ECE 604, Lecture 14

Mon, Feb 11, 2019

Contents

Printed on March 24, 2019 at 16 : 31: W.C. Chew and D. Jiao.

1 Reflection and Transmission—Single Interface Case

We will derive the reflection coefficients for the single interface case. These reflection coefficients are also called the Fresnel reflection coefficients because they were first derived by Austin-Jean Fresnel (1788-1827). Note that he lived before the completion of Maxwell's equations in 1865. But when Fresnel derived the reflection coefficients in 1823, they were based on the elastic theory of light; and hence, the formulas are not exactly the same as what we are going to derive (see Born and Wolf, Principles of Optics, p. 40).

1.1 TE Polarization (Perpendicular or E Polarization)

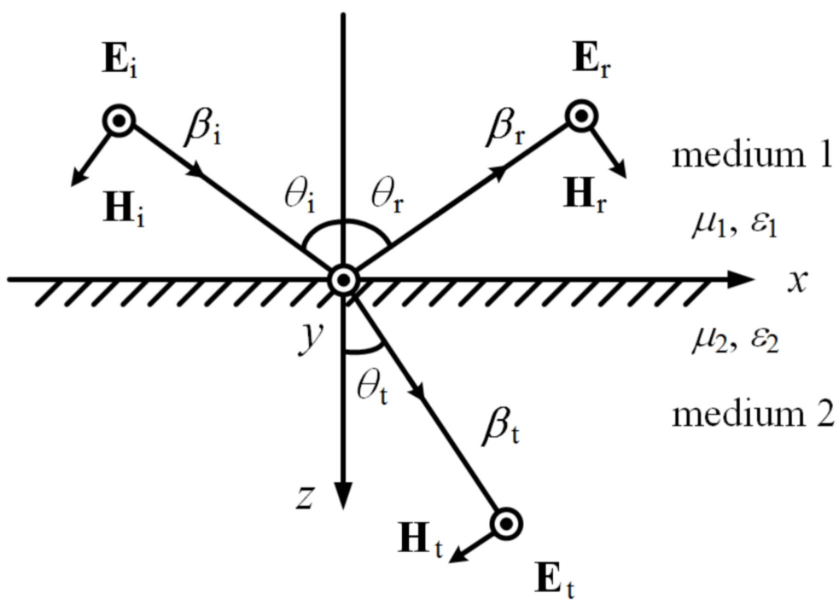

Figure 1:

To set up the above problem, the wave in Region 1 can be written as $\mathbf{E}_i + \mathbf{E}_r$. We assume plane wave polarized in the y direction where the wave vectors are $\boldsymbol{\beta}_i = \hat{x}\beta_{ix} + \hat{z}\beta_{iz}$, $\boldsymbol{\beta}_r = \hat{x}\beta_{rx} - \hat{z}\beta_{rz}$, $\boldsymbol{\beta}_t = \hat{x}\beta_{tx} + \hat{z}\beta_{tz}$, respectively for the incident, reflected, and transmitted waves. Then

$$\mathbf{E}_i = \hat{y}E_0 e^{-j\boldsymbol{\beta}_i\cdot\mathbf{r}} = \hat{y}E_0 e^{-j\beta_{ix}x - j\beta_{iz}z} \tag{1.1}$$

and

$$\mathbf{E}_r = \hat{y}R^{TE}E_0 e^{-j\boldsymbol{\beta}_r\cdot\mathbf{r}} = \hat{y}R^{TE}E_0 e^{-j\beta_{rx}x + j\beta_{rz}z} \tag{1.2}$$

In Region 2, we only have transmitted wave; hence

$$\mathbf{E}_t = \hat{y}T^{TE}E_0 e^{-j\boldsymbol{\beta}_t\cdot\mathbf{r}} = \hat{y}T^{TE}E_0 e^{-j\beta_{tx}x - j\beta_{tz}z} \tag{1.3}$$

In the above, the incident wave is known and hence, E_0 is known. From (1.2) and (1.3), R^{TE} and T^{TE} are unknowns yet to be sought. To find them, we need two boundary conditions to yield two equations. These are tangential $\mathbf{E}$ continuous and tangential $\mathbf{H}$ continuous, which are $\hat{n}\times\mathbf{E}$ continuous and $\hat{n}\times\mathbf{H}$ continuous conditions at the interface.

Imposing $\hat{n}\times\mathbf{E}$ continuous at $z = 0$, we get

$$E_0 e^{-j\beta_{ix}x} + R^{TE}E_0 e^{-j\beta_{rx}x} = T^{TE}E_0 e^{-j\beta_{tx}x}, \quad \forall x \tag{1.4}$$

In order for the above to be valid for all x, it is necessary that $\beta_{ix} = \beta_{rx} = \beta_{tx}$, which is also known as the phase matching condition.[1] From the above, by letting $\beta_{ix} = \beta_{rx} = \beta_1 \sin\theta_i = \beta_1 \sin\theta_r$, we obtain that $\theta_r = \theta_i$ or that the law of reflection that the angle of reflection is equal to the angle of incidence. By letting $\beta_{tx} = \beta_2 \sin\theta_t = \beta_{ix} = \beta_1 \sin\theta_i$, we obtain Snell's law that $\beta_1 \sin\theta_i = \beta_2 \sin\theta_t$, a law of refraction that was also known in the Islamic world in the 900 AD.

Now, canceling common terms on both sides of the equation (1.4), the above simplifies to

$$1 + R^{TE} = T^{TE} \tag{1.5}$$

To impose $\hat{n}\times\mathbf{H}$ continuous, one needs to find the $\mathbf{H}$ field using $\nabla\times\mathbf{E} = -j\omega\mu\mathbf{H}$, or that $\mathbf{H} = -j\boldsymbol{\beta}\times\mathbf{E}/(-j\omega\mu) = \boldsymbol{\beta}\times\mathbf{E}/(\omega\mu)$. By so doing

$$\mathbf{H}_i = \frac{\boldsymbol{\beta}_i\times\mathbf{E}_i}{\omega\mu_1} = \frac{\boldsymbol{\beta}_i\times\hat{y}}{\omega\mu_1}E_0 e^{-j\boldsymbol{\beta}_i\cdot\mathbf{r}} = \frac{\hat{z}\beta_{ix} - \hat{x}\beta_{iz}}{\omega\mu_1}E_0 e^{-j\boldsymbol{\beta}_i\cdot\mathbf{r}} \tag{1.6}$$

$$\mathbf{H}_r = \frac{\boldsymbol{\beta}_r\times\mathbf{E}_r}{\omega\mu_1} = \frac{\boldsymbol{\beta}_r\times\hat{y}}{\omega\mu_1}R^{TE}E_0 e^{-j\boldsymbol{\beta}_r\cdot\mathbf{r}} = \frac{\hat{z}\beta_{rx} + \hat{x}\beta_{rz}}{\omega\mu_2}R^{TE}E_0 e^{-j\boldsymbol{\beta}_r\cdot\mathbf{r}} \tag{1.7}$$

$$\mathbf{H}_t = \frac{\boldsymbol{\beta}_t\times\mathbf{E}_t}{\omega\mu_2} = \frac{\boldsymbol{\beta}_t\times\hat{y}}{\omega\mu_2}T^{TE}E_0 e^{-j\boldsymbol{\beta}_t\cdot\mathbf{r}} = \frac{\hat{z}\beta_{tx} - \hat{x}\beta_{tz}}{\omega\mu_2}T^{TE}E_0 e^{-j\boldsymbol{\beta}_t\cdot\mathbf{r}} \tag{1.8}$$

Imposing $\hat{n}\times\mathbf{H}$ continuous or H_x continuous at $z = 0$, we have

$$\frac{\beta_{iz}}{\omega\mu_1}E_0 e^{-j\beta_{ix}x} - \frac{\beta_{rz}}{\omega\mu_1}R^{TE}E_0 e^{-j\beta_{rx}x} = \frac{\beta_{tz}}{\omega\mu_2}T^{TE}E_0 e^{-j\beta_{tx}x} \tag{1.9}$$

As mentioned before, the phase-matching condition requires that $\beta_{ix} = \beta_{rx} = \beta_{tx}$. The dispersion relation for plane waves requires that

$$\beta_{ix}^2 + \beta_{iz}^2 = \beta_{rx}^2 + \beta_{rz}^2 = \omega^2\mu_1\varepsilon_1 = \beta_1^2 \tag{1.10}$$

$$\beta_{tx}^2 + \beta_{tz}^2 = \omega^2\mu_2\varepsilon_2 = \beta_2^2 \tag{1.11}$$

[1] The phase-matching condition can also be proved by taking the Fourier transform of the equation with respect to x.

Since $\beta_{ix} = \beta_{rx} = \beta_{tx} = \beta_x$, the above implies that $\beta_{iz} = \beta_{rz} = \beta_{1z}$. Moreover, $\beta_{tz} = \beta_{2z} \neq \beta_{1z}$ usually since $\beta_1 \neq \beta_2$. Then (1.9) simplifies to

$$\frac{\beta_{1z}}{\mu_1}(1 - R^{TE}) = \frac{\beta_{2z}}{\mu_2} T^{TE} \tag{1.12}$$

where $\beta_{1z} = \sqrt{\beta_1^2 - \beta_x^2}$, and $\beta_{2z} = \sqrt{\beta_2^2 - \beta_x^2}$.

Solving (1.5) and (1.12) yields

$$R^{TE} = \left(\frac{\beta_{1z}}{\mu_1} - \frac{\beta_{2z}}{\mu_2}\right) \bigg/ \left(\frac{\beta_{1z}}{\mu_1} + \frac{\beta_{2z}}{\mu_2}\right) \tag{1.13}$$

$$T^{TE} = 2\left(\frac{\beta_{1z}}{\mu_1}\right) \bigg/ \left(\frac{\beta_{1z}}{\mu_1} + \frac{\beta_{2z}}{\mu_2}\right) \tag{1.14}$$

1.2 TM Polarization (Parallel or H Polarization)

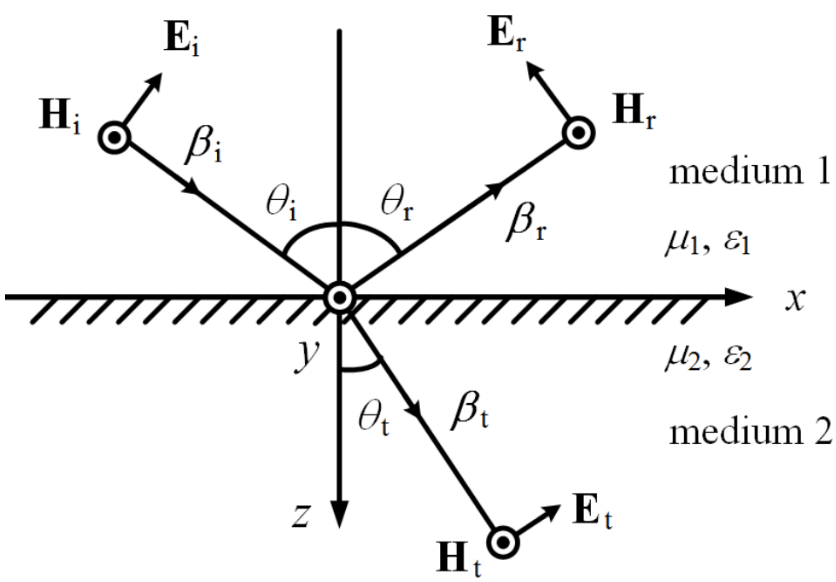

Figure 2:

The solution to the TM polarization case can be obtained by invoking duality principle where we do the substitution $\mathbf{E} \to \mathbf{H}$, $\mathbf{H} \to -\mathbf{E}$, and $\mu \rightleftharpoons \varepsilon$ as shown in Figure 2. The reflection coefficient for the TM magnetic field is then

$$R^{TM} = \left(\frac{\beta_{1z}}{\varepsilon_1} - \frac{\beta_{2z}}{\varepsilon_2}\right) \bigg/ \left(\frac{\beta_{1z}}{\varepsilon_1} + \frac{\beta_{2z}}{\varepsilon_2}\right) \tag{1.15}$$

$$T^{TM} = 2\left(\frac{\beta_{1z}}{\varepsilon_1}\right) \bigg/ \left(\frac{\beta_{1z}}{\varepsilon_1} + \frac{\beta_{2z}}{\varepsilon_2}\right) \tag{1.16}$$

Please remember that R^{TM} and T^{TM} are reflection and transmission coefficients for the magnetic fields, whereas R^{TE} and T^{TE} are those for the electric fields. Some textbooks may define these reflection coefficients based on electric field only, and they will look different, and duality principle cannot be applied.

2 Interesting Physical Phenomena

Three interesting physical phenomena emerge from the solutions of the single-interface problem. They are total internal reflection, Brewster angle effect, and surface plasmonic resonance. We will look at them next.

2.1 Total Internal Reflection

Total internal reflection comes about because of phase matching also called momentum matching. This phase-matching condition can be illustrated using β-surfaces (same as k-surfaces in some literature), as shown in Figure 3.

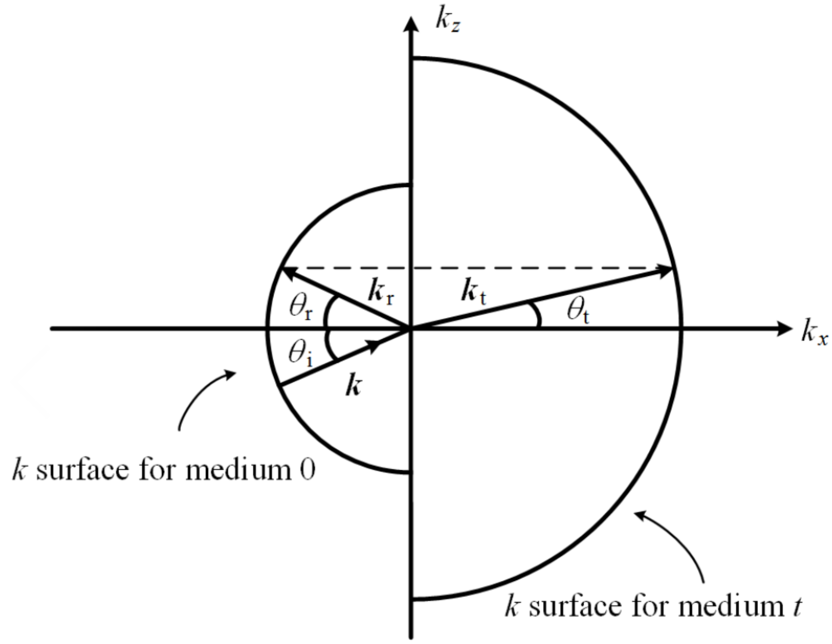

Figure 3: Courtesy of J.A. Kong, Electromagnetic Wave Theory. Here, k is synonymous with β. Also, the x axis is equivalent to the z axis in the previous figure.

It turns out that because of phase matching, for certain interfaces, β_{2z} becomes pure imaginary.

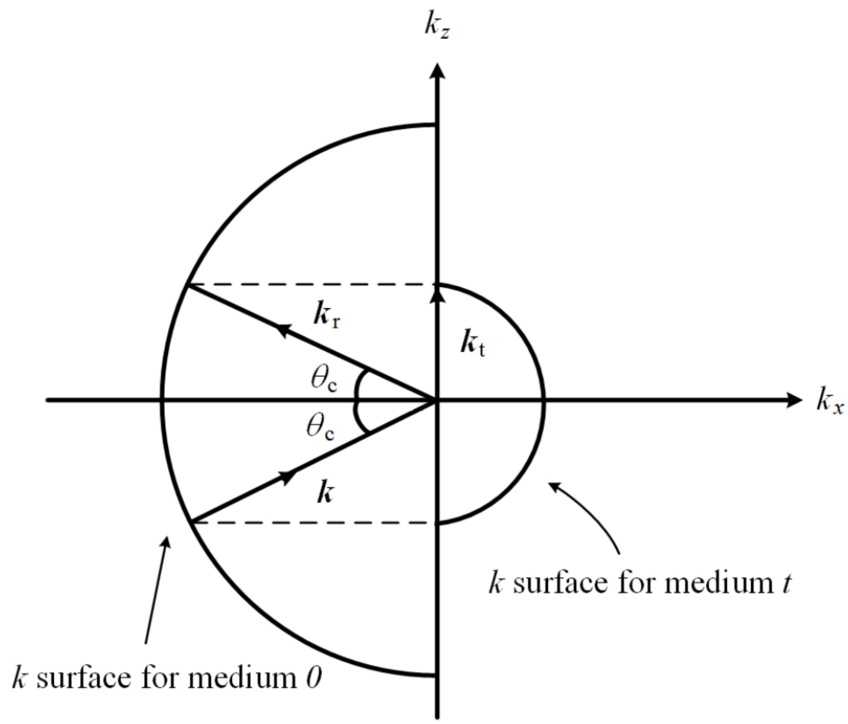

Figure 4: Courtesy of J.A. Kong, Electromagnetic Wave Theory. Here, k is synonymous with β, and x axis is the same as our z axis.

As shown in Figures 3 and 4, because of the dispersion relation that $\beta_{rx}^2 + \beta_{rz}^2 = \beta_{ix}^2 + \beta_{iz}^2 = \beta_1^2$, $\beta_{tx}^2 + \beta_{tz}^2 = \beta_2^2$, they are equations of two circles in 2D whose radii are β_1 and β_2, respectively. The tips of the $\boldsymbol{\beta}$ vectors for Regions 1 and 2 have to be on a spherical surface in the β_x, β_y, and β_z space in the general 3D case, but in this figure, we only show a cross section of the sphere assuming that $\beta_y = 0$.

Phase matching implies that the x-component of the $\boldsymbol{\beta}$ vectors are equal to each other as shown. One sees that $\theta_i = \theta_r$ in Figure 4, and also as θ_i increases, θ_t increases. For an optically less dense medium where $\beta_2 < \beta_1$, according to the Snell's law of refraction, the transmitted $\boldsymbol{\beta}$ will refract away from the normal, as seen in the figure. Therefore, eventually the vector $\boldsymbol{\beta}_t$ becomes parallel to the x axis when $\beta_{ix} = \beta_{rx} = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$ and $\theta_t = \pi/2$. The incident angle at which this happens is termed the critical angle θ_c.

Since $\beta_{ix} = \beta_1 \sin\theta_i = \beta_{rx} = \beta_1 \sin\theta_r = \beta_2$, or

$$\sin\theta_r = \sin\theta_i = \sin\theta_c = \frac{\beta_2}{\beta_1} = \frac{\sqrt{\mu_2\varepsilon_2}}{\sqrt{\mu_1\varepsilon_1}} = \frac{n_2}{n_1} \tag{2.1}$$

where n_1 is the reflective index defined as $c_0/v_i = \sqrt{\mu_i\varepsilon_i}/\sqrt{\mu_0\varepsilon_0}$ where v_i is the phase velocity of the wave in Region i. Hence,

$$\theta_c = \sin^{-1}(n_2/n_1) \tag{2.2}$$

When $\theta_i > \theta_c$. $\beta_x > \beta_2$ and $\beta_{2z} = \sqrt{{\beta_2}^2 - {\beta_x}^2}$ becomes pure imaginary. When β_{2z} becomes pure imaginary, the wave cannot propagate in Region 2, or $\beta_{2z} = -j\alpha_{2z}$, and the wave becomes evanescent. The reflection coefficient (1.13) becomes of the form

$$R^{TE} = (A - jB)/(A + jB) \tag{2.3}$$

It is clear that $|R^{TE}| = 1$ and that $R^{TE} = e^{j\theta_{TE}}$. Therefore, a total internally reflected wave suffers a phase shift. A phase shift in the frequency domain corresponds to a time delay in the time domain. Such a time delay is achieved by the wave traveling laterally in Region 2 before being refracted back to Region 1. Such a lateral shift is called the Goos-Hanschen shift as shown in Figure 5.

Please be reminded that total internal reflection comes about entirely due to the phase-matching condition when Region 2 is a faster medium than Region 1. Hence, it will occur with all manner of waves, such as elastic waves, sound waves, seismic waves, quantum waves etc.

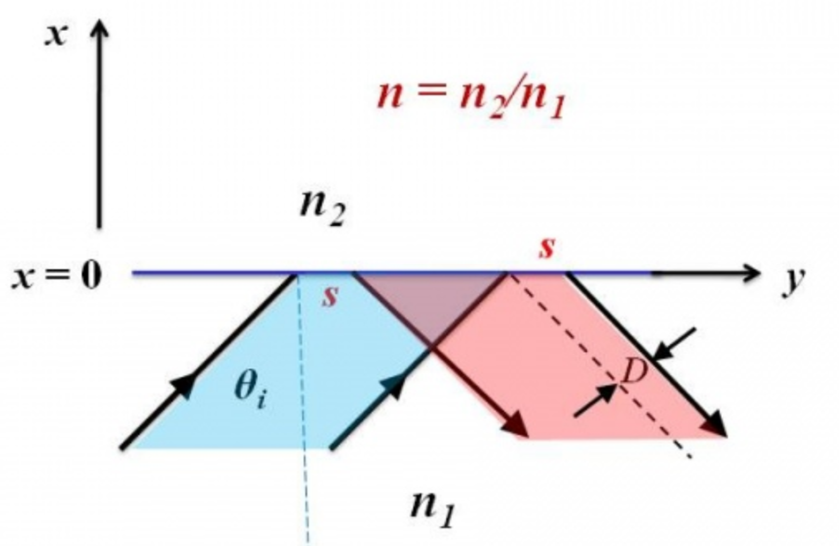

Figure 5: Goos-Hanschen Shift. Courtesy of Paul R. Berman (2012), Scholarpedia, 7(3):11584.

The guidance of a wave in a dielectric slab is due to total internal reflection at the dielectric-to-air interface. The wave bounces between the two interfaces of the slab, and creates evanescent waves outside, as shown in Figure 6. The guidance of waves in an optical fiber works by similar mechanism, as shown in Figure 7.

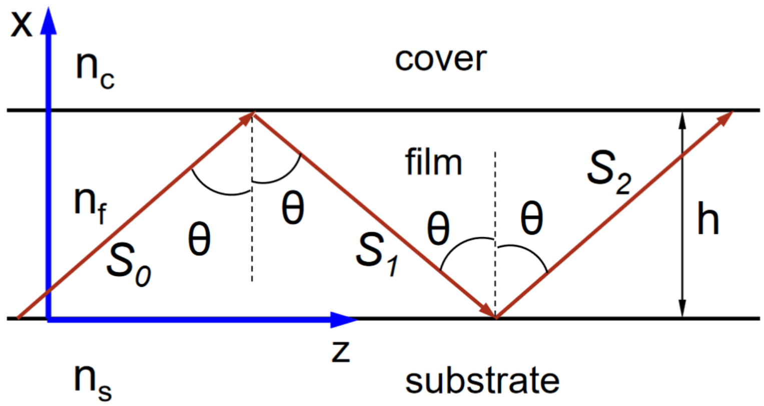

Figure 6: Courtesy of E.N. Glytsis, NTUA, Greece.

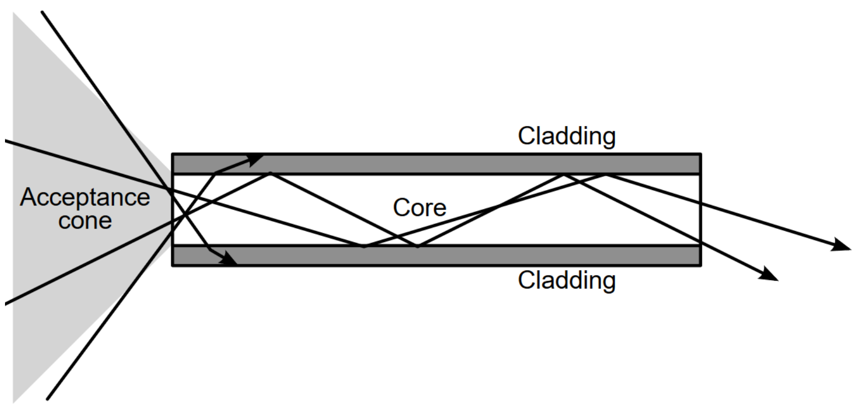

Figure 7: Courtesy of Wikepedia.