ECE 604, Lecture 24

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Contents

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1 Circuit Theory Revisited

Circuit theory is one of the most successful and often used theories in electrical engineering. Its success is mainly due to its simplicity: it can capture the physics of highly complex circuits and structures, which is very important in the micro-chip industry. Now, having understood electromagnetic theory in its full glory, it is prudent to revisit circuit theory and study its relationship to electromagnetic theory.

The two most important laws in circuit theory are Kirchoff current law (KCL) and Kirchhoff voltage law (KVL). These two laws are derivable from the current continuity equation and from Faraday's law.

1.1 Kirchhoff Current Law

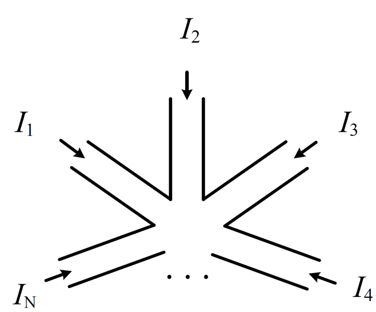

Figure 1:

Kirchhoff current law (KCL) is a consequence of the current continuity equation, or that

$$\nabla \cdot \mathbf{J} = -j\omega\varrho \tag{1.1}$$

It is a consequence of charge conservation. Some authors will say that this equation is derivable from Maxwell's equations, for example, from generalized Ampere's law and Gauss' law for charge. However, one should think that charge conservation is more fundamental, and that Gauss' law and Ampere's law are consistent with charge conservation and the current continuity equation.

First, we assume that all currents are flowing into a node as shown in Figure 1, and that the node is non-charge accumulating with $\omega \to 0$. Then the charge continuity equation becomes

$$\nabla \cdot \mathbf{J} = 0 \tag{1.2}$$

By integrating the above current continuity equation over a volume containing the node, it is easy to show that

$$\sum_i^N I_i = 0 \tag{1.3}$$

which is the statement of KCL. This is shown for the schematic of Figure 1.

1.2 Kirchhoff Voltage Law

Kirchhoff voltage law is the consequence of Faraday's law. For the truly static case when $\omega = 0$, it is

$$\nabla \times \mathbf{E} = 0 \tag{1.4}$$

The above implies that $\mathbf{E} = -\nabla\Phi$, from which we can deduce that

$$-\oint_C \mathbf{E} \cdot \mathbf{dl} = 0 \tag{1.5}$$

For statics, the statement that $\mathbf{E} = -\nabla\Phi$ also implies that we can define a voltage drop between two points, a and b to be

$$V_{ba} = -\int_a^b \mathbf{E} \cdot \mathbf{dl} = \int_a^b \nabla\Phi \cdot \mathbf{dl} = \Phi(\mathbf{r}_b) - \Phi(\mathbf{r}_a) = V_b - V_a \tag{1.6}$$

As has been shown before, this concept is only valid in the low frequency or long wavelength limit, or that the dimension over which the above is applied very small so that retardation effect can be ignored.

A good way to remember the above formula is that if $V_b > V_a$, then the electric field points from point a to point b. Electric field always points from the point of higher potential to point of lower potential. Faraday's law when applied to the static case for a closed loop of resistors shown in Figure 2 gives Kirchhoff voltage law (KVL), or that

$$\sum_i^N V_j = 0 \tag{1.7}$$

If one of the voltage drops is a voltage source, it can be modeled by a negative resistor as shown in Figure 3. The voltage drop across a negative resistor is opposite to that of a positive resistor. As we have learn from the Poynting's theorem, negative resistor gives out energy instead of dissipates energy.

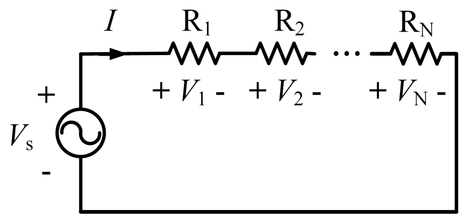

Figure 2:

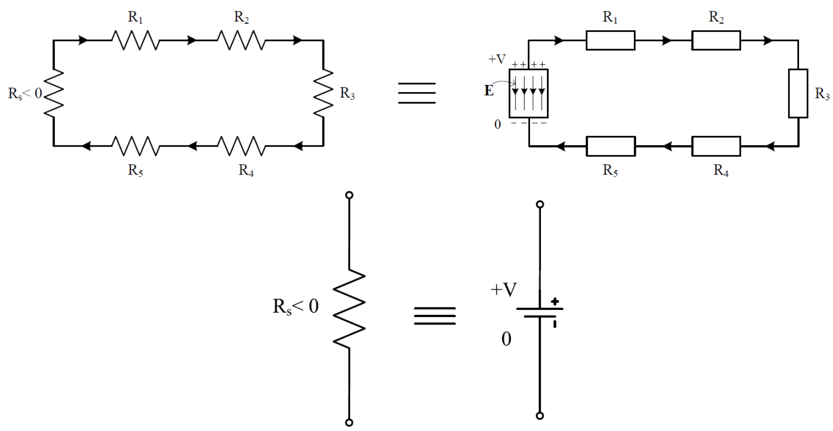

Figure 3:

Faraday's law for the time-varying case is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.8}$$

Writing the above in integral form, one gets

$$-\oint_C \mathbf{E} \cdot \mathbf{dl} = \frac{d}{dt} \int_s \mathbf{B} \cdot \mathbf{dS} \tag{1.9}$$

We can apply the above to a loop shown in Figure 4(a), or a loop C that goes from a to b to c to d to a. We can further assume that this loop is very small compared to wavelength so that potential theory that $\mathbf{E} = -\nabla\Phi$ can be applied. Furthermore, we assume that this loop C does not have any magnetic flux through it so that the right-hand side of the above can be set to zero.

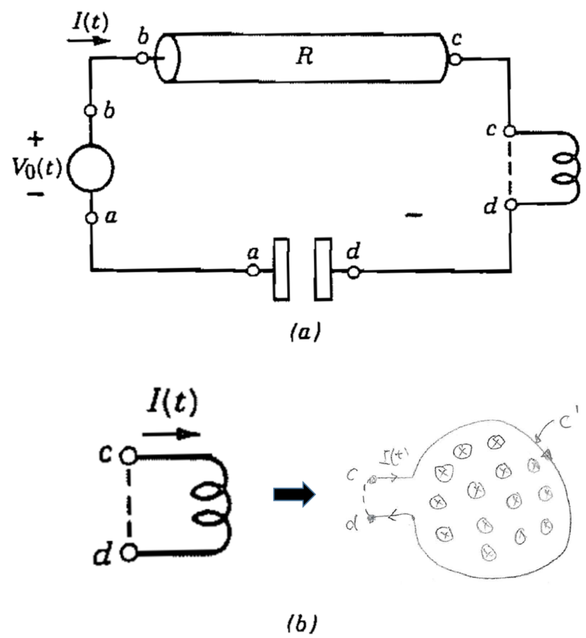

Figure 4:

Notice that this loop does not go through the inductor, but goes directly from c to d. Then there is no flux linkage in this loop and thus

$$-\int_a^b \mathbf{E}\cdot\mathbf{dl} - \int_b^c \mathbf{E}\cdot\mathbf{dl} - \int_c^d \mathbf{E}\cdot\mathbf{dl} - \int_d^a \mathbf{E}\cdot\mathbf{dl} = 0 \tag{1.10}$$

Inside the source or the battery, it is assumed that the electric field points opposite to the direction of integration $\mathbf{dl}$, and hence the first term on the left-hand side of the above is positive while the other terms are negative. Writing out the above more explicitly, we have

$$V_0(t) + V_{cb} + V_{dc} + V_{ad} = 0 \tag{1.11}$$

We will study the contributions to each of the terms, the inductor, the capacitor, and the resistor more carefully next.

1.3 Inductor

To find the voltage current relation of an inductor, we apply Faraday's law to a closed loop C' formed by dc and the inductor coil shown in the Figure 4(b). Assume that the inductor is made of a very good conductor, so that the electric field in the wire is small or zero. Then the only contribution to the left-hand side of Faraday's law is the integration from point d to point c. We assume that outside the loop, potential theory applies, and hence, $\mathbf{E} = -\nabla\Phi$. Now, we can connect V_{dc} in the previous equation to the flux linkage to the inductor. When the voltage source attempts to drive an electric current into the loop, Lenz's law (1834)[1] comes into effect, essentially, generating an opposing voltage. The opposing voltage gives rise to charge accumulation at d and c, and hence, a low frequency electric field at the gap.

To this end, we form a new C' that goes from d to c, and then continue onto the wire that leads to the inductor. But this new loop will contain the flux $\mathbf{B}$ generated by the inductor current. Thus

$$\oint_{C'} \mathbf{E} \cdot \mathbf{dl} = \int_d^c \mathbf{E} \cdot \mathbf{dl} = -V_{dc} = -\frac{d}{dt}\int_{S'} \mathbf{B} \cdot \mathbf{dS} \tag{1.12}$$

The inductance L is defined as the flux linkage per unit current, or

$$L = \left[\int_{S'} \mathbf{B} \cdot \mathbf{dS}\right] / I \tag{1.13}$$

So the voltage in (1.12) is then

$$V_{dc} = \frac{d}{dt}(LI) = L\frac{dI}{dt} \tag{1.14}$$

Had there been a finite resistance in the wire of the inductor, then the electric field is non-zero inside the wire. Taking this into account, we have

$$\oint \mathbf{E} \cdot \mathbf{dl} = R_L I - V_{dc} = -\frac{d}{dt}\int_s \mathbf{B} \cdot \mathbf{dS} \tag{1.15}$$

Consequently,

$$V_{dc} = R_L I + L\frac{dI}{dt} \tag{1.16}$$

Thus, to account for the loss of the coil, we add a resistor in the equation. The above becomes simpler in the frequency domain, namely

$$V_{dc} = R_L I + j\omega L I \tag{1.17}$$

1.4 Capacitance

The capacitance is the proportionality constant between the charge Q stored in the capacitor, and the voltage V applied across the capacitor, or $Q = CV$. Then

$$C = \frac{Q}{V} \tag{1.18}$$

[1] Lenz's law can also be explained from Faraday's law (1831).

From the current continuity equation, one can easily show that

$$I = \frac{dQ}{dt} = \frac{d}{dt}(CV_{da}) = C\frac{dV_{da}}{dt} \tag{1.19}$$

Integrating the above equation, one gets

$$V_{da}(t) = \frac{1}{C}\int^{t} Idt' \tag{1.20}$$

The above looks quite cumbersome in the time domain, but in the frequency domain, it becomes

$$I = j\omega CV_{da} \tag{1.21}$$

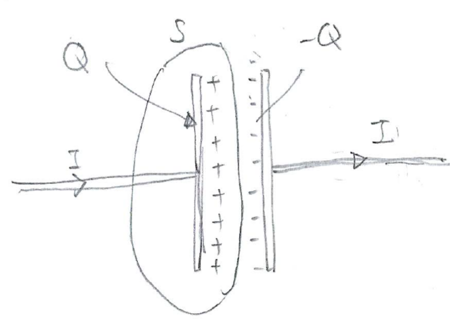

Figure 5:

1.5 Resistor

The electric field is not zero inside the resistor as electric field is needed to push electrons through it. As is well known,

$$\mathbf{J} = \sigma\mathbf{E} \tag{1.22}$$

From this, we deduce that $V_{cb} = V_c - V_b$ is a negative number given by

$$V_{cb} = -\int_b^c \mathbf{E}\cdot d\mathbf{l} = -\int_b^c \frac{\mathbf{J}}{\sigma}\cdot d\mathbf{l} \tag{1.23}$$

where we assume a uniform current $\mathbf{J} = \hat{l}I/A$ in the resistor where $\hat{l}$ is a unit vector pointing in the direction of current flow in the resistor. We can assumed that I is a constant along the length of the resistor, and thus,

$$V_{cb} = -\int_b^c \frac{Idl}{\sigma A} = -I\int_b^c \frac{dl}{\sigma A} = -IR \tag{1.24}$$

and

$$R = \int_b^c \frac{dl}{\sigma A} \tag{1.25}$$

Again, for simplicity, we assume long wavelength or low frequency in the above derivation.

2 Some Remarks

In this course, we have learnt that given the sources ϱ and $\mathbf{J}$ of an electromagnetic system, one can find Φ and $\mathbf{A}$, from which we can find $\mathbf{E}$ and $\mathbf{H}$. This is even true at DC or statics. We have also looked at the definition of inductor L and capacitor C. But clever engineering is driven by heuristics: it is better, at times, to look at inductors and capacitors as energy storage devices, rather than flux linkage and charge storage devices.

Another important remark is that even though circuit theory is simpler that Maxwell's equations in its full glory, not all the physics is lost in it. The physics of the induction term in Faraday's law and the displacement current term in generalized Ampere's law are still retained. In fact, wave physics is still retained in circuit theory: one can make slow wave structure out a series of inductors and capacitors. Since the wave is slow, it has a smaller wavelength, and resonators can be made smaller: We see this in the LC tank circuit which is a much smaller resonator in wavelength compared to a microwave cavity resonator for instance. The only short coming is that inductors and capacitors generally have higher losses than air or vacuum.

2.1 Energy Storage Method for Inductor and Capacitor

The energy stored in an inductor is due to its energy storage in the magnetic field, and it is alternatively written, according to circuit theory, as

$$W_m = \frac{1}{2} L I^2 \tag{2.1}$$

Therefore, it is simpler to think that an inductance exists whenever there is stray magnetic field to store magnetic energy. A piece of wire carries a current that produces a magnetic field enabling energy storage in the magnetic field. Hence, a piece of wire in fact behaves like a small inductor, and it is non-negligible at high frequencies: Stray inductances occur whenever there are stray magnetic fields.

By the same token, a capacitor can be thought of as an electric energy storage device rather than a charge storage device. The energy stored in a capacitor, from circuit theory, is

$$W_e = \frac{1}{2} C V^2 \tag{2.2}$$

Therefore, whenever stray electric field exists, one can think of stray capacitances as we have seen in the case of fringing field capacitances in a microstrip line.

Finding closed form solutions for inductors and capacitors is a difficult endeavor. Only certain geometries are amenable to closed form solutions. Even a simple circular loop does not have a closed form solution for its inductance L. If we assume a uniform current on a circular loop, in theory, the magnetic field can be calculated using Bio-Savart law that we have learnt before, namely that

$$\mathbf{H}(\mathbf{r}) = \int \frac{I(\mathbf{r}')\mathbf{dl}' \times \hat{R}}{4\pi R^2} \tag{2.3}$$

But the above cannot be evaluated in closed form save in terms of complicate elliptic integrals.

However, if we have a solenoid as shown in Figure 6, an approximate formula for the inductance L can be found if the fringing field at the end of the solenoid can be ignored. The inductance can be found using the flux linkage method. Figure 7 shows the schematic used to find the approximate inductance of this inductor.

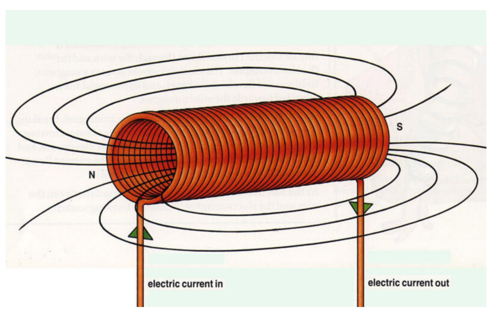

Figure 6: Courtesy of SolenoidSupplier.Com.

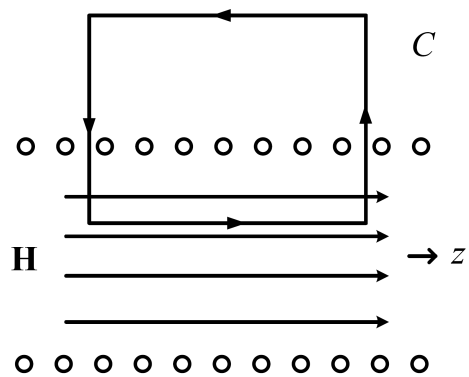

Figure 7:

The capacitance of a parallel plate capacitor can be found by solving a boundary value problem (BVP) for electrostatics, and also by the energy storage method. The electrostatic BVP for capacitor involves Poisson's equation and Laplace equation which are scalar equations.

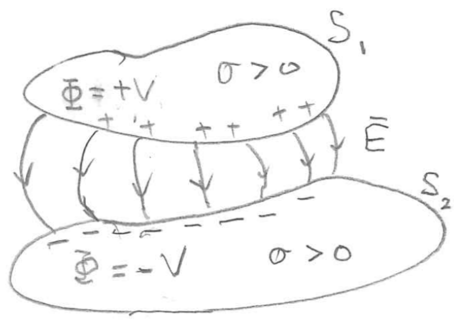

Figure 8:

Assume a geometry of two conductors charged to $+V$ and $-V$ volts as shown in Figure 8. Surface charges will accumulate on the surfaces of the conductors. Using Poisson's equations, and Green's function for Poisson's equation, one can express the potential in between the two conductors as due to the surface charges density $\sigma(\mathbf{r})$. It can be expressed as

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon}\int_S dS' \frac{\sigma(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{2.4}$$

where S is the union of two surfaces S_1 and S_2. Since Φ has values of $+V$ and

$-V$ on the two conductors, we require that

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon} \int_S dS' \frac{\sigma(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} = \begin{cases} +V, & \mathbf{r} \in S_1 \\ -V, & \mathbf{r} \in S_2 \end{cases} \tag{2.5}$$

In the above, $\sigma(\mathbf{r}')$, the surface charge density, is the unknown yet to be sought and it is embedded in an integral. But the right-hand side of the equation is known. Hence, this equation is also known as an integral equation. The integral equation can be solved by numerical methods.

Having found $\sigma(\mathbf{r})$, then it can be integrated to find Q, the total charge on one of the conductors. Since the voltage difference between the two conductors is known, the capacitance can be found as $C = Q/(2V)$.

2.2 Importance of Circuit Theory in IC Design

The clock rate of computer circuits has peaked at about 3 GHz due to the resistive loss, or the I^2R loss. At this frequency, the wavelength is about 10 cm. Since transistors and circuit components are shrinking due to the compounding effect of Moore's law, most components, which are of nanometer dimensions, are much smaller than the wavelength. Thus, most of the physics of electromagnetic signal in a circuit can be captured using circuit theory.

Figure 9 shows the schematic and the cross section of a computer chip at different levels: the transistor level at the bottom-most. The signals are taken out of a transistor by XY lines at the middle level that are linked to the ball-grid array at the top-most level of the chip. And then, the signal leaves the chip via a package. Since these nanometer-size structures are much smaller than the wavelength, they are usually modeled by lumped R, L, and C elements. If retardation effect is needed, it is usually modeled by a transmission line. This is important at the package level where the dimensions of the components are larger.

A process of parameter extraction where computer software or field solvers (software that solve Maxwell's equations numerically) are used to extract these lumped parameters. Finally, a computer chip is modeled as a network involving a large number of transistors, diodes, and R, L, and C elements. Subsequently, a very useful commercial software called SPICE (Simulation Program with Integrated-Circuit Emphasis), which is a computer-aided software, solves for the voltages and currents in this network.

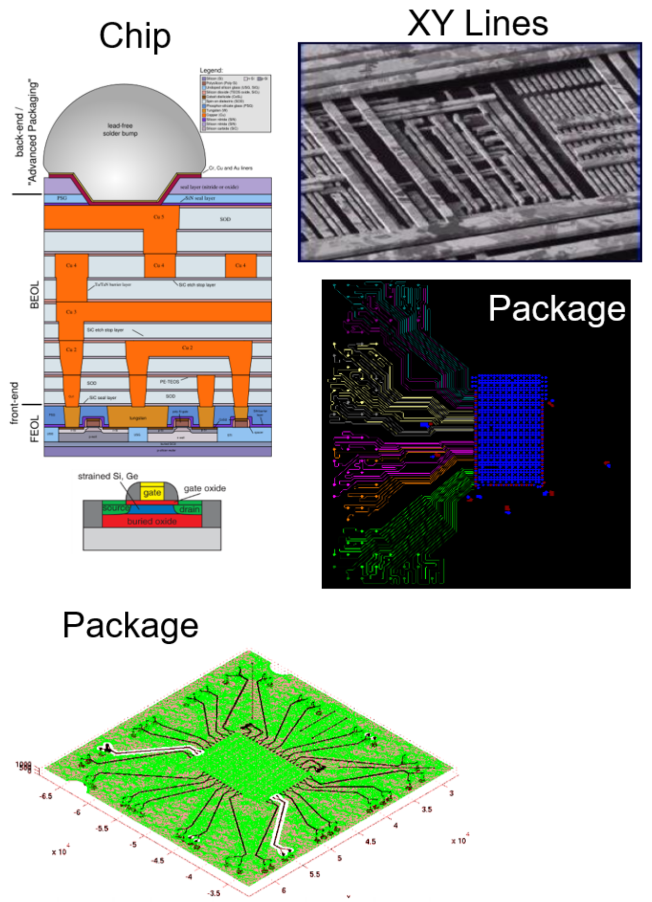

Figure 9:

The SPICE software has many capabilities, including modeling of transmission lines for microwave engineering. Figure 10 shows an interface of an RF-SPICE that allows the modeling of transmission line with a Smith chart interface.

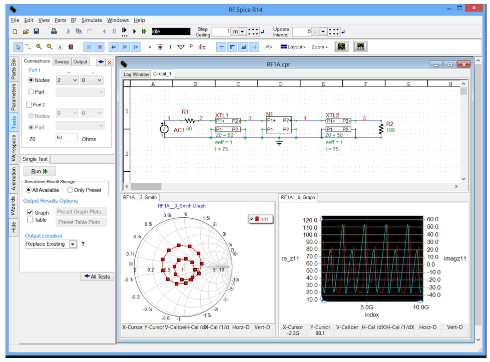

Figure 10:

2.3 Decoupling Capacitors and Spiral Inductors

Decoupling capacitors are an important part of modern computer chip design. They can regulate voltage supply on the power delivery network of the chip as they can remove high-frequency noise and voltage fluctuation from a circuit as shown in Figure 11. Figure 12 shows a 3D IC computer chip where decoupling capacitors are integrated into its design.

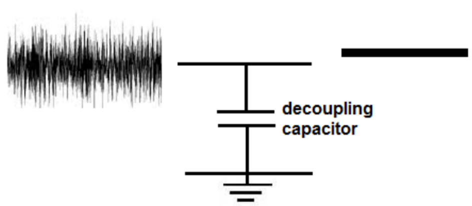

Figure 11: Courtesy of Learning About Electroncs.

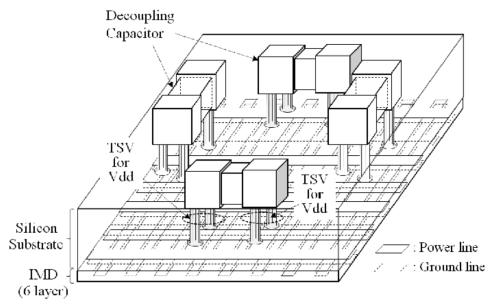

Figure 12: Modern computer chip design is 3D and is like a jungle. There are different levels in the chip and they are connected by through silicon vias (TSV). IMD stands for inter-metal dielectrics. One can see different XY lines serving as power and ground lines. (Courtesy of Semantic Scholars.)

Inductors are also indispensable in IC design, as they can be used as a high frequency choke. However, designing compact inductor is still a challenge. Spiral inductors are used because of their planar structure and ease of fabrication.

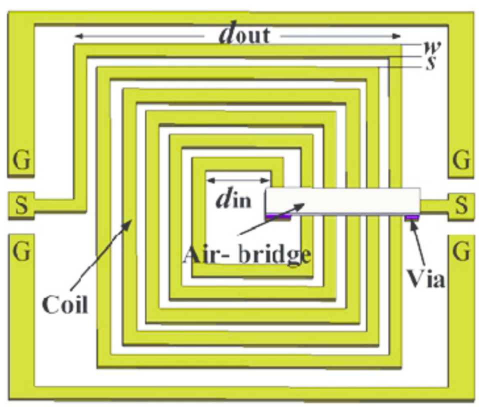

Figure 13: Courtesy of Quan Yuan, Research Gate.