ECE 604, Lecture 22

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1 The Quality Factor of a Cavity

The quality factor of a cavity or its Q measures how ideal or lossless a cavity resonator is. An ideal lossless cavity resonator will sustain free oscillations forever, while most resonators sustain free oscillations for a finite time due to losses coming from radiation, or dissipation in the dielectric material filling the cavity, or resistive loss of the metallic part of the cavity. Because of losses, the free oscillation in a cavity has electromagnetic field with time dependence as follows:

$$\mathbf{E} \propto e^{-\alpha t}\cos(\omega t + \phi_1), \quad \mathbf{H} \propto e^{-\alpha t}\cos(\omega t + \phi_2) \tag{1.1}$$

The total time-average stored energy, which is proportional to $\frac{1}{4}\epsilon\left|\mathbf{E}\right|^2 + \frac{1}{4}\mu\left|\mathbf{H}\right|^2$ is of the form

$$\langle W_T\rangle = \langle W_E\rangle + \langle W_H\rangle = W_0 e^{-2\alpha t} \tag{1.2}$$

If there is no loss, $\langle W_T\rangle$ will remain constant. However, with loss, the average stored energy will decrease to e^{-1} of its original value at $t = \frac{1}{2\alpha}$. The Q of a cavity is a measure of the number of free oscillations the field would have before the energy stored decreases to e^{-1} of its original value. In a time interval $t = \frac{1}{2\alpha}$, the number of free oscillations in radians is ωt or $\frac{\omega}{2\alpha}$; hence, the Q is defined to be

$$Q = \frac{\omega}{2\alpha} \tag{1.3}$$

Furthermore, by energy conservation, the decrease in stored energy per unit time must be equal to the total power dissipated in the losses of a cavity, in other words,

$$\langle P_d\rangle = -\frac{d\langle W_T\rangle}{dt} \tag{1.4}$$

By further assuming that W_T has to be of the form in (1.2), then

$$-\frac{d\langle W_T\rangle}{dt} = 2\alpha W_0 e^{-2\alpha t} = 2\alpha\langle W_t\rangle \tag{1.5}$$

Hence, we can write equation (1.3) as

$$Q = \frac{\omega\langle W_T\rangle}{\langle P_d\rangle} \tag{1.6}$$

By further letting $\omega = 2\pi/T$, we lent further physical interpretation to express Q as

$$Q = 2\pi\frac{\langle W_T\rangle}{\langle P_d\rangle T} = 2\pi\frac{\text{total energy stored}}{\text{Energy dissipated/cycle}} \tag{1.7}$$

In a cavity, the energy can dissipate in either the dielectric loss or the wall loss of the cavity due to the finiteness of the conductivity.

If the cavity is filled with air, then, the loss comes mainly from the metallic loss or copper-loss from the cavity wall. In this case, the power dissipated on the wall is given by

$$\langle P_d \rangle = \frac{1}{2}\Re e \oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot \hat{n}\, ds = \frac{1}{2}\Re e \oint_S (\hat{n} \times \mathbf{E}) \cdot \mathbf{H}^*\, dS \tag{1.8}$$

where S is the surface of the cavity wall. Here, $(\hat{n} \times \mathbf{E})$ is the tangential component of the electric field which would have been zero if the cavity is ideal PEC. Also, $\hat{n}$ is taken to be the outward pointing normal at the surface S. However, for metallic walls, $\hat{n} \times \mathbf{E} = \mathbf{H}_t Z_m$ where Z_m is the intrinsic impedance for the metallic conductor, $Z_m = \sqrt{\frac{\mu}{\epsilon_m}} = \sqrt{\frac{\mu}{-j\frac{\sigma}{\omega}}} = \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$,[1] and $\mathbf{H}_t$ is the tangential magnetic field. This relation between $\mathbf{E}$ and $\mathbf{H}$ will ensure that power is flowing into the metallic surface. Hence,

$$\langle P_d \rangle = \frac{1}{2}\Re e \oint_S \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)\left|\mathbf{H}_t\right|^2\, dS = \frac{1}{2}\sqrt{\frac{\omega\mu}{2\sigma}} \oint_S \left|\mathbf{H}_t\right|^2\, dS \tag{1.9}$$

By further assuming the the stored electric and magnetic energies of a cavity are equal to each other at resonance, the stored energy can be obtained by

$$\langle W_T \rangle = \frac{1}{2}\mu \oint_V |\mathbf{H}|^2 dV \tag{1.10}$$

Written explicitly, the Q becomes

$$Q = \sqrt{2\omega\mu\sigma}\frac{\oint_V \left|\mathbf{H}\right|^2\, dV}{\oint_S \left|\mathbf{H}_t\right|^2\, dS} = \frac{2}{\delta}\frac{\oint_V \left|\mathbf{H}\right|^2\, dV}{\oint_S \left|\mathbf{H}_t\right|^2\, dS} \tag{1.11}$$

In the above, δ is the skin depth of the metallic wall. Hence, the more energy stored we can have with respect to the power dissipated, the higher the Q of a resonating system. The lower the metal loss, or the smaller the skin depth, the high the Q would be.

1.1 Example: The Q of TM_{110} Mode

For the TM_{110} mode, as can be seen from the previous lecture, the only electric field is $\mathbf{E} = \hat{z}E_z$, where

$$E_z = E_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) \tag{1.12}$$

[1] When an electromagnetic wave enters a conductive region with a large β, it can be shown that the wave is refracted to propagate normally to the surface, and hence, this formula can be applied.

The magnetic field can be derived from the electric field using Maxwell's equation or Faraday's law, and

$$H_x = \frac{j\omega\epsilon}{\omega^2\mu\epsilon}\frac{\partial}{\partial y}E_Z = \frac{j\left(\frac{\pi}{b}\right)}{\omega\mu}E_0 \sin\left(\frac{\pi x}{a}\cos\right)\left(\frac{\pi y}{b}\right) \tag{1.13}$$

$$H_y = \frac{-j\omega\epsilon}{\omega^2\mu\epsilon}\frac{\partial}{\partial x}E_Z = -\frac{j\left(\frac{\pi}{a}\right)}{\omega\mu}E_0 \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) \tag{1.14}$$

Therefore

$$\begin{aligned}\oint_V |\mathbf{H}|^2\, dV &= \int_{-d}^{0}\int_0^b\int_0^a dx\, dy\, dz \left[|H_x|^2 + |H_y|^2\right] \\ &= \frac{|E_0|^2}{\omega^2\mu^2}\int_{-d}^{0}\int_0^b\int_0^a dx\, dy\, dz \\ &\quad \left[\left(\frac{\pi}{b}\right)^2 \sin^2\left(\frac{\pi x}{a}\right)\cos^2\left(\frac{\pi y}{b}\right) + \left(\frac{\pi}{a}\right)^2 \cos^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi y}{b}\right)\right] \\ &= \frac{|E_0|^2}{\omega^2\mu^2}\frac{\pi^2}{4}\left[\frac{a}{b}+\frac{b}{a}\right]d \end{aligned} \tag{1.15}$$

A cavity has six faces, finding the tangential exponent at each face and integrate

$$\begin{aligned}\oint_S |\mathbf{H}_t|\, dS &= 2\int_0^b\int_0^a dx\, dy \left[|H_x|^2 + |H_y|^2\right] \\ &\quad + 2\int_{-d}^{0}\int_0^a dx\, dz\, |H_x(y=0)|^2 + 2\int_{-d}^{0}\int_0^b dy\, dz\, |H_y(x=0)|^2 \\ &= \frac{2|E_0|^2}{\omega^2\mu^2}\frac{\pi^2 ab}{4}\left[\frac{1}{a^2}+\frac{1}{b^2}\right] + \frac{2\left(\frac{\pi}{b}\right)^2}{\omega^2\mu^2}|E_0|^2\frac{ad}{2} + \frac{2\left(\frac{\pi}{a}\right)^2}{\omega^2\mu^2}|E_0|^2\frac{bd}{2} \\ &= \frac{\pi^2|E_0|^2}{\omega^2\mu^2}\left[\frac{b}{2a}+\frac{a}{2b}+\frac{ad}{b^2}+\frac{bd}{a^2}\right] \end{aligned} \tag{1.16}$$

Hence the Q is

$$Q = \frac{2}{\delta}\frac{\left(\frac{ad}{b}+\frac{bd}{a}\right)}{\left(\frac{b}{2a}+\frac{a}{2b}+\frac{ad}{b^2}+\frac{bd}{a^2}\right)} \tag{1.17}$$

The result shows that the larger the cavity, the higher the Q. This is because the Q is the ratio of the energy stored in a volume to the energy dissipated over the surface of the cavity.

2 Mode Orthogonality and Matrix Eigenvalue Problem

It turns out the the modes of a waveguide or a resonator are orthogonal to each other. This is intimately related to the orthogonality of eigenvectors of a matrix

operator. Thus, it is best to understand this by the homomorphism between the electromagnetic mode problem and the matrix eigenvalue problem. Because of this similarity, electromagnetic modes are also called eigenmodes. Thus it is prudent that we revisit the matrix eigenvalue problem (BVP).

2.1 Matrix Eigenvalue Problem (EVP)

It is known in matrix theory that if a matrix is hermitian, then its eigenvalues are all real. Furthermore, their eigenvectors with distinct eigenvalues are orthogonal to each other. Assume that an eigenvalue and an eigenvector exists for the hermitian matrix $\overline{\mathbf{A}}$. Then

$$\overline{\mathbf{A}} \cdot \mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{2.1}$$

Dot multiplying the above from the left by $\mathbf{v}_i^\dagger$ where $\dagger$ indicates conjugate transpose, then the above becomes

$$\mathbf{v}_i^\dagger \cdot \overline{\mathbf{A}} \cdot \mathbf{v}_i = \lambda_i \mathbf{v}_i^\dagger \cdot \mathbf{v}_i \tag{2.2}$$

Since $\overline{\mathbf{A}}$ is hermitian, then the quantity $\mathbf{v}_i^\dagger \cdot \overline{\mathbf{A}} \cdot \mathbf{v}_i$ is purely real. Moreover, the quantity $\mathbf{v}_i^\dagger \cdot \mathbf{v}_i$ is positive real. So in order for the above to be satisfied, λ_i has to be real.

To prove orthogonality of eigenvectors, now, assume that $\overline{\mathbf{A}}$ has two eigenvectors with distinct eigenvalues such that

$$\overline{\mathbf{A}} \cdot \mathbf{v}_i = \lambda_i \mathbf{v}_i \tag{2.3}$$

$$\overline{\mathbf{A}} \cdot \mathbf{v}_j = \lambda_j \mathbf{v}_j \tag{2.4}$$

Left dot multiply the first equation with $\mathbf{v}_j^\dagger$ and do the same to the second equation with $\mathbf{v}_i^\dagger$, one gets

$$\mathbf{v}_j^\dagger \cdot \overline{\mathbf{A}} \cdot \mathbf{v}_i = \lambda_i \mathbf{v}_j^\dagger \cdot \mathbf{v}_i \tag{2.5}$$

$$\mathbf{v}_i^\dagger \cdot \overline{\mathbf{A}} \cdot \mathbf{v}_j = \lambda_j \mathbf{v}_i^\dagger \cdot \mathbf{v}_j \tag{2.6}$$

Taking the conjugate transpose of (2.5) in the above, and since $\overline{\mathbf{A}}$ is hermitian, their left-hand sides (2.5) and (2.6) are the same. Subtracting the two equations, we arrive at

$$0 = (\lambda_i - \lambda_j) \mathbf{v}_j^\dagger \cdot \mathbf{v}_i \tag{2.7}$$

For distinct eigenvalues, $\lambda_i \neq \lambda_j$, the only way for the above to be satisfied is that

$$\mathbf{v}_j^\dagger \cdot \mathbf{v}_i = C_i \delta_{ij} \tag{2.8}$$

Hence, eigenvectors of a hermitian matrix with distinct eigenvalues are orthogonal to each other. The eigenvalues are also real.

2.2 Homomorphism with the Waveguide Mode Problem

We shall next show that the problem for finding the waveguide modes or eigenmodes is analogous to the matrix eigenvalue problem. The governing equation for a waveguide mode is BVP involving the reduced wave equation previously derived, or

$$\nabla_s^2 \psi_i(\mathbf{r}_s) + \beta_{is}^2 \psi_i(\mathbf{r}_s) = 0 \tag{2.9}$$

with the pertinent homogeneous Dirichlet or Neumann boundary condition depending on if TE or TM modes are considered. In the above, the differential operator ∇_s^2 is analogous to the matrix operator $\overline{\mathbf{A}}$, the eigenfunction $\psi_i(\mathbf{r}_s)$ is analogous to the eigenvector $\mathbf{v}_i$, and β_{is}^2 is analogous to the eigenvalue λ_i.

2.2.1 Discussion on Functional Space

To think of a function $\psi(\mathbf{r}_s)$ as a vector, one has to think in the discrete world. If one needs to display the function $\psi(\mathbf{r}_s)$, on a computer, one will evaluate the function $\psi(\mathbf{r}_s)$ at discrete N locations $\mathbf{r}_{ls}$, where $l = 1, 2, 3, \ldots N$. For every $\mathbf{r}_{ls}$ or every l, there is a scalar number $\psi(\mathbf{r}_{ls})$. These scalar numbers can be stored in a column vector in a computer indexed by l. The larger N is, the better is the discrete approximation of $\psi(\mathbf{r}_s)$. In theory, one needs N to be infinite to describe this function exactly.

From the above discussion, a function is analogous to a vector and a functional space is analogous to a vector space. However, a functional space is infinite dimensional. But in order to compute on a computer with finite resource, such functions are approximated with vectors. Infinite dimensional vector spaces are replaced with finite dimensional ones to make the problem computable. Such infinite dimensional functional space is also called Hilbert space.

It is also necessary to define the inner product between two vectors in a functional space just as inner product between two vectors in an matrix vector space. The inner product (or dot product) in matrix vector is

$$\mathbf{v}_i^t \cdot \mathbf{v}_j = \sum_{l=1}^{N} v_{i,l} v_{j,l} \tag{2.10}$$

The analogous inner product between two vectors in function space is [2]

$$\langle \psi_i, \psi_j \rangle = \int_S d\mathbf{r}_s \psi_i(\mathbf{r}_s) \psi_j(\mathbf{r}_s) \tag{2.11}$$

where S denotes the cross-sectional area of the waveguide over which the integration is performed. The left-hand side is the shorthand notation for inner product in functional space or the infinite dimensional Hilbert space.

[2] In many math books, the conjugation of the first function ψ_i is implied, but here, we follow the electromagnetic convention that the conjugation of ψ_i is not implied unless explicitly stated.

Another requirement for a vector in a functional Hilbert space is that it contains finite energy or that

$$E_f = \int_S d\mathbf{r}_s |\psi_i(\mathbf{r}_s)|^2 \tag{2.12}$$

is finite. The above is analogous to that for a matrix vector $\mathbf{v}$ as

$$E_m = \sum_{l=1}^{N} |v_l|^2 \tag{2.13}$$

The square root of the above is often used to denote the "length" or the "metric" of the vector. Finite energy also implies that the vectors are of finite length. This length is also called the "norm" of a vector.

2.3 Proof of Orthogonality of Waveguide Modes

Because of the aforementioned discussion, we see the similarity between a function Hilbert space, and the matrix vector space. In order to use the result of the matrix EVP, one key step is to prove that the operator ∇_s^2 is hermitian. In matrix algebra, a matrix operator is hermitian if

$$\mathbf{x}_i^\dagger \cdot \overline{\mathbf{A}} \cdot \mathbf{x}_j = \left(\mathbf{x}_j^\dagger \cdot \overline{\mathbf{A}} \cdot \mathbf{x}_i\right)^\dagger = \left(\mathbf{x}_j^\dagger \cdot \overline{\mathbf{A}} \cdot \mathbf{x}_i\right)^* \tag{2.14}$$

The last equality follows because the quantity in the parenthesis is a scalar, and hence, its conjugate transpose is just its conjugate.

By the same token, a functional operator ∇^2 is hermitian if

$$\langle \psi_i^*, \nabla_s^2 \psi_j \rangle = \int_S d\mathbf{r}_s \psi_i^*(\mathbf{r}_s) \nabla_s^2 \psi_j(\mathbf{r}_s) = (\langle \psi_j^*, \nabla_s^2 \psi_i \rangle)^* = \int_S d\mathbf{r}_s \psi_j^*(\mathbf{r}_s) \nabla_s^2 \psi_i(\mathbf{r}_s) \tag{2.15}$$

Using integration by parts for higher dimensions and with the appropriate boundary condition for the function $\psi(\mathbf{r}_s)$, the above equality can be proved.

To this end, one uses the identity that

$$\nabla_s \cdot [\psi_i^*(\mathbf{r}_s) \nabla_s \psi_j(\mathbf{r}_s)] = \psi_i^*(\mathbf{r}_s) \nabla_s^2 \psi_j(\mathbf{r}_s) + \nabla_s \psi_i^*(\mathbf{r}_s) \cdot \nabla_s \psi_j(\mathbf{r}_s) \tag{2.16}$$

Integrating the above over the cross sectional area S, and invoking Gauss divergence theorem in 2D, one gets that

$$\begin{aligned} \int_C dl \hat{n} \cdot (\psi_i^*(\mathbf{r}_s) \nabla_s \psi_j(\mathbf{r}_s)) = &\int_S d\mathbf{r}_s \left(\psi_i^*(\mathbf{r}_s) \nabla_s^2 \psi_j(\mathbf{r}_s)\right) \\ &+ \int_S d\mathbf{r}_s \left(\nabla \psi_i^*(\mathbf{r}_s) \cdot \nabla_s \psi_j(\mathbf{r}_s)\right) \end{aligned} \tag{2.17}$$

where C the the contour bounding S or the waveguide wall. By applying the boundary condition that $\psi_i(\mathbf{r}_s) = 0$ or that $\hat{n} \cdot \nabla_s \psi_j(\mathbf{r}_s) = 0$, then the left-hand side of the above is zero. This will be the case be it TE or TM modes.

$$0 = \int_S d\mathbf{r}_s \left(\psi_i^*(\mathbf{r}_s)\nabla_s^2 \psi_j(\mathbf{r}_s)\right) + \int_S d\mathbf{r}_s \left(\nabla \psi_i^*(\mathbf{r}_s) \cdot \nabla_s \psi_j(\mathbf{r}_s)\right) \tag{2.18}$$

Applying the same treatment to the last term (2.15), we get

$$0 = \int_S d\mathbf{r}_s \left(\psi_j(\mathbf{r}_s)\nabla_s^2 \psi_i^*(\mathbf{r}_s)\right) + \int_S d\mathbf{r}_s \left(\nabla \psi_i^*(\mathbf{r}_s) \cdot \nabla_s \psi_j(\mathbf{r}_s)\right) \tag{2.19}$$

The above indicates that

$$\langle \psi_i^*, \nabla_s^2 \psi_j \rangle = (\langle \psi_j^*, \nabla_s^2 \psi_i \rangle)^* \tag{2.20}$$

proving that the operator ∇_s^2 is hermitian. One can then use the above property to prove the orthogonality of the eigenmodes when they have distinct eigenvalues, the same way we have proved the orthogonality of eigenvectors.