ECE 604, Lecture 10

Fri, Feb 1, 2019

Contents

[1]The derivation here is inspired by H.A. Haus, Electromagnetic Noise and Quantum Optical Measurements. Generalization to anisotropic media is given by W.C. Chew, Lectures on Theory of Microwave and Optical Waveguides.

Printed on March 24, 2019 at 16 : 18: W.C. Chew and D. Jiao.

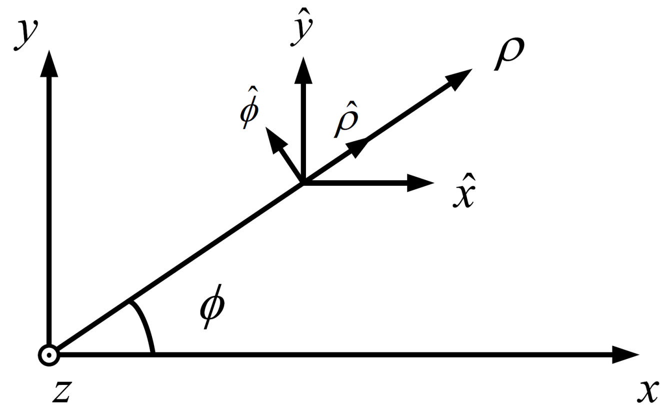

Figure 1:

1 Spin Angular Momentum and Cylindrical Vector Beam

In this section, we will study the spin angular momentum of a circularly polarized wave. It is to be noted that in cylindrical coordinates, as shown in Figure 1, $\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$, $\hat{y} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi$, then

$$(\hat{x} \pm j\hat{y}) = \hat{\rho}e^{\pm j\phi} \pm j\hat{\phi}e^{\pm j\phi} = e^{\pm j\phi}(\hat{\rho} \pm \hat{\phi}) \tag{1.1}$$

Therefore, the $\hat{\rho}$ and $\hat{\phi}$ of a CP is also an azimuthal traveling wave in the $\hat{\phi}$ direction in addition to being a traveling wave $e^{-j\beta z}$ in the $\hat{z}$ direction. This can be obviated by writing

$$e^{-j\phi} = e^{-jk_\phi \rho\phi} \tag{1.2}$$

where $k_\phi = 1/\rho$ is the azimuthal wave number, and $\rho\phi$ is the arc length traversed by the azimuthal wave.

Thus, the wave possesses angular momentum called the spin angular momentum (SAM), just as a traveling wave $e^{-j\beta z}$ possesses linear angular momentum in the $\hat{z}$ direction.

In optics research, the generation of cylindrical vector beam is in vogue. Figure 2 shows a method to generate such a beam. A CP light passes through a radial analyzer that will only allow the radial component of (1.1) to be transmitted. Then a spiral phase element (SPE) compensates for the $\exp(\pm j\phi)$ phase shift in the azimuthal direction. Finally, the light is a cylindrical vector beam which is radially polarized without spin angular momentum. Such a beam has been found to have nice focussing property, and hence, has aroused researchers in the optics community.

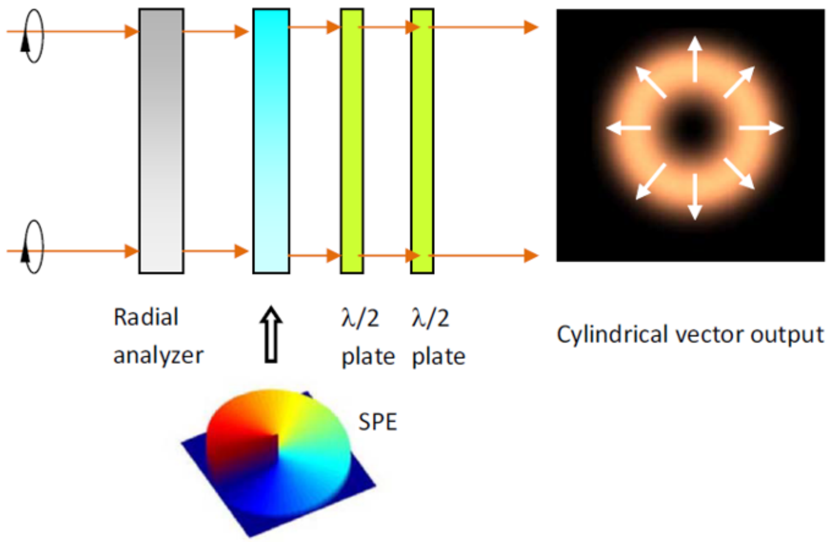

Figure 2: Courtesy of Zhan, Q. (2009). Cylindrical vector beams: from mathematical concepts to applications.Advances in Optics and Photonics,1(1), 1-57.

2 Complex Poynting's Theorem and Lossless Conditions

2.1 Complex Poynting's Theorem

It has been previously shown that the vector $\mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)$ has a dimension of watts/m^2 which is that of power density. Therefore, it is associated with the direction of power flow. As has been shown for time-harmonic field, a time average of this vector can be defined as

$$\langle \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)\, dt. \tag{2.1}$$

Given the phasors of time harmonic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$, namely, $\mathbf{E}(\mathbf{r},\omega)$ and $\mathbf{H}(\mathbf{r},\omega)$ respectively, we can show that

$$\langle \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t) \rangle = \frac{1}{2} \Re e\{\mathbf{E}(\mathbf{r},\omega) \times \mathbf{H}^*(\mathbf{r},\omega)\}. \tag{2.2}$$

Here, the vector $\mathbf{E}(\mathbf{r},\omega) \times \mathbf{H}^*(\mathbf{r},\omega)$, as previously discussed, is also known as the complex Poynting vector. Moreover, because of its aforementioned property, and its dimension of power density, we will study its conservative property. To do so, we take its divergence and use the appropriate vector identity to obtain[2]

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^*. \tag{2.3}$$

Next, using Maxwell's equations for $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{H}^*$, namely

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B} \tag{2.4}$$

$$\nabla \times \mathbf{H}^* = -j\omega \mathbf{D}^* + \mathbf{J}^* \tag{2.5}$$

and the constitutive relations for anisotropic media that

$$\mathbf{B} = \overline{\boldsymbol{\mu}} \cdot \mathbf{H}, \quad \mathbf{D}^* = \overline{\varepsilon}^* \cdot \mathbf{E}^* \tag{2.6}$$

we have

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) &= -j\omega\, \mathbf{H}^* \cdot \mathbf{B} + j\omega\, \mathbf{E} \cdot \mathbf{D}^* - \mathbf{E} \cdot \mathbf{J}^* && (2.7) \\ &= -j\omega\, \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} + j\omega\, \mathbf{E} \cdot \overline{\varepsilon}^* \cdot \mathbf{E}^* - \mathbf{E} \cdot \mathbf{J}^*. && (2.8) \end{aligned}$$

The above is also known as the complex Poynting's theorem. It can also be written in an integral form using Gauss' divergence theorem, namely,

$$\int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = -j\omega \int_V dV (\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \overline{\varepsilon}^* \cdot \mathbf{E}^*) - \int_V dV \mathbf{E} \cdot \mathbf{J}^*. \tag{2.9}$$

where S is the surface bounding the volume V.

2.2 Lossless Conditions

For a region V that is lossless and source-free, $\mathbf{J} = 0$. There should be no net time-averaged power-flow out of or into this region V. Therefore,

$$\Re e \int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = 0, \tag{2.10}$$

Because of energy conservation, the real part of the right-hand side of (2.8), without the $\mathbf{E} \cdot \mathbf{J}^*$ term, must be zero. In other words, the right-hand side of (2.8) should be purely imaginary. Thus

$$\int_V dV (\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \overline{\varepsilon}^* \cdot \mathbf{E}^*) \tag{2.11}$$

must be a real quantity.

Other than the possibility that the above is zero, the general requirement for (2.11) to be real for arbitrary $\mathbf{E}$ and $\mathbf{H}$, is that $\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ and $\mathbf{E} \cdot \overline{\varepsilon}^* \cdot \mathbf{E}^*$ are real

[2]We will drop the argument $\mathbf{r}, \omega$ for the phasors in our discussion next as they will be implied.

quantities. Notice that they are also scalar numbers. But since the conjugate transpose of a real scalar number is itself, we have $(\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H})^\dagger = \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ where $\dagger$ implies conjugate transpose. The above, in detail, using the rule of matrix algebra that $(\overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \cdot \overline{\mathbf{C}})^t = \overline{\mathbf{C}}^t \cdot \overline{\mathbf{B}}^t \cdot \overline{\mathbf{A}}^t$, implies that[3]

$$(\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H})^\dagger = (\mathbf{H} \cdot \overline{\boldsymbol{\mu}}^* \cdot \mathbf{H}^*)^t = \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}}^\dagger \cdot \mathbf{H} = \mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}. \tag{2.12}$$

The last equality in the above is possible only if $\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^\dagger$ or that $\overline{\boldsymbol{\mu}}$ is hermitian. Therefore, the conditions for anisotropic media to be lossless are

$$\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^\dagger, \qquad \overline{\boldsymbol{\varepsilon}} = \overline{\boldsymbol{\varepsilon}}^\dagger, \tag{2.13}$$

requiring the permittivity and permeability tensors to be hermitian. If this is the case, (2.11) is always real for arbitraty $\mathbf{E}$ and $\mathbf{H}$, and (2.10) is true, implying a lossless region V. Notice that for an isotropic medium, this lossless conditions reduce simply to that $\Im m(\mu) = 0$ and $\Im m(\varepsilon) = 0$, or that μ and ε are pure real quantities. Hence, many of the effective permittivities or dielectric constants that we have derived using the Drude-Lorentz-Sommerfeld model cannot be lossless when the friction term is involved.

If a medium is source-free, but lossy, then $\Re e \int d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) < 0$. In other words, time-average power must flow inward to the volume V. Consequently, from (2.9) without the source term $\mathbf{J}$, this implies

$$\Im m \int_V dV (\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \mathbf{E}^*) < 0. \tag{2.14}$$

But the above, using that $\Im m(Z) = 1/(2j)(Z - Z^*)$, is the same as

$$-j \int_V dV [\mathbf{H}^* \cdot (\overline{\boldsymbol{\mu}}^\dagger - \overline{\boldsymbol{\mu}}) \cdot \mathbf{H} + \mathbf{E}^* \cdot (\overline{\boldsymbol{\varepsilon}}^\dagger - \overline{\boldsymbol{\varepsilon}}) \cdot \mathbf{E}] > 0. \tag{2.15}$$

Therefore, for a medium to be lossy, $-j(\overline{\boldsymbol{\mu}}^\dagger - \overline{\boldsymbol{\mu}})$ and $-j(\overline{\boldsymbol{\varepsilon}}^\dagger - \overline{\boldsymbol{\varepsilon}})$ must be hermitian, positive definite matrices, to ensure the inequality in (2.15). Similarly, for an active medium, $-j(\overline{\boldsymbol{\mu}}^\dagger - \overline{\boldsymbol{\mu}})$ and $-j(\overline{\boldsymbol{\varepsilon}}^\dagger - \overline{\boldsymbol{\varepsilon}})$ must be hermitian, negative definite matrices.

For a lossy medium which is conductive, we may define $\mathbf{J} = \overline{\boldsymbol{\sigma}} \cdot \mathbf{E}$ where $\overline{\boldsymbol{\sigma}}$ is a conductivity tensor. In this case, equation (2.9), after combining the last two terms, may be written as

$$\int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = -j\omega \int_V dV \left[\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \left(\overline{\boldsymbol{\varepsilon}}^* + \frac{j\overline{\boldsymbol{\sigma}}^*}{\omega} \right) \cdot \mathbf{E}^* \right] \tag{2.16}$$

$$= -j\omega \int dV [\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \tilde{\overline{\boldsymbol{\varepsilon}}}^* \cdot \mathbf{E}^*], \tag{2.17}$$

[3] In physics notation, the transpose of a vector is implied in a dot product.

where $\tilde{\overline{\varepsilon}} = \overline{\varepsilon} - \frac{j\overline{\sigma}}{\omega}$ which is the complex permittivity tensor. In this manner, (2.17) has the same structure as the source-free Poynting's theorem. Notice here that the complex permittivity tensor $\tilde{\overline{\varepsilon}}$ is clearly non-hermitian corresponding to a lossy medium.

For a lossless medium without the source term, by taking the imaginary part of (2.9), we arrive at

$$\Im m \int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}^*) = -\omega \int_V dV (\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H} - \mathbf{E} \cdot \overline{\varepsilon}^* \cdot \mathbf{E}^*), \tag{2.18}$$

The left-hand side of the above is the reactive power coming out of the volume V, and hence, the right-hand side can be interpreted as reactive power as well. It is to be noted that $\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ and $\mathbf{E} \cdot \overline{\varepsilon}^* \cdot \mathbf{E}^*$ are not to be interpreted as stored energy density when the medium is dispersive. The correct expressions for stored energy density will be derived in the next section.

But, the quantity $\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \mathbf{H}$ for lossless, dispersionless media is associated with the time-averaged energy density stored in the magnetic field, while the quantity $\mathbf{E} \cdot \overline{\varepsilon}^* \cdot \mathbf{E}^*$ for lossless dispersionless media is associated with the time-averaged energy density stored in the electric field. Then, for lossless, dispersionless, source-free media, then the right-hand side of the above can be interpreted as stored energy density. Hence, the reactive power is proportional to the time rate of change of the difference of the time-averaged energy stored in the magnetic field and the electric field.

3 Energy Density in Dispersive Media[4]

A dispersive medium alters our concept of what energy density is. To this end, we assume that the field has complex ω dependence in $e^{j\omega t}$, where $\omega = \omega' - j\omega''$, rather than real ω dependence. We take the divergence of the complex power for fields with such time dependence, and let $e^{j\omega t}$ be attached to the field. So $\mathbf{E}(t)$ and $\mathbf{H}(t)$ are complex field but not exactly like phasors since they are not truly time harmonic. Hence,

$$\begin{aligned} \nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] &= \mathbf{H}^*(t) \cdot \nabla \times \mathbf{E}(t) - \mathbf{E}(t) \cdot \nabla \times \mathbf{H}^*(t) \\ &= -\mathbf{H}^*(t) \cdot j\omega\mu\mathbf{H}(t) + \mathbf{E}(t) \cdot j\omega^*\varepsilon^*\mathbf{E}^* \end{aligned} \tag{3.1}$$

where Maxwell's equations have been used to substitute for $\nabla \times \mathbf{E}(t)$ and $\nabla \times \mathbf{H}^*(t)$. The space dependence of the field is implied, and we assure a source-free medium so that $\mathbf{J} = 0$.

If $\mathbf{E}(t) \sim e^{j\omega t}$, then, due to ω being complex, now $\mathbf{H}^*(t) \sim e^{-j\omega^* t}$, and the term like

$$\mathbf{E}(t) \times \mathbf{H}^*(t) \sim e^{j(\omega - \omega^*)t} = e^{2\omega'' t} \tag{3.2}$$

[4] The derivation here is inspired by H.A. Haus, Electromagnetic Noise and Quantum Optical Measurements. Generalization to anisotropic media is given by W.C. Chew, Lectures on Theory of Microwave and Optical Waveguides.

And each of the term above will have similar time dependence. Writing (3.1) more explicitly, by letting $\omega = \omega' - j\omega''$, we have

$$\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] = -j(\omega' - j\omega'')\mu(\omega)|\mathbf{H}(t)|^2 + j(\omega' + j\omega'')\varepsilon^*(\omega)|\mathbf{E}(t)|^2 \tag{3.3}$$

Assuming that $\omega'' \ll \omega'$, or that the field is quasi-time-harmonic, we can let, after using Taylor series approximation, that

$$\mu(\omega' - j\omega'') \cong \mu(\omega') - j\omega'' \frac{\partial \mu(\omega')}{\partial \omega'}, \quad \varepsilon(\omega' - j\omega'') \cong \varepsilon(\omega') - j\omega'' \frac{\partial \varepsilon(\omega')}{\partial \omega'} \tag{3.4}$$

Using (3.4) in (3.3), and collecting terms of the same order gives

$$\begin{aligned}\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] = &- j\omega'\mu(\omega')|\mathbf{H}(t)|^2 + j\omega'\varepsilon^*(\omega')|\mathbf{E}(t)|^2 \\ &- \omega''\mu(\omega')|\mathbf{H}(t)|^2 - \omega'\omega''\frac{\partial \mu}{\partial \omega'}|\mathbf{H}(t)|^2 \\ &- \omega''\varepsilon^*(\omega')|\mathbf{E}(t)|^2 - \omega'\omega''\frac{\partial \varepsilon^*}{\partial \omega'}|\mathbf{E}(t)|^2\end{aligned} \tag{3.5}$$

The above can be rewritten as

$$\begin{aligned}\nabla \cdot [\mathbf{E}(t) \times \mathbf{H}^*(t)] = &-j\omega' \left[\mu(\omega')|\mathbf{H}(t)|^2 - \varepsilon^*(\omega')|\mathbf{E}(t)|^2\right] \\ &- \omega'' \left[\frac{\partial \omega'\mu(\omega')}{\partial \omega'}|\mathbf{H}(t)|^2 + \frac{\partial \omega'\varepsilon^*(\omega')}{\partial \omega'}|\mathbf{E}(t)|^2\right]\end{aligned} \tag{3.6}$$

For a lossless medium, $\varepsilon(\omega')$ is purely real, and the first term of the right-hand side is purely imaginary while the second term is purely real. In the limit when $\omega'' \to 0$, when we take half the imaginary part of the above equation, we have

$$\nabla \cdot \frac{1}{2}\Im m\,[\mathbf{E} \times \mathbf{H}^*] = -\omega' \left[\frac{1}{2}\mu|\mathbf{H}|^2 - \frac{1}{2}\varepsilon|\mathbf{E}|^2\right] \tag{3.7}$$

which has the physical interpretation of reactive power as has been previously discussed. When we take half the real part of (3.6), we obtain

$$\nabla \cdot \frac{1}{2}\Re e[\mathbf{E} \times \mathbf{H}^*] = -\frac{\omega''}{2} \left[\frac{\partial \omega'\mu}{\partial \omega'}|\mathbf{H}|^2 + \frac{\partial \omega'\varepsilon}{\partial \omega'}|\mathbf{E}|^2\right] \tag{3.8}$$

Since the right-hand side has time dependence of $e^{2\omega'' t}$, it can be written as

$$\nabla \cdot \frac{1}{2}\Re e[\mathbf{E} \times \mathbf{H}^*] = -\frac{\partial}{\partial t}\frac{1}{4} \left[\frac{\partial \omega'\mu}{\partial \omega'}|\mathbf{H}|^2 + \frac{\partial \omega'\varepsilon}{\partial \omega'}|\mathbf{E}|^2\right] = -\frac{\partial}{\partial t}\langle W_T \rangle \tag{3.9}$$

Therefore, the time-average stored energy density can be identified as

$$\langle W_T \rangle = \frac{1}{4} \left[\frac{\partial \omega'\mu}{\partial \omega'}|\mathbf{H}|^2 + \frac{\partial \omega'\varepsilon}{\partial \omega'}|\mathbf{E}|^2\right] \tag{3.10}$$

For a non-dispersive medium, the above reduces to

$$\langle W_T \rangle = \frac{1}{4}\left[\mu|\mathbf{H}|^2 + \varepsilon|\mathbf{E}|^2\right] \tag{3.11}$$

which is what we have derived before. In the above analysis, we have used a quasi-time-harmonic signal with $\exp(j\omega t)$ dependence. In the limit when $\omega'' \to 0$, this signal reverts back to a time-harmonic signal, and our interpretation of complex power. However, by assuming the frequency ω to have a small imaginary part ω'', it forces the stored energy to grow very slightly, and hence, power has to be supplied to maintain the growth of this stored energy. By so doing, it allows us to identify the expression for energy density for a dispersive medium.