ECE 604, Lecture 18

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1 Hollow Waveguides

1.1 Introduction

Hollow waveguides are useful for high-power microwaves. Air has a higher breakdown voltage compared to most materials, and hence, could be a good medium for propagating high power microwave. Also, they are sufficiently shielded from the rest of the world so that interference from other sources is minimized. Furthermore, for radio astronomy, they can provide a low-noise system immune to interference. Air generally has less loss than materials, and loss is often the source of thermal noise. A low loss waveguide is also a low noise waveguide.

Many waveguide problems can be solved in closed form. An example is the coaxial waveguide previously discussed. But there are many other waveguide problems that have closed form solutions. Closed form solutions to Laplace and Helmholtz equations are obtained by the separation of variables method. The separation of variables method works only for separable coordinate systems. There are 11 separable coordinates for Helmholtz equations, but 13 for Laplace equation. Some examples of separable coordinate systems are cartesian, cylindrical, and spherical coordinates. But these three coordinates are about all we need to know for solving many engineering problems. More complicated cases are now handled with numerical methods using computers.

When a waveguide has a center conductor or two conductors like a coaxial cable, it can support a TEM wave where both the electric field and the magnetic field are orthogonal to the direction of propagation. The uniform plane wave is a TEM wave, for instance.

However, when the waveguide is hollow or is filled completely with a homogeneous medium, without a center conductor, it cannot support a TEM mode as we shall prove next.

1.2 Absence of TEM Mode in a Hollow Waveguide

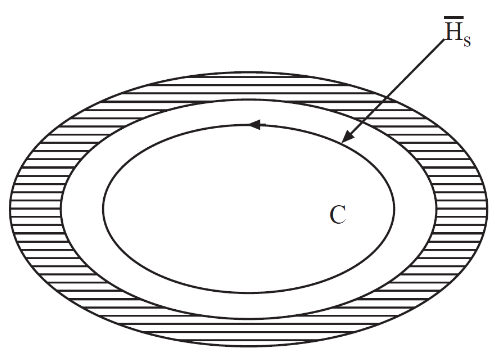

Figure 1: Absence of TEM mode in a hollow, enclosed waveguide.

We would like to prove by contradiction (reductio ad absurdum) that a hollow waveguide (i.e. without a center conductor) cannot support a TEM mode as follows. If we assume that TEM mode does exist, then the magnetic field has to end on itself due to the absence of magnetic charges. It is clear that $\oint_C \mathbf{H}_s \cdot d\mathbf{l} \neq 0$ about any closed contour following the magnetic field lines. But Ampere's law states that the above is equal to

$$\oint_C \mathbf{H}_s \cdot d\mathbf{l} = j\omega \int_S \mathbf{D} \cdot d\mathbf{S} + \int_S \mathbf{J} \cdot d\mathbf{S} \tag{1.1}$$

Hence, this equation cannot be satisfied unless there are E_z component, or that $\mathbf{J}_z \neq 0$ inside the waveguide. This implies that E_z cannot be zero unless a center conductor carrying a current $\mathbf{J}$ is there. This result contradicts the absence of E_z, and is contradictory implying the absence of a TEM mode in a hollow waveguide.

Therefore, in a hollow waveguide filled with homogeneous medium, only TE_z or TM_z modes can exist like the case of a layered medium. For a TE_z wave (or TE wave), $E_z = 0$, $H_z \neq 0$ while for a TM_z wave (or TM wave), $H_z = 0$, $E_z \neq 0$. These classes of problems can be decomposed into two scalar problems like the layerd medium case, by using the pilot potential method. However, when the hollow waveguide is filled with a center conductor, the TEM mode can exist in addition to TE and TM modes.

1.3 TE Case ($E_z = 0$, $H_z \neq 0$)

In this case, the field inside the waveguide is TE to z or TE_z. To ensure a TE field, we can write the $\mathbf{E}$ field as

$$\mathbf{E}(\mathbf{r}) = \nabla \times \hat{z}\Psi_h(\mathbf{r}) \tag{1.2}$$

Equation (1.2) will guarantee that $E_z = 0$ due to its construction. Here, $\Psi_h(\mathbf{r})$ is a scalar potential and $\hat{z}$ is the pilot vector.[1]

The waveguide is assumed source free and filled with a lossless, homogeneous material. Eq. (1.2) also satisfies the source-free condition since $\nabla \cdot \mathbf{E} = 0$. And hence, from Maxwell's equations, it follows that

$$(\nabla^2 + \beta^2)\mathbf{E}(\mathbf{r}) = 0 \tag{1.3}$$

where $\beta^2 = \omega^2\mu\varepsilon$. Substituting (1.2) into (1.3), we get

$$(\nabla^2 + \beta^2)\nabla \times \hat{z}\Psi_h(\mathbf{r}) = 0 \tag{1.4}$$

In the above, we assume that $\nabla^2\nabla \times \hat{z}\Psi = \nabla \times \hat{z}(\nabla^2\Psi)$, or that these operators commute.[2] Then it follows that

$$\nabla \times \hat{z}(\nabla^2 + \beta^2)\Psi_h(\mathbf{r}) = 0 \tag{1.5}$$

[1] It "pilots" the field so that it is transverse to z.

[2] This is a mathematical parlance, and a commutator is defined to be $[A, B] = AB - BA$ for two operators A and B. If these two operators commute, then $[A, B] = 0$.

Thus, if

$$(\nabla^2 + \beta^2)\Psi_h(\mathbf{r}) = 0 \tag{1.6}$$

then (1.5) is satisfied, and so is (1.3). Hence, the $\mathbf{E}$ field constructed with (1.2) satisfies Maxwell's equations, if $\Psi_h(\mathbf{r})$ satisfies (1.6).

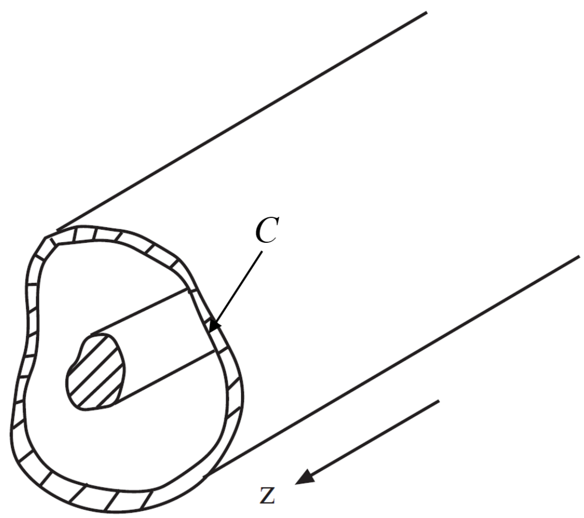

Figure 2:

Next, we look at the boundary condition for $\Psi_h(\mathbf{r})$. The boundary condition for $\mathbf{E}$ is that $\hat{n} \times \mathbf{E} = 0$ on C, the wall of the waveguide. But from (1.2), using the back-of-the-cab (BOTC) formula,

$$\hat{n} \times \mathbf{E} = \hat{n} \times (\nabla \times \hat{z}\Psi_h) = -\hat{n} \cdot \nabla\Psi_h = 0 \tag{1.7}$$

In applying the BOTC formula, one has to be mindful that ∇ operates on a function to its right, and the function Ψ_h is always placed to the right of the ∇ operator.

In the above $\hat{n}\cdot\nabla = \hat{n}\cdot\nabla_s$ where $\nabla_s = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$ since $\hat{n}$ has no z component. The boundary condition (1.7) then becomes

$$\hat{n} \cdot \nabla_s \Psi_h = \partial_n \Psi_h = 0 \text{ on } C \tag{1.8}$$

which is also known as the homogeneous Neumann boundary condition.

Furthermore, in a waveguide, just as in a transmission line case, we are looking for traveling solutions of the form $\exp(\mp j\beta_z z)$ for (1.6), or that

$$\Psi_h(\mathbf{r}) = \Psi_{hs}(\mathbf{r}_s)e^{\mp j\beta_z z} \tag{1.9}$$

where $\mathbf{r}_s = \hat{x}x + \hat{y}y$, or in short, $\Psi_{hs}(\mathbf{r}_s) = \Psi_{hs}(x, y)$. Thus, $\partial_n \Psi_h = 0$ implies that $\partial_n \Psi_{hs} = 0$. With this assumption, $\frac{\partial^2}{\partial z^2} \to -{\beta_z}^2$, and (1.6) becomes even simpler, namely,

$$({\nabla_s}^2 + \beta^2 - {\beta_z}^2)\Psi_{hs}(\mathbf{r}_s) = ({\nabla_s}^2 + \beta_s^2)\Psi_{hs}(\mathbf{r}_s) = 0\ , \quad \partial_n \Psi_{hs}(\mathbf{r}_s) = 0,\ \text{on } C \tag{1.10}$$

where $\beta_s^2 = \beta^2 - \beta_z^2$. The above is a boundary value problem for a 2D waveguide problem. The above 2D wave equation is also called the reduced wave equation.

1.4 TM Case ($E_z \neq 0$, $H_z = 0$)

Repeating similar treatment for TM waves, the TM magnetic field is then

$$\mathbf{H} = \nabla \times \hat{z}\Psi_e(\mathbf{r}) \tag{1.11}$$

where

$$(\nabla^2 + \beta^2)\Psi_e(\mathbf{r}) = 0 \tag{1.12}$$

The corresponding $\mathbf{E}$ field is obtained by taking the curl of the magnetic field in (1.11), and thus the $\mathbf{E}$ field is proportional to

$$\mathbf{E} \sim \nabla \times \nabla \times \hat{z}\Psi_e(\mathbf{r}) = \nabla\nabla \cdot (\hat{z}\Psi_e) - \nabla^2 \hat{z}\Psi_e = \nabla \frac{\partial}{\partial z}\Psi_e + \hat{z}\beta^2\Psi_e \tag{1.13}$$

Taking the z component of the above, we get

$$E_z \sim \frac{\partial^2}{\partial z^2}\Psi_e + \beta^2\Psi_e \tag{1.14}$$

Assuming that

$$\Psi_e \sim e^{\mp j{\beta_z}^z} \tag{1.15}$$

then in (1.14), $\partial^2/\partial z^2 \to -\beta_z^2$, and

$$E_z \sim (\beta^2 - \beta_z^2)\Psi_e \tag{1.16}$$

Therefore, if

$$\Psi_e(\mathbf{r}) = 0 \text{ on } C, \tag{1.17}$$

then,

$$E_z(\mathbf{r}) = 0 \text{ on } C \tag{1.18}$$

Equation (1.16) is also called the homogeneous Dirichlet boundary condition. One can further show from (1.13) that the homogeneous Dirichlet boundary condition also implies that the other components of tangential $\mathbf{E}$ are zero, namely $\hat{n} \times \mathbf{E} = 0$ on the waveguide wall C.

Thus, with some manipulation, the boundary value problem related to equation (1.12) reduces to a simpler 2D problem, i.e.,

$$(\nabla_s{}^2 + \beta_s^2)\Psi_{es}(\mathbf{r}_s) = 0 \tag{1.19}$$

with the homogeneous Dirichlet boundary condition that

$$\Psi_{es}(\mathbf{r}_s) = 0, \ \mathbf{r}_s \text{ on } C \tag{1.20}$$

where we have assumed that

$$\Psi_e(\mathbf{r}) = \Psi_{es}(\mathbf{r}_s)e^{\mp j\beta_z{}^z} \tag{1.21}$$

To illustrate the above theory, we can solve some simple waveguides problems.

2 Rectangular Waveguides

Rectangular waveguides are among the simplest waveguides to analyze because closed form solutions exist in cartesian coordinates. One can imagine traveling waves in the xy directions bouncing off the walls of the waveguide causing standing waves to exist inside the waveguide.

As shall be shown, it turns out that not all electromagnetic waves can be guided by a hollow waveguide. Only when the wavelength is short enough, or the frequency is high enough that an electromagnetic wave can be guided by a waveguide.

2.1 TE Modes (H Mode or $H_z \neq 0$ Mode)

For this mode, the scalar potential $\Psi_{hs}(\mathbf{r}_s)$ satisfies

$$(\nabla_s{}^2 + \beta_s{}^2)\Psi_{hs}(\mathbf{r}_s) = 0, \quad \frac{\partial}{\partial n}\Psi_{hs}(\mathbf{r}_s) = 0 \quad \text{on } C \tag{2.1}$$

where $\beta_s{}^2 = \beta^2 - \beta_z{}^2$. A viable solution using separation of variables[3] for $\Psi_{hs}(x, y)$ is then

$$\Psi_{hs}(x, y) = A\cos(\beta_x x)\cos(\beta_y y) \tag{2.2}$$

where $\beta_x{}^2 + \beta_y^2 = \beta_s^2$. One can see that the above is the representation of standing waves in the xy directions. It is quite clear that $\Psi_{hs}(x, y)$ satisfies equation (2.1). Furthermore, cosine functions, rather than sine functions are chosen with the hindsight that the above satisfies the homogenous Neumann boundary condition at $x = 0$, and $y = 0$ surfaces.

[3]For those who are not familiar with this topic, please consult p. 385 of Kong.

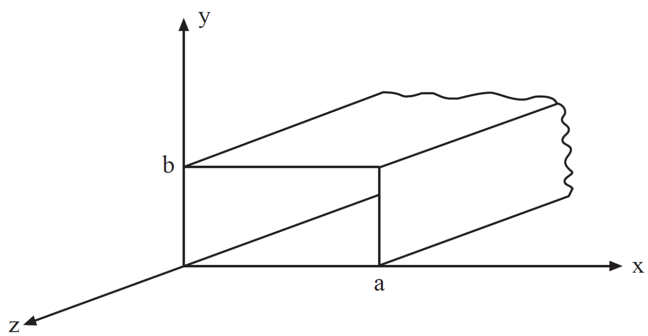

Figure 3:

To further satisfy the boundary condition at $x = a$, and $y = b$ surfaces, it is necessary that the boundary condition for eq. (1.8) is satisfied or that

$$\partial_x \Psi_{hs}(x,y)\big|_{x=a} \sim \sin(\beta_x a)\cos(\beta_y y) = 0, \tag{2.3}$$

$$\partial_y \Psi_{hs}(x,y)\big|_{y=b} \sim \cos(\beta_x x)\sin(\beta_y b) = 0, \tag{2.4}$$

The above puts constraints on β_x and β_y, implying that $\beta_x a = m\pi$, $\beta_y b = n\pi$ where m and n are integers. Hence (2.2) becomes

$$\Psi_{hs}(x,y) = A\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \tag{2.5}$$

where

$$\beta_x^2 + \beta_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \beta_s^2 = \beta^2 - {\beta_z}^2 \tag{2.6}$$

Clearly, (2.5) satisfies the requisite homogeneous Neumann boundary condition at the entire waveguide wall.

The above condition on β_s^2 is the guidance condition for the modes in the waveguide. Furthermore,

$$\beta_z = \sqrt{\beta^2 - \beta_s^2} = \sqrt{\beta^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{2.7}$$

Furthermore, from (2.7), when

$$\beta_s^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 > \beta^2 = \omega^2\mu\varepsilon \tag{2.8}$$

β_z becomes pure imaginary and the mode cannot propagate or become evanescent in the z direction.[4] For fixed m and n, the frequency at which the above

[4]We have seen this happening in a plasma medium earlier.

happens is called the cutoff frequency of the TE_{mn} mode of the waveguide. It is given by

$$\omega_{mn,c} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{2.9}$$

When $\omega < \omega_{mn,c}$, the TE_{mn} mode is evanescent and cannot propagate inside the waveguide. A corresponding cutoff wavelength is then

$$\lambda_{mn,c} = \frac{2}{[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2]^{1/2}} \tag{2.10}$$

So when $\lambda > \lambda_{mn,c}$, the mode cannot propagate inside the waveguide.

When $m = n = 0$, then $\Psi_h(\mathbf{r}) = \Psi)hs(x,y)\exp(\mp j\beta_z z)$ is a function independent of x and y. Then $\mathbf{E}(\mathbf{r}) = \nabla \times \hat{z}\Psi_h(\mathbf{r}) = \nabla_s \times \hat{z}\Psi_h(\mathbf{r}) = 0$. It turns out the only way for $H_z \neq 0$ is for $\mathbf{H}(\mathbf{r}) = \hat{z}H_0$ which is a static field in the waveguide. This is not a very interesting mode, and thus TE_{00} propagating mode is assumed not to exist and not useful. So the TE_{mn} modes cannot have both $m = n = 0$. As such, the TE_{10} mode, when $a > b$, is the mode with the lowest cutoff frequency or longest cutoff wavelength.

For the TE_{10} mode, for the mode to propagate, from (2.10), it is needed that

$$\lambda < \lambda_{10,c} = 2a \tag{2.11}$$

The above has the nice physical meaning that the wavelength has to be smaller than $2a$ in order for the mode to fit into the waveguide. As a mnemonic, we can think that photons have "sizes", corresponding to its wavelength. Only when its wavelength is small enough can the photons go into (or be guided by) the waveguide. The TE_{10} mode, when $a > b$, is also the mode with the lowest cutoff frequency or longest cutoff wavelength.

It is seen with the above analysis, when the wavelength is short enough, or frequency is high enough, many modes can be guided. Each of these modes has a different group and phase velocity. But for most applications, a single guided mode only is desirable. Hence, the knowledge of the cutoff frequencies of the fundamental mode (the mode with the lowest cutoff frequency) and the next higher mode is important. This allows one to pick a frequency window within which only a single mode can propagate in the waveguide.