin\theta \tag{1.12}$$

and

$$\mathbf{H} \cong \hat{\phi} j\beta \frac{Il}{4\pi r} e^{-j\beta r} \sin\theta \tag{1.13}$$

Note that $\frac{E_\theta}{H_\phi} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\epsilon}} = \eta_0$. **E** and **H** are orthogonal to each other and are both orthogonal to the direction of propagation, as in the case of a plane wave. A spherical wave resembles a plane wave in the far field approximation.

1.5 Radiation, Power, and Directive Gain Patterns

The time average power flow is given by

$$\langle \mathbf{S} \rangle = \frac{1}{2} \Re e[\mathbf{E} \times \mathbf{H}^*] = \hat{r} \frac{1}{2} \eta_0 |H_\phi|^2 = \hat{r} \frac{\eta_0}{2} \left(\frac{\beta Il}{4\pi r} \right)^2 \sin^2\theta \tag{1.14}$$

The **radiation field pattern** of a Hertzian dipole is the plot of $|\mathbf{E}|$ as a function of θ at a constant $\mathbf{r}$. Hence, it is proportional to $\sin\theta$, and it can be proved that it is a circle.

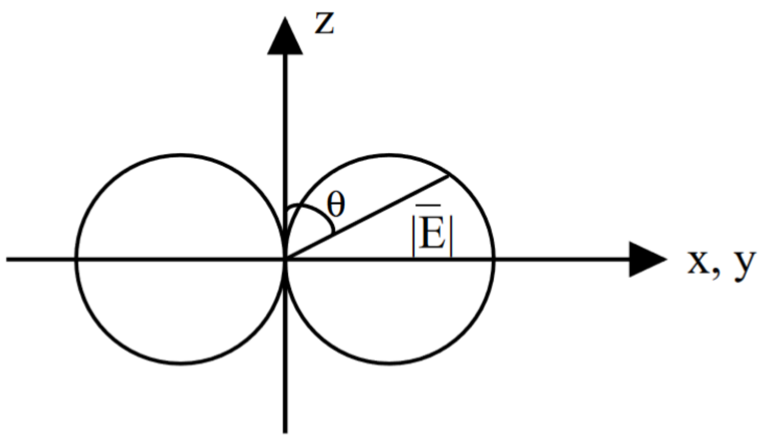


Figure 5:

The **radiation power pattern** is the plot of $\langle S_r \rangle$ at a constant r.

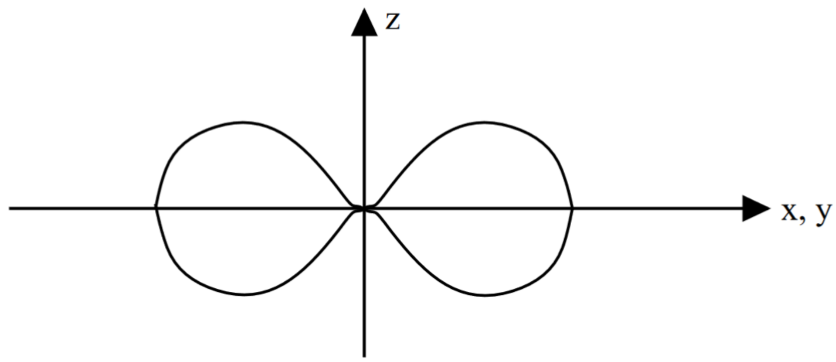


Figure 6:

The total power radiated by a Hertzian dipole is given by

$$P = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta r^2 \sin\theta \langle S_r \rangle = 2\pi \int_0^{\pi} d\theta \frac{\eta_0}{2} \left(\frac{\beta I l}{4\pi} \right)^2 \sin^3\theta \tag{1.15}$$

Since

$$\int_0^{\pi} d\theta \sin^3\theta = -\int_1^{-1} (d\cos\theta)[1 - \cos^2\theta] = \int_{-1}^{1} dx(1 - x^2) = \frac{4}{3} \tag{1.16}$$

then

$$P = \frac{4}{3}\pi\eta_0 \left(\frac{\beta I l}{4\pi} \right)^2 \tag{1.17}$$

The **directive gain** of an antenna, $D(\theta, \phi)$, is defined as

$$D(\theta, \phi) = \frac{\langle S_r \rangle}{\frac{P}{4\pi r^2}} \tag{1.18}$$

where

$$\frac{P}{4\pi r^2}$$

is the power density if the power P were uniformly distributed over a sphere of radius r. Substituting (1.14) and (1.17) into the above, we have

$$D(\theta, \phi) = \frac{\frac{\eta_0}{2} \left(\frac{\beta I l}{4\pi r} \right)^2 \sin^2\theta}{\frac{1}{4\pi r^2} \frac{4}{3} \eta_0 \pi \left(\frac{\beta I l}{4\pi} \right)^2} = \frac{3}{2}\sin^2\theta \tag{1.19}$$

The peak of $D(\theta, \phi)$ is known as the **directivity** of an antenna. It is 1.5 in this case. If an antenna is radiating isotropically, its directivity is 1. Therefore, the lowest possible values for the directivity of an antenna is 1, whereas it can be over 100 for some antennas like reflector antennas. A **directive gain pattern** is a plot of the above function $D(\theta, \phi)$ and it resembles the radiation power pattern.

If the total power fed into the antenna instead of the total radiated power is used in the denominator of (1.18), the ratio is known as the **power gain** or just bf gain. The total power fed into the antenna is not equal to the total radiated power because there could be some loss in the antenna system like metallic loss.

1.6 Radiation Resistance

Defining a **radiation resistance** R_r by $P = \frac{1}{2} I^2 R_r$, we have

$$R_r = \frac{2P}{I^2} = \eta_0 \frac{(\beta l)^2}{6\pi}, \qquad \text{where } \eta_0 = 377\Omega \tag{1.20}$$

For example, for a Hertzian dipole with $l = 0.1\lambda$, $R_r \approx 8\Omega$.

The above assumes that the current is uniformly distributed over the length of the Hertzian dipole. This is true if there are two charge reservoirs at its two ends. For a small dipole with no charge reservoir at the two ends, the currents have to vanish at the tip of the dipole as shown in Figure 7.

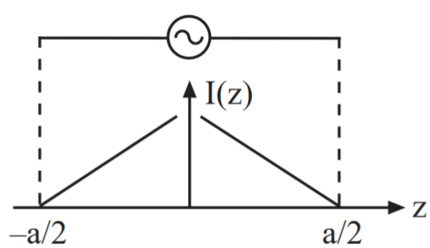


Figure 7:

The effective length of the dipole is **half** of its actual length due to the manner the currents are distributed. For example, for a half-wave dipole, $a = \frac{\lambda}{2}$, and if we use $l_{\text{eff}} = \frac{\lambda}{4}$ in (1.20), we have

$$R_r \approx 50\Omega \tag{1.21}$$

However, a half-wave dipole is not much smaller than a wavelength and does not qualify to be a Hertzian dipole. Furthermore, the current distribution on the half-wave dipole is not triangular in shape as above. A more precise calculation shows that $R_r = 73\Omega$ for a half-wave dipole.