ECE 604, Lecture 16

Fri, Feb 15, 2019

Contents

Printed on March 24, 2019 at 16 : 36: W.C. Chew and D. Jiao.

1 Waves in Layered Media

1.1 Generalized Reflection Coefficient for Layered Media

Because of the homomorphism between transmission line problems and plane waves in layered medium problems, one can capitalize on using the multi-section transmission line formulas for generalized reflection coefficient, which is

$$\tilde{\Gamma}_{12} = \frac{\Gamma_{12} + \tilde{\Gamma}_{23} e^{-2j\beta_2 l_2}}{1 + \Gamma_{12}\tilde{\Gamma}_{23} e^{-2j\beta_2 l_2}} \tag{1.1}$$

This reflection coefficient includes multiple reflections from the right of the 12 junction. It can be used to study electromagnetic waves in layered media shown in Figures 1 and 2.

Using the result from the multi-junction transmission line, we can write down the generalized reflection coefficient for a layered medium with an incident wave at the 12 interface, including multiple reflections from the right. It is given by

$$\tilde{R}_{12} = \frac{R_{12} + \tilde{R}_{23} e^{-2j\beta_{2z} l_2}}{1 + R_{12}\tilde{R}_{23} e^{-2j\beta_{2z} 2 l_2}} \tag{1.2}$$

where l_2 is now the thickness of the region 2. In the above, we assume that the wave is incident from medium 1 which is semi-infinite, the generalized reflection coefficient above is defined at the media 1 and 2 interface. It is assumed that there are multiple reflection coming from the 23 interface, so that the 23 reflection coefficient is the generalized reflection coefficient $\tilde{R}_{23}$.

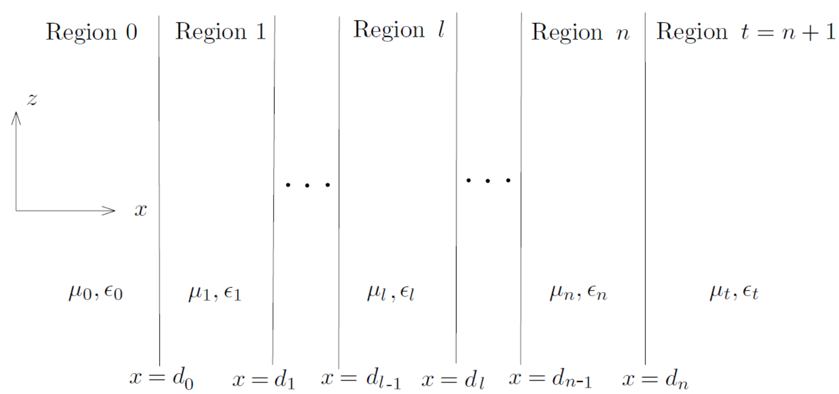

Figure 1: Courtesy of J.A. Kong, Electromagnetic Wave Theory. Replace x with z and vice versa.

Figure 2 shows the case of a normally incident wave into a layered media. For this case, the wave impedance becomes the intrinsic impedance.

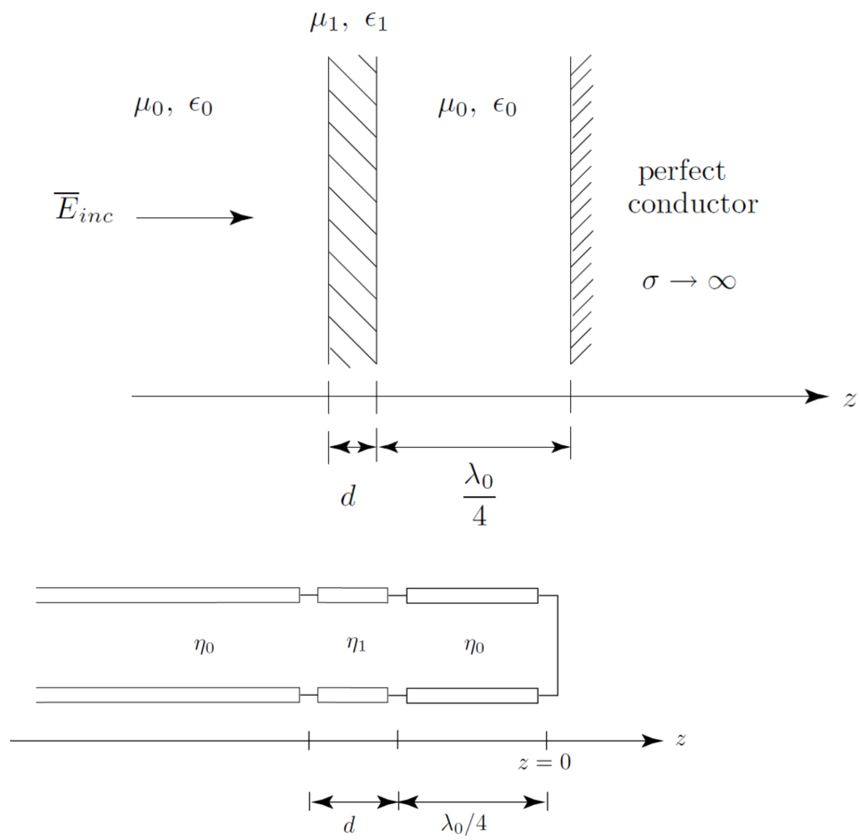

Figure 2: For normal incidence, the wave impedances becomes intrinsic impedances (Courtesy of J.A. Kong, Electromagnetic Wave Theory).

We shall discuss finding guided waves in a layered medium next using the generalized reflection coefficient. But it will be prudent to understand phase and group velocity better before doing this.

2 Phase Velocity and Group Velocity

Now that we know how a medium can be frequency dispersive in the Drude-Lorentz-Sommerfeld (DLS) model, we are ready to distinguish the difference between the phase velocity and the group velocity

2.1 Phase Velocity

The phase velocity is the velocity of the phase of a wave. It is only defined for a mono-chromatic signal (also called time-harmonic, CW (constant wave), or sinusoidal signal) at one given frequency. A sinusoidal wave signal, e.g., the voltage signal on a transmission line, can take the form

$$V(z,t) = V_0 \cos(\omega t - kz + \alpha) \tag{2.1}$$

This sinusoidal signal moves with a velocity

$$v_{ph} = \frac{\omega}{k} \tag{2.2}$$

where, for example, $k = \omega\sqrt{\mu\varepsilon}$, inside a simple coax. Hence,

$$v_{ph} = 1/\sqrt{\mu\varepsilon} \tag{2.3}$$

But a dielectric medium can be frequency dispersive, or $\varepsilon(\omega)$ is not a constant but a function of ω as has been shown with the Drude-Lorentz-Sommerfeld model. Therefore, signals with different ω's will travel with different phase velocity.

More bizarre still, what if the coax is filled with a plasma medium where

$$\varepsilon = \varepsilon_0 \left(1 - \frac{{\omega_p}^2}{\omega^2}\right) \tag{2.4}$$

Then, $\varepsilon < \varepsilon_0$ always meaning that the phase velocity given by (2.3) can be larger than the velocity of light in vacuum (assuming $\mu = \mu_0$). Also, $\varepsilon = 0$ when $\omega = \omega_p$, implying that $k = 0$; then in accordance to (2.2), $v_{ph} = \infty$. These ludicrous observations can be justified or understood only if we can show that information can only be sent by using a wave packet.[1] The same goes for energy which can only be sent by wave packets, but not by CW signal; only in this manner can a finite amount of energy be sent. These wave packets can only travel at the group velocity as shall be shown, which is always less than the velocity of light.

[1] In information theory, according to Shannon, the basic unit of information is a bit, which can only be sent by a digital signal, or a wave packet.

2.2 Group Velocity

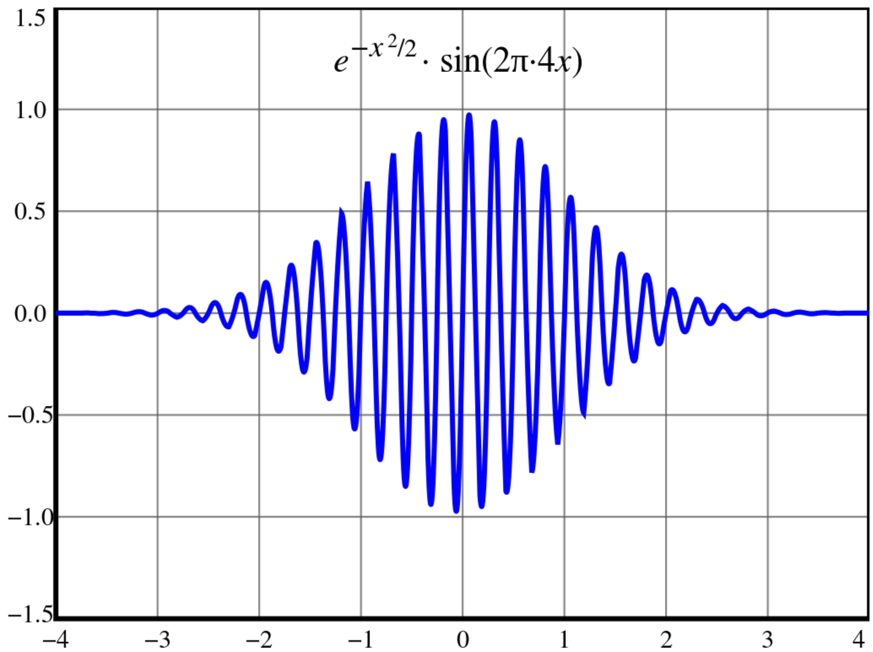

Figure 3: A Gaussian wave packet. (Courtesy of Wikimedia.)

Now, consider a narrow band wave packet as shown in Figure 3. It cannot be mono-chromatic, but can be written as a linear superposition of many frequencies. One way to express this is to write this wave packet as an integral in terms of Fourier transform, or a summation over many frequencies, namely

$$V(z,t) = \int_{-\infty}^{\infty} d\omega V(z,\omega) e^{j\omega t} \tag{2.5}$$

Assume that $V(z,t)$ is the solution to the dispersive transmission line equations with $\varepsilon(\omega)$, then it can be shown that $V(z,\omega)$ is the solution to the one-dimensional Helmholtz equation

$$\frac{d^2}{dz^2} V(z,\omega) + k^2(\omega) V(z,\omega) = 0 \tag{2.6}$$

When the dispersive transmission line is filled with dispersive material, then $k^2 = \omega^2 \mu_0 \varepsilon(\omega)$. Thus, upon solving the above equation, one obtains that

$V(z,\omega) = V_0(\omega)e^{-jkz}$, and

$$V(z,t) = \int_{-\infty}^{\infty} d\omega V_0(\omega) e^{j(\omega t - kz)} \tag{2.7}$$

In the general case, k is a complicated function of ω as shown in Figure 4.

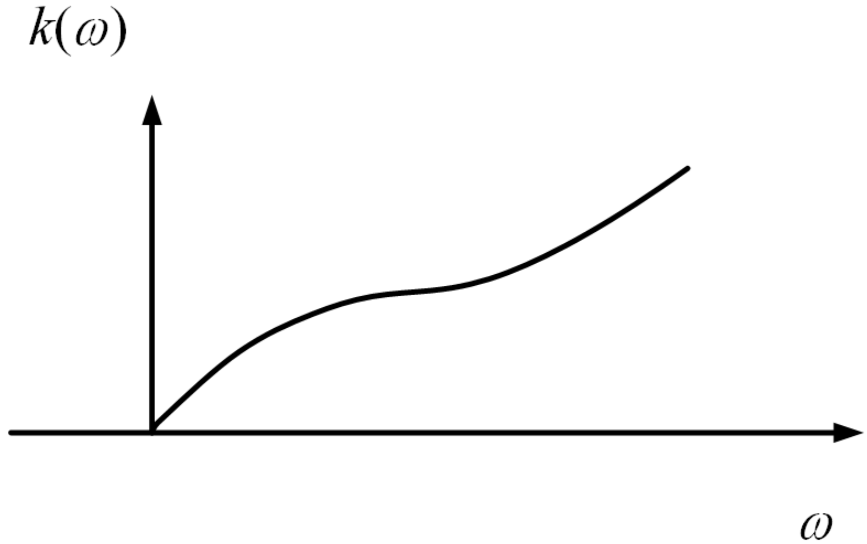

Figure 4:

Since this is a wave packet, we assume that $V_0(\omega)$ is narrow band centered about a frequency ω_0, the carrier frequency as shown in Figure 5. Therefore, when the integral in (2.7) is performed, it needs only be summed over a narrow range of frequencies in the vicinity of ω_0.

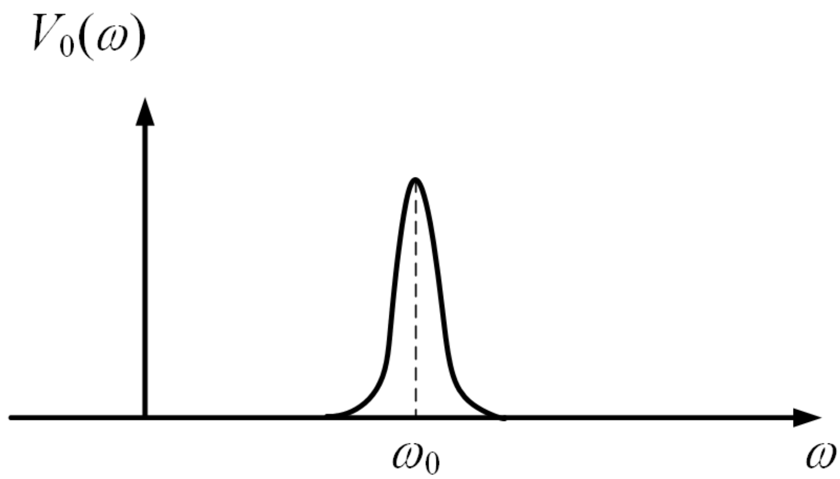

Figure 5:

Thus, we can approximate the integrand in the vicinity of $\omega = \omega_0$, and let

$$k(\omega) \cong k(\omega_0) + (\omega - \omega_0)\frac{dk(\omega_0)}{d\omega} + \frac{1}{2}(\omega - \omega_0)^2\frac{d^2k(\omega_0)}{d\omega^2} + \cdots \tag{2.8}$$

To ensure the real-valuedness of (2.5), one can ensure that $-\omega$ part of the integrand is exactly the complex conjugate of the $+\omega$ part. Another way is to sum over only the $+\omega$ part of the integral and take twice the real part of the integral. So, for simplicity, we write (2.5) as

$$V(z,t) = 2\Re e \int_0^\infty d\omega V_0(\omega) e^{j(\omega t - kz)} \tag{2.9}$$

Since we need to integrate over $\omega \approx \omega_0$, we can substitute (2.8) into (2.9) and rewrite it as

$$V(z,t) \cong 2\Re e \left[e^{j[\omega_0 t - k(\omega_0)z]} \underbrace{\int_0^\infty d\omega V_0(\omega) e^{j(\omega-\omega_0)t} e^{-j(\omega-\omega_0)\frac{dk}{d\omega}z}}_{F\left(t - \frac{dk}{d\omega}z\right)} \right] \tag{2.10}$$

where more specifically,

$$F\left(t - \frac{dk}{d\omega}z\right) = \int_0^\infty d\omega V_0(\omega) e^{j(\omega-\omega_0)t} e^{-j(\omega-\omega_0)\frac{dk}{d\omega}z} \tag{2.11}$$

It can be seen that the above integral now involves the integral summation over a small range of ω in the vicinity of ω_0. By a change of variable by letting $\Omega = \omega - \omega_0$, it becomes

$$F\left(t - \frac{dk}{d\omega}z\right) = \int_{-\Delta}^{+\Delta} d\Omega V_0(\Omega + \omega_0) e^{j\Omega\left(t - \frac{dk}{d\omega}z\right)} \tag{2.12}$$

The above itself is a Fourier transform integral that involves only the low frequencies of the Fourier spectrum. Hence, F is a slowly varying function. Moreover, this function F moves with a velocity

$$v_g = \frac{d\omega}{dk} \tag{2.13}$$

Here, $F(t - \frac{z}{v_g})$ in fact is the velocity of the envelope in Figure 3. In (2.10), the envelope function $F(t - \frac{z}{v_g})$ is multiplied by the rapidly varying function

$$e^{j[\omega_0 t - k(\omega_0)z]} \tag{2.14}$$

before one takes the real part of the entire function. Hence, this rapidly varying part represents the rapidly varying carrier frequency shown in Figure 3. More importantly, this carrier, the rapidly varying part of the signal, moves with the velocity

$$v_{ph} = \frac{\omega_0}{k(\omega_0)} \tag{2.15}$$

which is the phase velocity.

3 Wave Guidance in a Layered Media

We have seen that in the case of a surface plasmonic resonance, the wave is guided by an interface because the Fresnel reflection coefficient becomes infinite. This physically means that a reflected wave exists even if an incident wave is absent or vanishingly small. This condition can be used to find a guided mode in a layered medium, namely, to find the condition under which the generalized reflection coefficient (1.2) will become infinite.

3.1 Transverse Resonance Condition

Therefore, to have a guided mode exist in a layered medium, the denominator of (1.2) is zero, or that

$$1 + R_{12}\tilde{R}_{23} e^{-2j\beta_{2z} l_2} = 0 \tag{3.1}$$

where t is the thickness of the dielectric slab. Since $R_{12} = -R_{21}$, the above can be written as

$$1 = R_{21}\tilde{R}_{23} e^{-2j\beta_{2z} l_2} \tag{3.2}$$

The above has the physical meaning that the wave, after going through two reflections at the two interfaces, 21, and 23 interfaces, which are R_{21} and R_{23}, plus a phase delay given by $e^{-2j\beta_{2z}l_2}$, becomes itself again. This is also known as the transverse resonance condition. When specialized to the case of a dielectric slab with two interfaces and three regions, the above becomes

$$1 = R_{21}R_{23}e^{-2j\beta_{2z}l_2} \tag{3.3}$$

The above can be generalized to finding the guided mode in a general layered medium. It can also be specialized to finding the guided mode of a dielectric slab.