W.C.Chew
ECE 350 Lecture Notes

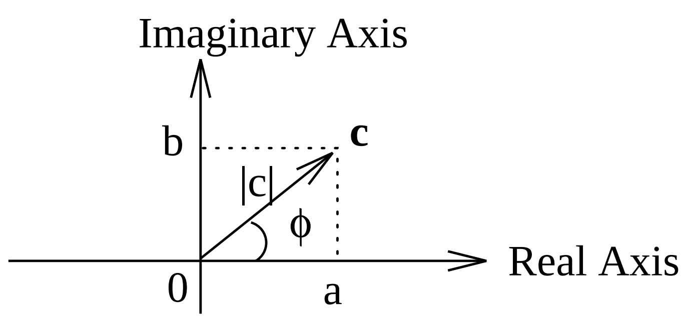
1. Elements of Complex Algebra

Complex numbers are extensions of real numbers, and they make the number fields complete in the sense that an n-th order polynomial has n-th roots in a complex field while it is not always true in the real field. Complex numbers are also very useful in time harmonic analysis of engineering and physical systems, because they considerably simplify the analysis.

A complex number can be represented in cartesian form as

$$c = a + jb \tag{1}$$

where $j = \sqrt{-1}$. a is the ***real part*** of c while b is the ***imaginary part*** of c. On the complex plane, c is represented by a point c or sometimes an arrow oc as shown.



Sometimes it is more convenient to represent c in polar form, i.e.

$$c = a + jb = |c|\, e^{j\phi} = |c| \cos\phi + j\, |c| \sin\phi \tag{2}$$

where $|c| = \sqrt{a^2 + b^2}$ is the magnitude or the absolute value of c.
From (2), it is seen that

$$\tan\phi = \frac{b}{a} \quad \Rightarrow \quad \phi = \tan^{-1}\frac{b}{a} \tag{3}$$

where ϕ is the phase of c.

Addition and Subtraction

Addition and subtraction of complex numbers are carried out in Cartesian forms.

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2. Review of Vector Analysis

A vector $\mathbf{A}$ can be written as

$$\mathbf{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z. \tag{1}$$

Similarly, a vector $\mathbf{B}$ can be written as

$$\mathbf{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z. \tag{2}$$

In the above, $\hat{x}, \hat{y}, \hat{z}$ are unit vectors pointing in the x, y, z directions respectively. A_x, A_y and A_z are the components of the vector $\mathbf{A}$ in the x, y, z directions respectively. The same statement applies to B_x, B_y, and B_z.

Addition

$$\mathbf{A} + \mathbf{B} = \hat{x}(A_x + B_x) + \hat{y}(A_y + B_y) + \hat{z}(A_z + B_z). \tag{3}$$

Multiplication

(a) Dot Product (scalar product)

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z, \tag{4}$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}, \qquad \text{commutative property} \tag{5}$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}, \qquad \text{distributive property} \tag{6}$$

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta. \tag{7}$$

In (7), θ is the angle between vectors $\mathbf{A}$ and $\mathbf{B}$.

(b) Cross Product (vector product)

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(A_yB_z - A_zB_y) + \hat{y}(A_zB_x - A_xB_z) + \hat{z}(A_xB_y - A_yB_x), \tag{8}$$

$$\mathbf{A} \times \mathbf{B} = \hat{u}\,|\mathbf{A}|\,|\mathbf{B}|\sin\theta, \tag{9}$$

where $\hat{u}$ is a unit vector obtained from $\mathbf{A}$ and $\mathbf{B}$ via the right hand rule.

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \qquad \text{distributive property} \tag{10}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}, \quad \text{non-associative property} \tag{11}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \qquad \text{anti-commutative property} \tag{12}$$

Vector Derivatives

$$\text{Del} \quad \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}, \tag{13}$$

$$\text{Gradient} \quad \nabla\phi = \hat{x}\frac{\partial}{\partial x}\phi + \hat{y}\frac{\partial}{\partial y}\phi + \hat{z}\frac{\partial}{\partial z}\phi, \tag{14}$$

$$\text{Divergent} \quad \nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z, \tag{15}$$

$$\begin{aligned}
\text{Curl} \quad \nabla \times \mathbf{A} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
&= \hat{x}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) + \hat{y}\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) \\
&\quad + \hat{z}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_z\right).
\end{aligned} \tag{16}$$

Divergence Theorem

$$\oint_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot \hat{n} dS. \tag{17}$$

Stokes Theorem

$$\oint_S (\nabla \times \mathbf{A}) \cdot \hat{n} dS = \oint_C \mathbf{A} \cdot d\mathbf{l}. \tag{18}$$

Some Useful Vector Identities

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}), \tag{19}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\,(\mathbf{a}\cdot\mathbf{b}), \tag{20}$$

$$\mathbf{a}\times\mathbf{a} = 0, \tag{21}$$

$$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0, \tag{22}$$

$$\nabla\times(\nabla\phi) = 0, \tag{23}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0, \tag{24}$$

$$\nabla\cdot(\psi\mathbf{A}) = \mathbf{A}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{A}, \tag{25}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B}, \tag{26}$$

$$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\cdot\nabla\mathbf{A}, \tag{27}$$

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{28}$$

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3. Wave Equation from Maxwell's Equations

Lossless Medium

In a source free region, Maxwell's equations are

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t,} \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\nabla \cdot \mathbf{D} = 0, \tag{4}$$

where $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = \epsilon \mathbf{E}$. Taking the curl of (2), we have

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H}. \tag{5}$$

Substituting (1) into (5), we obtain

$$\nabla \times \nabla \times \mathbf{E} = -\mu\epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}. \tag{6}$$

Making use of the vector identity that

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}, \tag{7}$$

we have

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu\epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}. \tag{8}$$

Since the region is source free, and $\nabla \cdot \mathbf{E} = 0$, we have

$$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2}{\partial t^2} \mathbf{E}, \tag{9}$$

which is the vector wave equation in freespace where $\nabla \cdot \mathbf{E} = 0$.
Similarly, we can show that

$$\nabla^2 \mathbf{H} = \mu\epsilon \frac{\partial^2}{\partial t^2} \mathbf{H} \tag{10}$$

if $\nabla \cdot \mathbf{H} = 0$, which is, of course, true in free space.

Plane Wave Solutions to the Vector Wave Equations

The condition for arriving at Equation (9) is that $\nabla \cdot \mathbf{E} = 0$. We can have solutions of the form

$$\mathbf{E} = \hat{x} E_x(z, t), \tag{11}$$

$$\mathbf{E} = \hat{y} E_y(z, t), \tag{12}$$

but not

$$\mathbf{E} = \hat{z} E_z(z, t), \tag{13}$$

because (13) violates $\nabla \cdot \mathbf{E} = 0$ unless E_z is independent of z. If $\mathbf{E}$ is of the form (11), then

$$\nabla^2 \mathbf{E} = \hat{x} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_x(z, t) = \hat{x} \frac{\partial^2}{\partial z^2} E_x, \tag{14}$$

with both $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$ equal to zero. Then (9) becomes

$$\frac{\partial^2}{\partial z^2} E_x(z, t) - \mu\epsilon \frac{\partial^2}{\partial t^2} E_x(z, t) = 0. \tag{15}$$

Similarly, if $\mathbf{H} = \hat{y} H_y(z, t)$, (10) becomes

$$\frac{\partial^2}{\partial z^2} H_y(z, t) - \mu\epsilon \frac{\partial^2}{\partial t^2} H_y(z, t) = 0. \tag{16}$$

Equations (15) and (16) are scalar, one dimensional wave equations of the form

$$\frac{\partial^2}{\partial z^2} y(z, t) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y(z, t) = 0, \tag{17}$$

where $v = 1/\sqrt{\mu\epsilon}$. The solution to (17) is of the form $y = f(z + at)$. We can show that

$$\frac{\partial}{\partial z} f = f'(z + at), \qquad \frac{\partial f}{\partial t} = a f'(z + at), \tag{18}$$

$$\frac{\partial^2}{\partial z^2} f = f''(z + at), \qquad \frac{\partial^2 f}{\partial t^2} = a^2 f''(z + at). \tag{19}$$

Substituting (19) into (17), we have

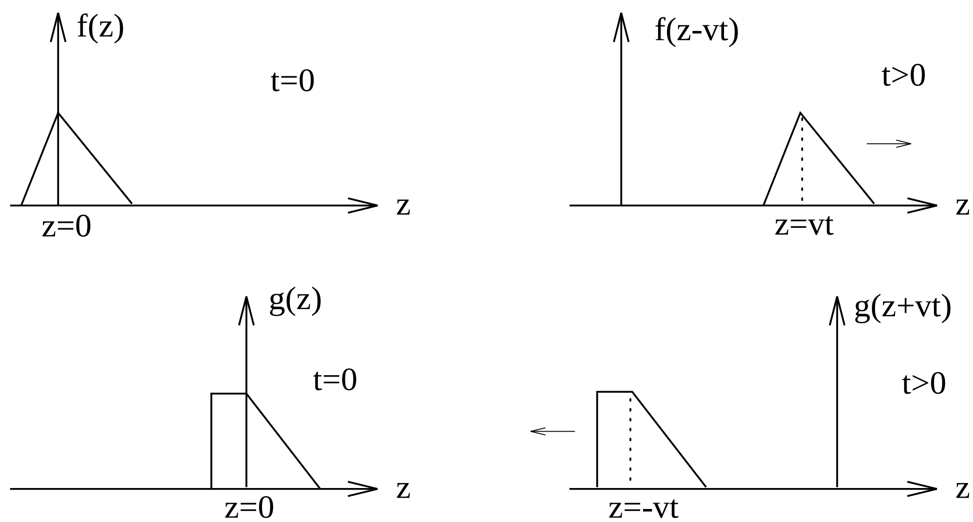
$$f''(z + at) - \frac{a^2}{v^2} f''(z + at) = 0, \tag{20}$$

which is possible only if $a = \pm v$. Hence, the general solution to the wave equation is

$$y = f(z - vt) + g(z + vt), \tag{21}$$

where f and g are arbitrary functions.

The solution $f(z - vt)$ moves in the positive z-direction for increasing t.



The solution $g(z + vt)$ moves in the negative z-direction for increasing t.

The shapes of the functions f and g are undistorted as they move along. We can observe wavelike behavior in a pond when we drop a pebble into it. Solutions to (9) and (10) that correspond to a plane wave is of the form

$$\mathbf{E} = \hat{x} f_1(z - vt), \qquad \mathbf{H} = \hat{y} f_2(z - vt). \tag{22}$$

The wave is propagating in the z-direction, but the electric and magnetic fields are transverse to the direction of propagation. Such a wave is known as the **T**ransverse **E**lectro **M**agnetic wave or TEM wave.

If one substitutes (22) into Equation (2), one has

$$\nabla \times \mathbf{E} = \hat{y} \frac{\partial}{\partial z} E_x = -\mu \frac{\partial}{\partial t} \mathbf{H}, \tag{23}$$

or

$$\frac{\partial}{\partial z} f_1(z - vt) = -\mu \frac{\partial}{\partial t} f_2(z - vt), \tag{24}$$

or

$$f_1'(z - vt) = \mu v f_2'(z - vt), \tag{25}$$

or

$$f_2(z - vt) = \sqrt{\frac{\epsilon}{\mu}} f_1(z - vt). \tag{26}$$

Hence, for a plane TEM wave,

$$\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\epsilon}} = 377\,\Omega, \quad \text{for free space.} \tag{27}$$

The quantity

$$Z = \sqrt{\frac{\mu}{\epsilon}} \tag{28}$$

is also known as the intrinsic impedance of free-space.

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4. Using Phasor Techniques to Solve Maxwell's Equations

For a time-harmonic (simple harmonic) signal, Maxwell's Equations can be easily solved using phasor techniques. For example, if we let

$$\mathbf{H} = \Re e[\tilde{\mathbf{H}} e^{j\omega t}], \tag{1}$$

$$\mathbf{E} = \Re e[\tilde{\mathbf{E}} e^{j\omega t}], \tag{2}$$

and substituting into (3.1), we have

$$\Re e[\nabla \times \tilde{\mathbf{H}} e^{j\omega t}] = \Re e\left[\frac{\partial}{\partial t} \epsilon \tilde{\mathbf{E}} e^{j\omega t}\right]. \tag{3}$$

We could replace $\frac{\partial}{\partial t}$ by $j\omega$ since the signal is time harmonic. Furthermore, we can remove the $\Re e$ operator and obtain

$$\nabla \times \tilde{\mathbf{H}} e^{j\omega t} = j\omega\epsilon \tilde{\mathbf{E}} e^{j\omega t}, \tag{4}$$

where $e^{j\omega t}$ cancels out on both sides.
Equation (4) implies Equation (3). Also, any time dependence cancels out in the problem. Hence,

$$\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon \tilde{\mathbf{E}}. \tag{5}$$

Similarly,

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu \tilde{\mathbf{H}}, \tag{6}$$

$$\nabla \cdot \mu \tilde{\mathbf{H}} = 0, \tag{7}$$

$$\nabla \cdot \epsilon \tilde{\mathbf{E}} = 0. \tag{8}$$

Taking the curl of (6) and substituting (5) into it, we have

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = -j\omega\mu \nabla \times \tilde{\mathbf{H}} = \omega^2 \mu\epsilon \tilde{\mathbf{E}}. \tag{9}$$

Again, making use of the identity $\nabla \times \nabla \times \tilde{\mathbf{E}} = \nabla(\nabla \cdot \tilde{\mathbf{E}}) - \nabla^2 \tilde{\mathbf{E}}$, and $\nabla \cdot \tilde{\mathbf{E}} = 0$, we have

$$\nabla^2 \tilde{\mathbf{E}} = -\omega^2 \mu\epsilon \tilde{\mathbf{E}}. \tag{10}$$

Similarly,

$$\nabla^2 \tilde{\mathbf{H}} = -\omega^2 \mu\epsilon \tilde{\mathbf{H}}. \tag{11}$$

These are the Helmholtz's wave equations.

Lossy Medium (Conductive Medium)

Phasor technique is particularly appropriate for solving Maxwell's equations in a lossy medium. In a lossy medium, Equation (3.1) becomes

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}, \tag{12}$$

where $\mathbf{J}$ is the induced currents in the medium, and hence,

$$\mathbf{J} = \sigma \mathbf{E}. \tag{13}$$

Applying phasor technique to (12), we have

$$\begin{aligned} \nabla \times \tilde{\mathbf{H}} &= j\omega\epsilon\tilde{\mathbf{E}} + \sigma\tilde{\mathbf{E}} \\ &= j\omega \left(\epsilon - j\frac{\sigma}{\omega} \right) \tilde{\mathbf{E}}. \end{aligned} \tag{14}$$

We can define the quantity

$$\tilde{\epsilon} = \epsilon - j\frac{\sigma}{\omega} \tag{15}$$

to be the complex permittivity of the medium, and (14) becomes

$$\nabla \times \tilde{\mathbf{H}} = j\omega\tilde{\epsilon}\tilde{\mathbf{E}}. \tag{16}$$

Notice that the only difference between (16) and (5) is the complex permittivity versus the real permittivity. If one goes about deriving the Helmholtz wave equations for a lossy medium, the results are

$$\nabla^2 \tilde{\mathbf{E}} = -\omega^2 \mu \tilde{\epsilon} \tilde{\mathbf{E}}, \tag{17}$$

$$\nabla^2 \tilde{\mathbf{H}} = -\omega^2 \mu \tilde{\epsilon} \tilde{\mathbf{H}}. \tag{18}$$

Hence, a lossy medium is easily treated using phasor technique by replacing a real permittivity with a complex permittivity.

If we restrict ourselves to one dimension, Equation (17), for instance, becomes of the form

$$\frac{d^2}{dz^2}\tilde{E}_x(z) - \gamma^2 \tilde{E}_x(z) = 0, \tag{19}$$

where

$$\gamma = j\omega\sqrt{\mu\tilde{\epsilon}} = j\omega\sqrt{\mu\left(\epsilon - j\frac{\sigma}{\omega}\right)} = \alpha + j\beta. \tag{20}$$

The general solution to (19) is of the form

$$\tilde{E}_x(z) = C_1 e^{-\gamma z} + C_2 e^{+\gamma z}. \tag{21}$$

In real space time,

$$\begin{aligned} E_x(z,t) &= \Re e[\tilde{E}_x(z) e^{j\omega t}] \\ &= \Re e[C_1 e^{-\gamma z} e^{j\omega t}] + \Re e[C_2 e^{\gamma z} e^{j\omega t}] \end{aligned} \tag{23}$$

If $C_1 = |C_1| e^{j\phi_1}, \quad C_2 = |C_2| e^{j\phi_2}, \quad \gamma = \alpha + j\beta, \quad$ then

$$E_x(z,t) = |C_1| \cos(\omega t - \beta z + \phi_1) e^{-\alpha z} + \mid C_2 \mid \cos(\omega t + \beta z + \phi_2) e^{\alpha z}. \tag{24}$$

Note that one of the solutions in (24) is decaying with z while another solution is growing with z. The function $\cos(\omega t \pm \beta z + \phi)$ can be written as $\cos[\pm\beta(z \pm \frac{\omega}{\beta} t) + \phi]$. Hence, it moves with a velocity

$$v = \frac{\omega}{\beta}. \tag{25}$$

Depending on its sign, it moves either in the positive or negative z direction. In the above, γ is the ***propagation constant***, α is the ***attenuation constant*** while β is the ***phase constant***.

Intrinsic Impedance

The intrinsic impedance can be easily derived also in the phasor world. The phasor representation of Equation (3.23) is

$$\frac{d}{dz}\tilde{E}_x = -j\omega\mu \tilde{H}_y. \tag{26}$$

A corresponding one for $\tilde{H}_y$ is

$$\frac{d}{dz}\tilde{H}_y = -j\omega\epsilon \tilde{E}_x. \tag{27}$$

If we now let $\tilde{E}_x = E_0 e^{-\gamma z}$, $\tilde{H}_y = H_0 e^{-\gamma z}$, and using them in (26) yields

$$-\gamma E_0 e^{-\gamma z} = -j\omega\mu H_0 e^{-\gamma z}. \tag{28}$$

The above implies that

$$\eta = \frac{E_0}{H_0} = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{\mu}{\epsilon}}. \tag{29}$$

For a lossy medium, we replace ϵ by the complex permittivity and the intrinsic impedance becomes

$$\eta = \sqrt{\frac{\mu}{\tilde{\epsilon}}} = \sqrt{\frac{\mu}{\epsilon - j\frac{\sigma}{\omega}}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\mu}}. \tag{30}$$

The above is obviously a complex number.

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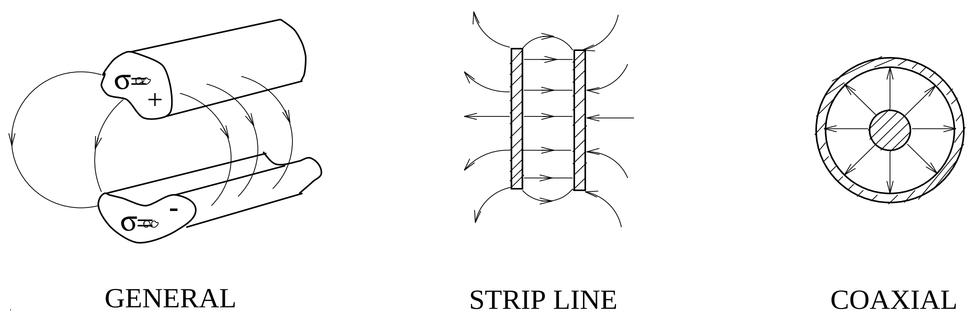
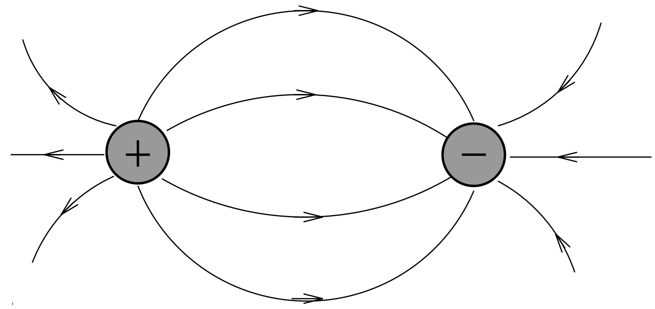
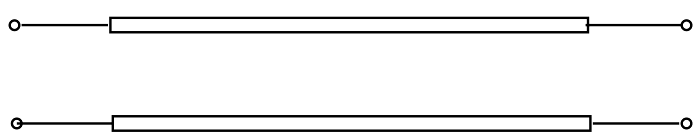
5. Transmission Lines



Examples of Transmission lines

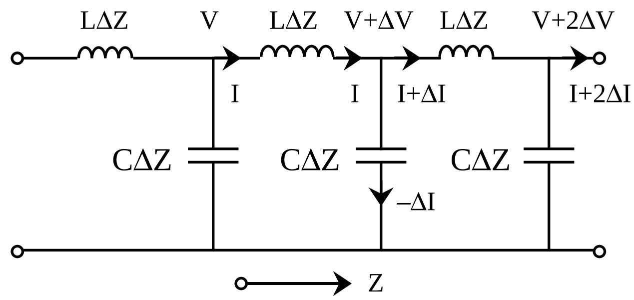


Symbol of a Transmission Line



Symbol of a Transmission Line

Another place where wave phenomenon is often encountered is on transmission lines. A transmission line consists of two parallel conductors of arbitrary cross-sections that can carry two opposite currents or two opposite charges. A transmission line has capacitances between the two conductors, and the conductors have inductances to them. We can characterize this capacitance by a line capacitance C which has the unit of farad m^{-1}, and the inductance by a line inductance L, which has the unit of henry m^{-1}. Hence a transmission line can be approximated by a lumped element equivalent as shown



We can derive the voltage equation between nodes (1) and (2) to get

$$V - (V + \Delta V) = L\Delta z \frac{\partial I}{\partial t}, \tag{1}$$

or

$$\Delta V = -L\Delta z \frac{\partial I}{\partial t}. \tag{2}$$

Similarly, the current relation at node (3) says that

$$-\Delta I = C\Delta z \frac{\partial (V + \Delta V)}{\partial t} \simeq C\Delta z \frac{\partial V}{\partial t}. \tag{3}$$

In the limit when we let our discrete or lumped element model become very small, or $\Delta z \to 0$, we have

$$\frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t}, \tag{4}$$

and

$$\frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t}. \tag{5}$$

The above are known as the telegrapher's equations. Wave equations can be easily derived from the above

$$\frac{\partial^2 V}{\partial z^2} - LC \frac{\partial^2 V}{\partial t^2} = 0, \tag{6}$$

and

$$\frac{\partial^2 I}{\partial z^2} - LC \frac{\partial^2 I}{\partial t^2} = 0. \tag{7}$$

Comparing with Equation (3.17), we deduce that the ***velocity*** of the current and voltage waves on a transmission line is

$$v = \frac{1}{\sqrt{LC}}. \tag{8}$$

The solution to (6) may be of the form

$$V(z,t) = f(z - vt). \tag{9}$$

Substituting into (4), we have

$$-L\frac{\partial I}{\partial t} = f'(z - vt) \tag{10}$$

or

$$I(z,t) = \frac{1}{Lv} f(z - vt). \tag{11}$$

Hence,

$$\frac{V(z,t)}{I(z,t)} = Lv = \sqrt{\frac{L}{C}} \tag{12}$$

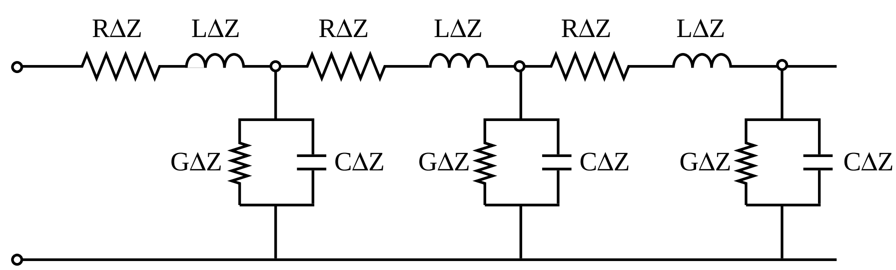
for a forward going wave. The quantity

$$Z_0 = \sqrt{\frac{L}{C}} \tag{13}$$

is the ***characteristic impedance*** of a transmission line.

Lossy Transmission Line

Often time, a transmission line has loss to it. For example, the conductor has a finite conductivity and hence is a little resistive. The insulation between the conductors may have current leakage, thus not forming an ideal capacitor. A more appropriate lumped element model is as follows.



The above circuit is more easily treated using phasor techniques. If we have applied phasor technique to (4) and (5), we would have obtained

$$\frac{d\tilde{V}}{dz} = -j\omega L\tilde{I}, \tag{14}$$

$$\frac{d\tilde{I}}{dz} = -j\omega C\tilde{V}. \tag{15}$$

Note that $j\omega L$ is the series impedance per unit length of the lossless line while $j\omega C$ is the shunt admittance per unit length of the lossless line. In the lossy line case, the series impedance per unit length becomes

$$Z = j\omega L + R \tag{16}$$

while the shunt admittance per unit length becomes

$$Y = j\omega C + G \tag{17}$$

where R and G are line resistance and line conductance respectively. The telegraphers equations become

$$\frac{d\tilde{V}}{dz} = -Z\tilde{I}, \tag{18}$$

$$\frac{d\tilde{I}}{dz} = -Y\tilde{V}, \tag{19}$$

and the corresponding Helmholtz wave equations are

$$\frac{d^2\tilde{V}}{dz^2} - ZY\tilde{V} = 0, \tag{20}$$

$$\frac{d^2\tilde{I}}{dz^2} - ZY\tilde{I} = 0. \tag{21}$$

Similarly, the characteristic impedance, is

$$Z_0 = \sqrt{\frac{j\omega L}{j\omega C}} \quad \Rightarrow Z_0 = \sqrt{\frac{j\omega L + R}{j\omega C + G}} = \sqrt{\frac{Z}{Y}}. \tag{22}$$

Equations (20) and (21) are of the same form as (4.22) or

$$\frac{d^2\tilde{V}}{dz^2} - \gamma^2\tilde{V} = 0, \tag{23}$$

$$\frac{d^2\tilde{I}}{dz^2} - \gamma^2\tilde{I} = 0, \tag{24}$$

where

$$\gamma = \sqrt{ZY} = \sqrt{(j\omega L + R)(j\omega C + G)} = \alpha + j\beta. \tag{25}$$

The general solution is of the form (4.23). For example,

$$\begin{aligned}\tilde{V}(z) &= V_+ e^{-\gamma z} + V_- e^{+\gamma z} \\ &= V_+ e^{-\alpha z - j\beta z} + V_- e^{\alpha z + j\beta z}.\end{aligned} \tag{26}$$

If $V_+ = |V_+|\, e^{j\phi_+}$, $\quad V_- = |V_-|\, e^{+j\phi_-}$, $\quad$ then the real time representation of V is

$$\begin{aligned}V(z,t) &= \Re e[\tilde{V}(z)e^{j\omega t}] \\ &= |V_+|\, e^{-\alpha z}\cos(\omega t - \beta z + \phi_1) + |V_-|\, e^{\alpha z}\cos(\omega t + \beta z + \phi_2).\end{aligned} \tag{27}$$

The first term corresponds to a decaying wave moving in the positive z-direction while the second term corresponds to a wave decaying and moving in the negative z-direction. Hence, $e^{-\gamma z}$ corresponds to a positive going wave, while $e^{+\gamma z}$ corresponds to a negative going wave.

If the transmission line is lossless, i.e., $R = G = 0$, then, the attenuation constant $\alpha = 0$, and the propagation constant γ becomes $\gamma = j\beta$. In this case, there is no attenuation, and (26) becomes

$$\tilde{V}(z) = V_+ e^{-j\beta z} + V_- e^{+j\beta z}, \tag{28}$$

and (27) becomes

$$V(z,t) = |V_+| \cos(\omega t - \beta z + \phi_1) + |V_-| \cos(\omega t + \beta z + \phi_2). \tag{29}$$

The wave propagates without attenuation or without decay in this case. The velocity of propagation is $v = \omega/\beta$.

Furthermore, we can derive the current that corresponds to the voltage in (26) using Equation (18). Hence

$$\tilde{I} = -\frac{1}{Z}\frac{d\tilde{V}}{dz} = \frac{\gamma}{Z} V_+ e^{-\gamma Z} - \frac{\gamma}{Z} V_- e^{+\gamma Z}. \tag{30}$$

But

$$\frac{\gamma}{Z} = \sqrt{\frac{Y}{Z}} = \frac{1}{Z_0} \tag{31}$$

where Z_0 is the characteristic impedance given by Equation (22). Hence,

$$\tilde{I} = \frac{V_+}{Z_0} e^{-\gamma Z} - \frac{V_-}{Z_0} e^{\gamma Z} = I_+ e^{-\gamma Z} + I_- e^{\gamma Z}, \tag{32}$$

where

$$\frac{V_+}{I_+} = Z_0, \qquad \frac{V_-}{I_-} = -Z_0. \tag{33}$$

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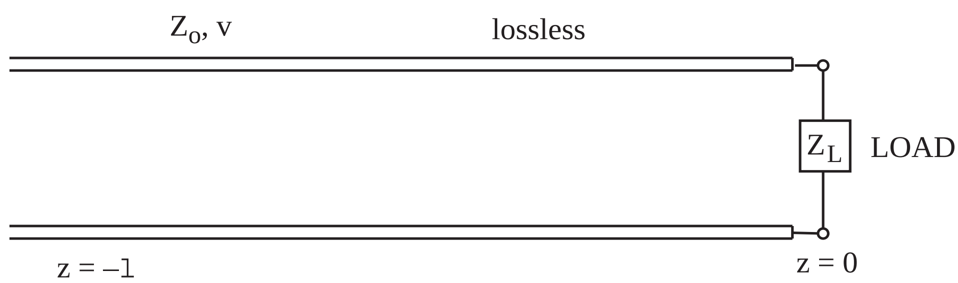
6. Terminated Uniform Lossless Transmission Lines



Consider a lossless transmission line terminated in a load of impedance Z_L. A wave traveling to the right will be reflected at the termination. In general, there will be both positive going and negative going waves on the line. Hence,

$$\tilde{V}(z) = V_0 e^{-j\beta z} + V_1 e^{+j\beta z}. \tag{1}$$

Here, $\gamma = j\beta$, $\alpha = 0$, because of no loss. The corresponding current, as in (5.32), is

$$\tilde{I}(z) = \frac{V_0}{Z_0} e^{-j\beta z} - \frac{V_1}{Z_0} e^{+j\beta z}, \tag{2}$$

where $Z_0 = \sqrt{\frac{L}{C}}$ and $\beta = \omega\sqrt{LC}$ for a lossless line.

At $z = 0$,

$$\frac{\tilde{V}(z=0)}{\tilde{I}(z=0)} = Z_L = \frac{V_0 + V_1}{V_0 - V_1} Z_0. \tag{3}$$

We can solve for V_1 in terms of V_0, i.e.

$$V_1 = \frac{Z_L - Z_0}{Z_L + Z_0} V_0. \tag{4}$$

If we define

$$\rho_v = \frac{Z_L - Z_0}{Z_L + Z_0}, \tag{5}$$

then $V_1 = \rho_v V_0$, and Equation (1) becomes

$$\tilde{V}(z) = V_0 e^{-j\beta z} + \rho_v V_0 e^{+j\beta z}. \tag{6}$$

In the above, ρ_v is the ratio of the negative going voltage amplitude to the positive going voltage amplitude at $z = 0$, and it is known as the ***voltage reflection coefficient***.

The ***current reflection coefficient*** is defined as the ratio of the negative going current to the positive going current at $z = 0$, and it is

$$\rho_i = \frac{I_1}{I_0} = -\frac{V_1}{V_0} = -\rho_v. \tag{7}$$

The current can be written as

$$\tilde{I}(z) = \frac{V_0}{Z_0}e^{-j\beta z} - \rho_v \frac{V_0}{Z_0}e^{j\beta z}. \tag{8}$$

The voltage and current in (6) and (8) are not constants of position. We can define a ***generalized impedance*** at position z to be

$$Z(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = Z_0 \frac{e^{-j\beta z} + \rho_v e^{+j\beta z}}{e^{-j\beta z} - \rho_v e^{+j\beta z}}. \tag{9}$$

At $z = -l$, this becomes

$$Z(-l) = Z_0 \frac{e^{j\beta l} + \rho_v e^{-j\beta l}}{e^{j\beta l} - \rho_v e^{-j\beta l}}. \tag{10}$$

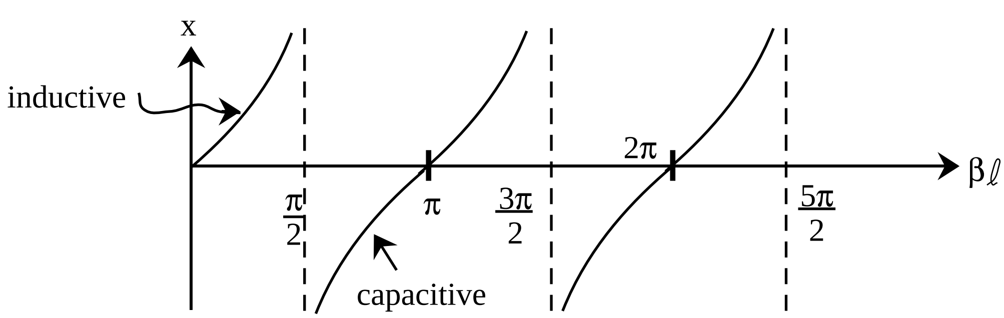
With ρ_v defined by (5), we can substitute it into (10) to give after some simplifications,

$$Z(-l) = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}. \tag{11}$$

Shorted Terminations

If Z_L is a short, or $Z_L = 0$, then,

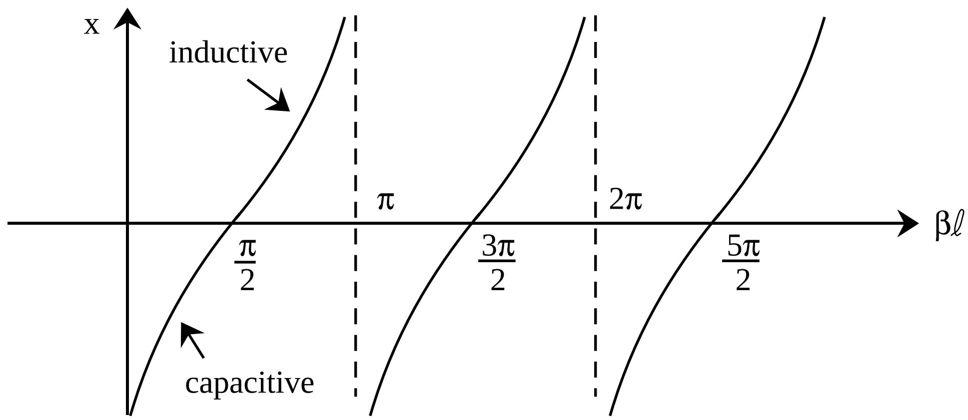
$$Z(-l) = jZ_0 \tan \beta l = jX. \tag{12}$$



Open-Circuit Terminations

If Z_L is an open circuit, $Z_L = \infty$, then

$$Z(-l) = -jZ_0 \cot \beta l = jX. \tag{13}$$



Standing Waves on a Lossless Transmission Line

The positive going wave in Equation (6) is

$$V_+(z) = V_0 e^{-j\beta z}, \tag{14}$$

and the negative going wave in Equation (6) is

$$V_-(z) = \rho_v V_0 e^{+j\beta z}. \tag{15}$$

We can define a ***generalized reflection coefficient*** to be the ratio of $V_+(z)$ to $V_-(z)$ at position z. Hence,

$$\Gamma(z) = \frac{V_-(z)}{V_+(z)} = \rho_v e^{2j\beta z}. \tag{16}$$

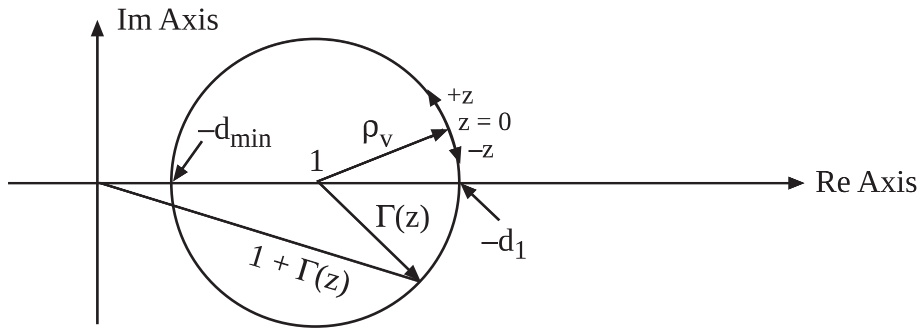
Hence,

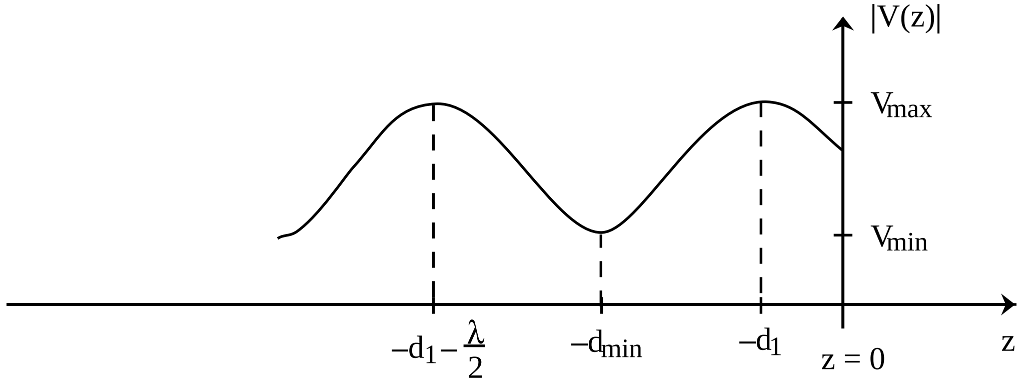
$$V(z) = V_0 e^{-j\beta z}[1 + \Gamma(z)]. \tag{17}$$

The magnitude of $V(z)$ is then

$$|V(z)| = |V_0|\,|1 + \Gamma(z)|\,. \tag{18}$$

A plot of $|V(z)|$ is as shown.





We can use the triangular inequality and show that

$$|V_0|\,(1-|\Gamma(z)|) \leq |V(z)| \leq |V_0|\,(1+|\Gamma(z)|). \tag{19}$$

From (16), $|\Gamma(z)| = |\rho_v|$, hence (19) becomes,

$$|V_0|\,(1-|\rho_v|) \leq |V(z)| \leq |V_0|\,(1+|\rho_v|). \tag{20}$$

The voltage standing wave ratio is defined to be V_{max}/V_{min}, and from (20), it is

$$\text{VSWR} = \frac{1+|\rho_v|}{1-|\rho_v|}. \tag{21}$$

If $\rho_v = 0$, then VSWR$=1$, and we have no reflected wave. We say that the load is matched to the transmission line. Note that $\rho_v = 0$ when $Z_L = Z_0$.

If $|\rho_v| =1$, then VSWR$=\infty$, and we have a badly matched transmission line. In a passive load,

$$0 \leq |\rho_v| \leq 1. \tag{22}$$

$|\rho_v| =1$ only when $Z_L = 0$, or $Z_L = \infty$ according to Equation (5). Hence,

$$1 \leq \ \text{VSWR} \ < \infty. \tag{23}$$

VSWR is an indicator of how well a load is being matched to the transmission line. We can solve (21) for $|\rho_v|$ in terms of VSWR, i.e.

$$|\rho_v| = \frac{\text{VSWR}-1}{\text{VSWR}+1}. \tag{24}$$

Therefore, given the measurement of $VSWR$ on a terminated transmission line, we can deduce the magnitude of ρ_v. Furthermore, if we know the phase of ρ_v, we would be able to derive Z_L from (5), or

$$Z_L = Z_0\frac{1+\rho_v}{1-\rho_v}, \tag{25}$$

or

$$Z_L = Z_0\frac{1+|\rho_v|\,e^{j\theta_v}}{1-|\rho_v|\,e^{j\theta_v}}, \tag{26}$$

where

$$\rho_v = |\rho_v| \, e^{j\theta_v}. \tag{27}$$

Determining θ_v from $|V(z)|$

θ_v can be determined from the voltage standing wave measured. The voltage standing wave pattern is proportional to $|1 + \Gamma(z)|$, but $\Gamma(z)$ is related to ρ_v as

$$\Gamma(z) = \rho_v e^{2j\beta z}. \tag{28}$$

Writing the polar representation of ρ_v, we have,

$$\Gamma(z) = |\rho_v| \, e^{j(2\beta z + \theta_v)}. \tag{29}$$

However, we know that the first minimum value of $V(z)$ occurs when $\Gamma(z)$ is purely negative, or the phase of $\Gamma(z)$ is $-\pi$. This occurs at $z = -d_{min}$ first. In other words,

$$-2\beta d_{min} + \theta_v = -\pi. \tag{30}$$

Since d_{min} can be obtained from the voltage standing wave pattern measurement, and that $\beta = 2\pi/\lambda$, we deduce that

$$\theta_v = -\pi + \frac{4\pi}{\lambda} d_{min}. \tag{31}$$

Transmission Coefficients

It is sometimes useful to define a transmission coefficient on a transmission line. The transmission coefficient may be defined as the ratio of the voltage on the load to the amplitude of the incident voltage. Since

$$V(z) = V_0 e^{-j\beta z} + \rho_v V_0 e^{+j\beta z}. \tag{32}$$

The voltage at the load is $V(z = 0)$, and it is given by

$$V(0) = V_0(1 + \rho_v). \tag{33}$$

Since the amplitude of the incident voltage is V_0, we have

$$\tau_v = \frac{V(0)}{V_0} = 1 + \rho_v = \frac{2Z_L}{Z_L + Z_0}. \tag{34}$$

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7. The Smith Chart

We have seen from Equation (6.9) that a generalized impedance can be defined as

$$Z(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = Z_0 \frac{e^{-j\beta z} + \rho_v e^{+j\beta z}}{e^{-j\beta z} - \rho_v e^{+j\beta z}}. \tag{1}$$

The above can be written as

$$Z(z) = Z_0 \frac{1 + \rho_v e^{2j\beta z}}{1 - \rho_v e^{2j\beta z}} = Z_0 \frac{1 + \Gamma(z)}{1 - \Gamma(z)}, \tag{2}$$

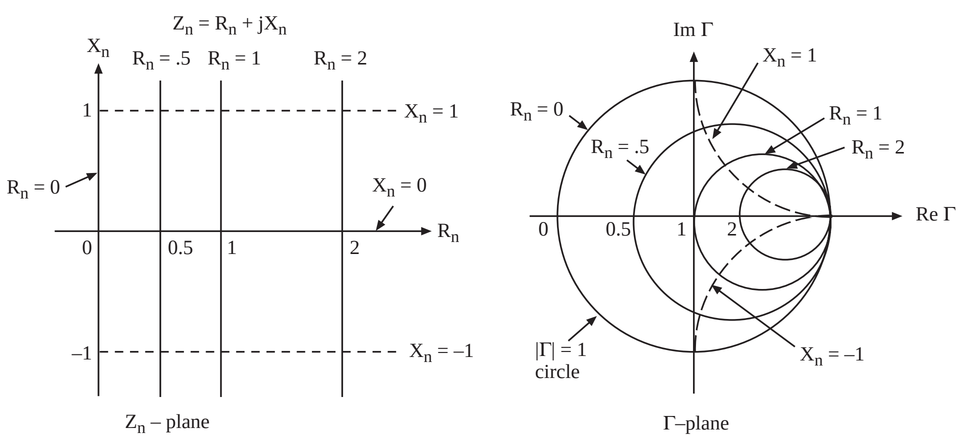
where $\Gamma(z)$ is as defined in (6.16). When $z = 0$, $Z(0) = Z_L$, and $\Gamma(0) = \rho_v$, and (2) becomes (6.25). Hence (6.25) is a special case of (2). We can introduce a ***normalized*** generalized impedance to be

$$Z_n(z) = \frac{Z(z)}{Z_0} = \frac{1 + \Gamma(z)}{1 - \Gamma(z)}. \tag{3}$$

Similarly,

$$\Gamma(z) = \frac{Z_n(z) - 1}{Z_n(z) + 1}. \tag{4}$$

Given $\Gamma(z)$, we can solve for $Z_n(z)$ in (3), and given $Z_n(z)$, we can solve for $\Gamma(z)$ in (4). It turns out that the mapping of $Z_n(z)$ to $\Gamma(z)$ and the mapping of $\Gamma(z)$ to $Z_n(z)$ are one-to-one. We shall next discuss a graphical method to solve (3) and (4) rapidly using the ***Smith Chart.***



Z_n is a complex number and can be represented by a point on the Z_n-plane, and Γ is a complex number and can be represented by a point on the complex Γ plane.

We noted that from Equation (4) that:

(i) When $Z_n = 0, \quad \Gamma = -1$.
(ii) When $Z_n = 1$, or $R_n = 1, X_n = 0, \quad \Gamma = 0$.
(iii) When $Z_n \to \infty$ in any direction, $\quad \Gamma \to 1$.
(iv) When $Z_n = jX_n, \quad |\Gamma| = 1$.
(v) When $Z_n = j$, or $R_n = 0, X_n = 1, \quad \Gamma = j$.
(vi) When $Z_n = -j$, or $R_n = 0, X_n = -1, \quad \Gamma = -j$.

If one works out the mapping from Z_n-plane to Γ-plane completely, one finds that the $R_n = 0$ line on Z_n-plane maps onto the unit-circle on the Γ-plane. Furthermore, the other $R_n = constant$ lines map into circles as shown. The $X_n = constant$ lines map into arcs like the $X_n \pm 1$ lines as shown. Hence, if one puts grids on the Γ-plane, one can read off the R_n and X_n associated with the corresponding Γ immediately, and, given the value of Γ, one can read off the values of R_n and X_n immediately.

The mappings (3) and (4) are known as bilinear transforms. A bilinear transform always maps a circle onto a circle.

Properties of a Smith Chart

(i) The normalized admittance $Y_n = 1/Z_n$, or the reciprocal of Z_n, can be found easily from a Smith Chart, because

$$\Gamma = \frac{Z_n - 1}{Z_n + 1} = \frac{1 - \frac{1}{Z_n}}{1 + \frac{1}{Z_n}} = \frac{1 - Y_n}{1 + Y_n} = -\frac{Y_n - 1}{Y_n + 1}. \tag{5}$$

(ii) The change of impedance along the line is obtained by adding or subtracting phase to $\Gamma(z)$ via the relationship

$$\Gamma(z) = \rho_v e^{2j\beta z}. \tag{6}$$

(iii)

$$\text{VSWR} = \frac{1 + |\rho_v|}{1 - |\rho_v|} = R_{n\,max}, \tag{7}$$

since the Smith Chart is a graphical tool to solve Equation (7), and $|\rho_v|$ is real, corresponding to a number on the $X_n = 0$ line. Notice that $1 <$ VSWR $< \infty$ always.

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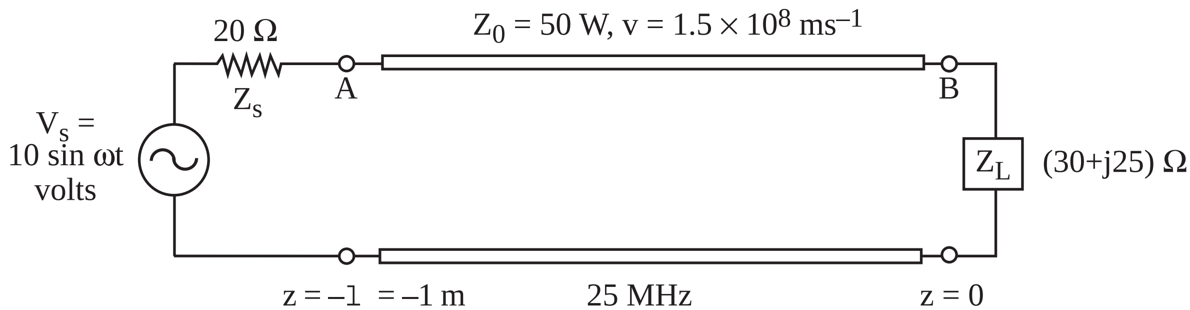
8. Examples on Using the Smith Chart

(a) Find the voltages at A on the transmission line.



The voltage source sets up a forward going and a backward going wave on the transmission lines. Hence,

$$V(z) = V_0 e^{-j\beta z} + \rho_v V_0 e^{j\beta z}. \tag{1}$$

The corresponding current is

$$I(z) = \frac{V_0}{Z_0} e^{-j\beta z} - \rho_v \frac{V_0}{Z_0} e^{j\beta z}. \tag{2}$$

In impedance at position Z is

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{e^{-j\beta z} + \rho_v e^{j\beta z}}{e^{-j\beta z} - \rho_v e^{j\beta z}} = Z_0 \frac{1+\Gamma(z)}{1-\Gamma(z)}, \tag{3}$$

where

$$\Gamma(z) = \rho_v e^{2j\beta z}, \quad \rho_v = \frac{Z_L - Z_0}{Z_l + Z_0}. \tag{4}$$

We can use the Smith Chart to find $Z(-l)$. To use the Smith Chart, we have to normalize all the impedances with respect to the characteristic impedance of the line. Hence,

$$Z_{nL} = \frac{Z_L}{Z_0} = \frac{30 + j25}{50} = 0.6 + j0.5. \tag{5}$$

We can locate Z_{nL} on the Smith Chart which is the complex Γ plane. $\Gamma(0)$ or ρ_v can also be deduced from the Smith Chart. Since $\Gamma(z)$ is given by (4), at $z = -l$, we have

$$\Gamma(-l) = \rho_v e^{-2j\beta l}. \tag{6}$$

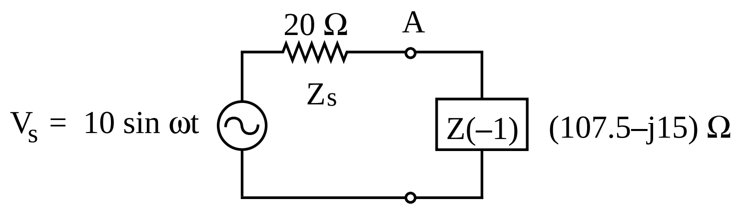
At $f = 25MHz$, and with $v = 1.5 \times 10^8$ ms^{-1}, $\lambda = v/f = 6$m. Then $\beta l = \frac{2\pi}{\lambda} l = \frac{\pi}{3} l$. Therefore,

$$\Gamma(-l) = \rho_v e^{-j\frac{2\pi}{3} l}. \tag{7}$$

At $z = -l = -1m, \Gamma(-1) = \rho_v e^{-j\frac{2\pi}{3}}$. From the Smith Chart, we can read

$$Z_n(-1) = 2.15 - j0.3, \qquad \text{or} \quad \mathrm{Z}(-1) = 107.5 - \mathrm{j}15\,\Omega. \tag{8}$$

So, an equivalent circuit for the point A is:



In phasor representation, $V_S = 10e^{-j\frac{\pi}{2}} = -j10$. Hence,

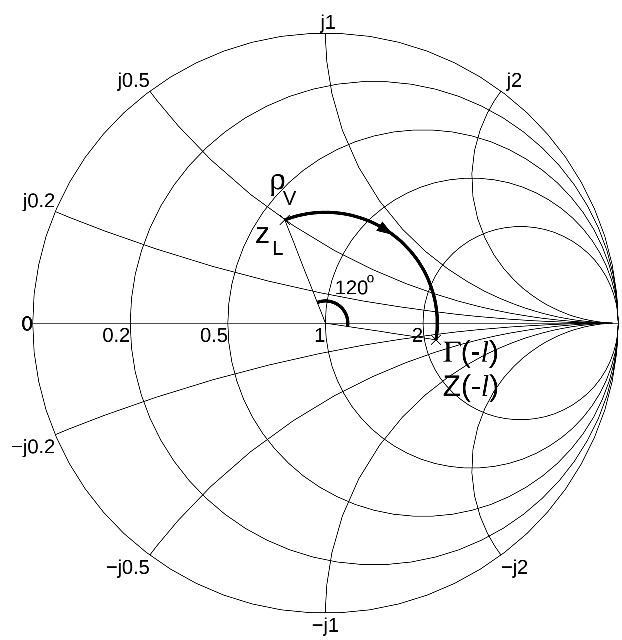
$$V_A = V_S \frac{Z(-1)}{Z_S + Z(-1)} = -j10\frac{107.5 - j15}{127.5 - j15} = \frac{108.54e^{-j7.9^\circ}}{128.38e^{-j6.7}} e^{-j90^\circ} 10\mathrm{V}$$
$$= 8.5e^{-j91.2^\circ}\mathrm{V}. \tag{9}$$

Since

$$V_A = V(-1) = V_0 e^{j\beta}[1 + \Gamma(-1)], \tag{10}$$

we can find V_o from the above. Once V_o is found, we can find V_B from

$$V_B = V(0) = V_o[1 + \rho_v]. \tag{11}$$



(b) Find Z_L from VSWR and d_{min} using a Smith Chart

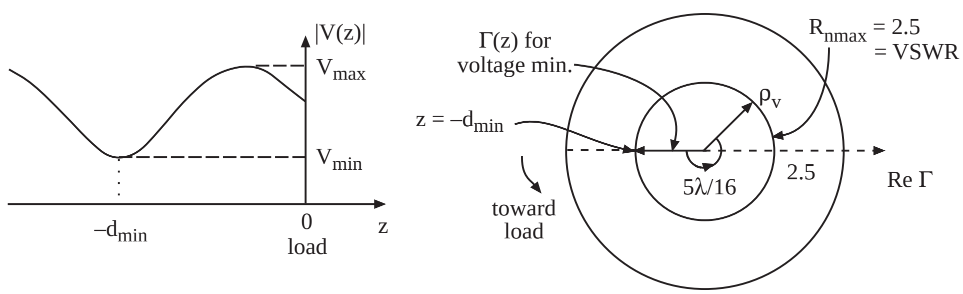
The voltage on the transmission line is

$$V(z) = V_o(e^{-j\beta z} + \rho_v e^{+j\beta z}) = V_o e^{-j\beta z}[1 + \Gamma(z)]. \tag{12}$$

If $V(z) = |V(z)|e^{j\theta(z)}$, the real time voltage can be written as

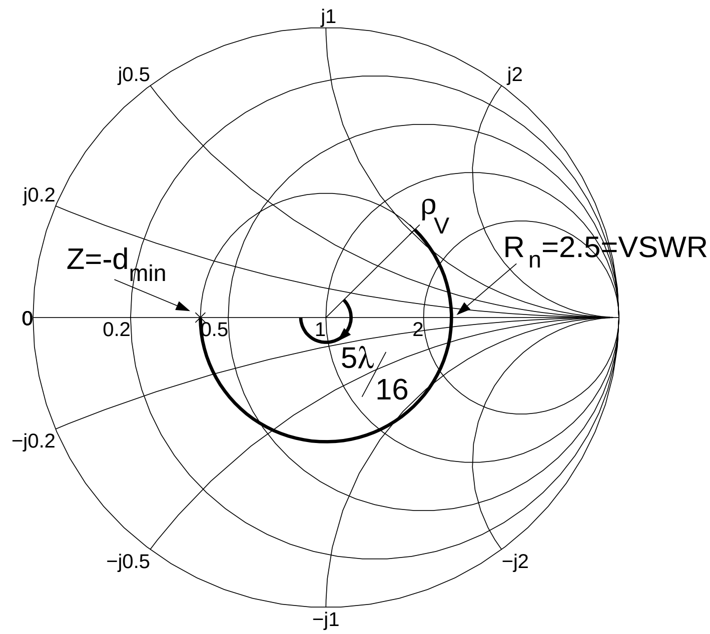
$$V(z,t) = \Re e[|V(z)|e^{j\theta(z)}e^{j\omega t}] = |V(z)|\cos[\omega t + \theta(t)]. \tag{13}$$

Hence the amplitude of the real time voltage is proportional to $|V(z)|$ which is the voltage standing wave pattern.



For example, we may be given that the $VSWR = 2.5$ on the line, $Z_o = 75\Omega$, and $d_{min} = 5\lambda/16$, in order to find Z_L.

First, we note that $|V(z)| \propto |1 + \Gamma(z)|$ where $\Gamma(z) = \rho_v e^{2j\beta z}$. Note that V_{min} occurs when $\Gamma(z)$ is purely negative. When z varies, $\Gamma(z)$ traces out a constant circle on the Smith Chart, since $|\Gamma(z)| = |\rho_v|$ is independent of z. Since the $|\Gamma(z)|$ circle must intersect the real Γ axis at $R_n = 2.5$ since the VSWR$= 2.5$, we can deduce that magnitude of $|\Gamma(z)| = |\rho_v|$. Since $z = -d_{min}$ point corresponds to $\Gamma(z)$ as shown above, and the load is $5\lambda/16$ from the d_{min} point, we can figure out ρ_v's location on the Smith Chart. We can read off $Z_{nL} = 1.4 + j1.1$ on the Smith Chart. Hence $Z_L = (105 - j82.5)\Omega$.


j1
j0.5
j2
j0.2
ρV
Rn=2.5=VSWR
Z=-dmin
0
0.2
0.5
1
2
5λ/16
−j0.2
−j0.5
−j2
−j1

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9. Complex Power on a Transmission Line

Complex Power

Since we are dealing with phasors, it is convenient to define a complex power which has an imaginary part as well as a real part. We shall define the meaning of complex power.

A complex power is defined as

$$\tilde{P} = \tilde{V}\tilde{I}^*, \tag{1}$$

i.e. the product of a voltage phasor and a current phasor at a given point. If

$$\tilde{V} = |\tilde{V}|e^{j\phi_V}, \quad \tilde{I} = |\tilde{I}|e^{j\phi_I}, \tag{2}$$

then

$$\tilde{P} = |\tilde{V}||\tilde{I}|[\cos(\phi_V - \phi_I) + \sin(\phi_V - \phi_I)]. \tag{3}$$

The corresponding real time voltage and current are

$$V(t) = |\tilde{V}|\cos(\omega t + \phi_V), \quad I(t) = |\tilde{I}|\cos(\omega t + \phi_I). \tag{4}$$

Then, the instantaneous power is

$$\begin{aligned} P(t) = V(t)I(t) &= |\tilde{V}||\tilde{I}|\cos(\omega t + \phi_V)\cos(\omega t + \phi_V + \phi_I - \phi_V) \\ &= |\tilde{V}||\tilde{I}|[\cos^2(\omega t + \phi_V)\cos(\phi_I - \phi_V) \\ &\qquad - \cos(\omega t + \phi_V)\sin(\omega t + \phi_V)\sin(\phi_I - \phi_V)]. \end{aligned} \tag{5}$$

The time average of $P(t)$, defined as

$$\begin{aligned} \langle P(t)\rangle = \langle V(t)I(t)\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_0^T dt P(t) \\ &= |\tilde{V}||\tilde{I}|[\langle\cos^2(\omega t + \phi_V)\rangle\cos(\phi_I - \phi_V) \\ &\qquad - \langle\cos(\omega t + \phi_V)\sin(\omega t + \phi_V)\rangle\sin(\phi_I - \phi_V)]. \end{aligned} \tag{6}$$

Since

$$\langle\cos^2(\omega t + \phi_V)\rangle = \frac{1}{2}, \quad \langle\cos(\omega t + \phi_V)\sin(\omega t + \phi_V)\rangle = 0, \tag{7}$$

we have

$$\langle P(t)\rangle = \frac{1}{2}|\tilde{V}\tilde{I}|\cos(\phi_I - \phi_V). \tag{8}$$

Comparing with (3), we see that

$$\langle P(t)\rangle = \frac{1}{2}\Re e[\tilde{P}]\,. \tag{9}$$

The imaginary part of the complex power is proportional to the second term in (5), and hence, the imaginary part of the complex power is proportional to a part of the instantaneous power that averages to zero. Consequently, the imaginary part of the complex power is called ***reactive power***. For example, a purely reactive device dissipates no power on the average, but instantaneous power is being constantly absorbed and released by a reactive device. The current and voltage through a reactive device is 90° out-of-phase, and the complex power is purely imaginary or purely reactive.

Complex Power on a Transmission Line

The voltage on a transmission line could be written as

$$\begin{aligned}\tilde{V}(z) &= V_0\left(e^{-j\beta z} + \rho_v e^{j\beta z}\right)\\ &= V_0 e^{-j\beta z}[1+\Gamma(z)]\,.\end{aligned} \tag{10}$$

The current on the line could be written as

$$\tilde{I}(z) = \frac{V_0}{Z_0}e^{-j\beta z}\left[1-\Gamma(z)\right]\,. \tag{11}$$

The complex power is given by

$$\tilde{P} = \tilde{V}\tilde{I}^* = \frac{|V|^2}{Z_0}[1+\Gamma(z)][1-\Gamma(z)^*]\,, \tag{12}$$

which reduces to

$$\tilde{P} = \frac{|V|^2}{Z_0}[1-|\Gamma(z)|^2+\Gamma(z)-\Gamma(z)^*]\,, \tag{13}$$

or

$$\tilde{P} = \tilde{V}\tilde{I}^* = \frac{|V|^2}{Z_0}[1-|\rho_v|^2 + j2\Im m\Gamma(z)]\,. \tag{14}$$

The time average power, defined to be

$$\langle P(z,t)\rangle = \frac{1}{2}\Re e[\tilde{P}(z)] = \frac{|V|^2}{2Z_0}(1-|\rho_v|^2)\,, \tag{15}$$

for a lossless transmission line. If $\rho_v = 0$, or when the load is ***matched*** to the transmission line, (i.e., $Z_L = Z_0$), all the power carried in the forward going wave is dumped into the load. Otherwise, part of the power is reflected. The power carried by the forward going wave is

$$\langle P_+\rangle = \frac{|V|^2}{2Z_0}, \tag{16}$$

and the power carried by the backward going wave is

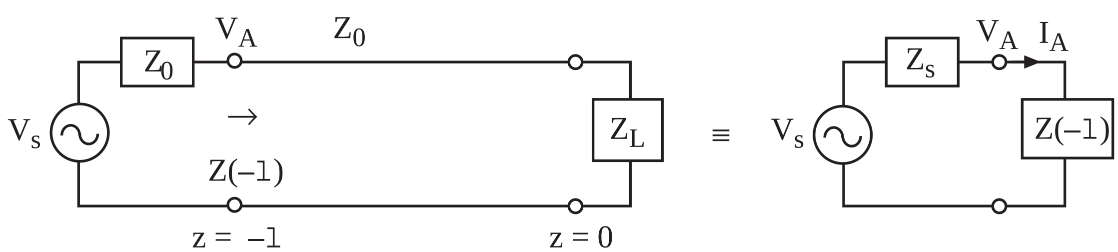
$$\langle P_-\rangle = \frac{|V|^2}{2Z_0}|\rho_v|^2 . \tag{17}$$

Note that $\langle P(z,t)\rangle$ is independent of z because of energy conservation.

$$\langle P\rangle = \langle P_+\rangle - \langle P_-\rangle, \tag{18}$$

is everywhere the same on the lossless transmission line because the total power leaving the source all arrive at the load end with no loss on the lossless transmission line. The transmission line can only absorb reactive power. Hence, the reactive power in (14) is not a constant of position.

Power Delivered to the Load on a Transmission Line



To find the power delivered to the load on a lossless transmission line, we can first find $Z(-l)$ using formula (6.11). Then, we can replace the transmission line circuit with the equivalent circuit for finding V_A, and I_A. The real power delivered to $Z(-l)$ would be the same as the real power delivered to Z_L.

$$\tilde{P} = V_A I_A^* = \frac{|V_A|^2}{Z^*(-l)} = \left|\frac{Z(-l)}{Z_s + Z(-l)}\right|^2 \frac{|V_S|^2}{Z^*(-l)} = \frac{Z(-l)|V_S|^2}{|Z_s + Z(-l)|^2} . \tag{19}$$

The time-average power delivered to the load is

$$\langle P\rangle = \frac{1}{2}\Re e[\tilde{P}] = \frac{1}{2}\frac{R(-l)|V_S|^2}{|R_s + jX_S + R(-l) + jX(-l)|^2}, \tag{20}$$

where we have assumed that $Z_S = R_S + jX_S$, and $Z(-l) = R(-l) + jX(-l)$. To optimize $\langle P\rangle$, with respect to $X(-l)$, we choose $X(-l) = -X_S$, hence,

$$\langle P\rangle = \frac{1}{2}\frac{R(-l)|V_S|^2}{|R_s + R(-l)|^2}. \tag{21}$$

The above is maximum when $R(-l) = R_S$. Hence, maximum power is delivered to the load when

$$Z(-l) = Z_S^* \,. \tag{22}$$

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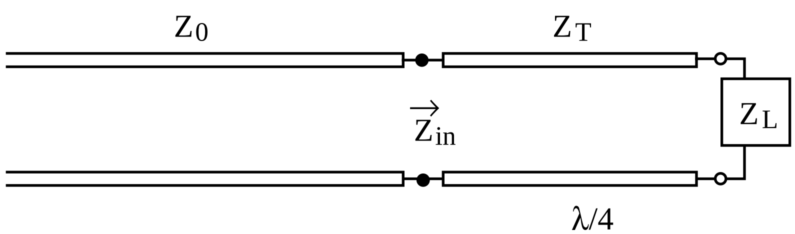
10. Impedance Matching on a Transmission Line.

We note that when the impedance of a load is the same as the characteristic impedance of the transmission line, there is no reflected wave, and all the forward going power is dissipated in the load. There are various ways to achieve this ***impedance matching*** and we will discuss some of them below.

(a) Quarter-Wave Transformer

A quarter wave transformer, like low-frequency transformers, changes the impedance of the load to another value so that matching is possible.



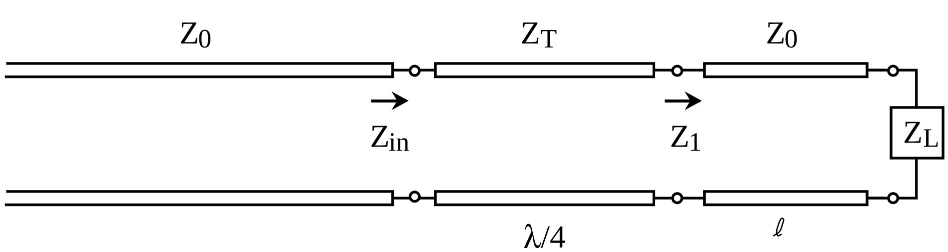
A quarter-wave transformer uses a section of line of characterstic impedance Z_T of $\frac{\lambda}{4}$ long. To have a matching condition, we want $Z_{in} = Z_0$. From Equation (6.11) we have

$$Z_{in} = Z_T \frac{Z_L + jZ_T \tan\frac{\pi}{2}}{Z_T + jZ_L \tan\frac{\pi}{2}} = \frac{Z_T^2}{Z_L}, \tag{1}$$

since $\tan\beta l = \tan\frac{2\pi}{\lambda}\frac{\lambda}{4} = \tan\frac{\pi}{2} = \infty$. In order for $Z_{in} = Z_0$, we need that

$$Z_T^2 = Z_0 Z_L \Rightarrow Z_T = \sqrt{Z_0 Z_l}. \tag{2}$$

If Z_0 and Z_L are both real, then Z_T is real, and we can use a lossless line to perform the matching. If Z_L is complex, it can be made real by adding a section of line to it.



Example

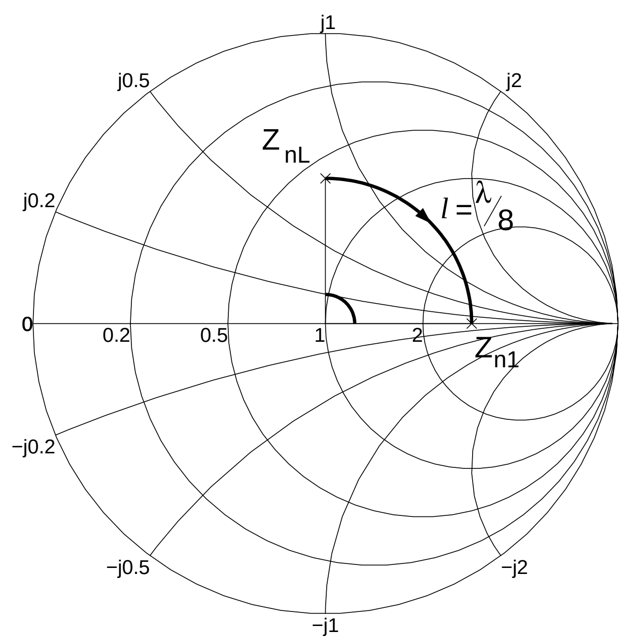
Given that $Z_L = (30 + j40)\Omega$, $Z_0 = 50\Omega$, find the shortest l and Z_T so that the above circuit is matched. Assume that Z_T is real and lossless.

We want Z_1 to be real and Z_{in} to be $Z_0 = 50\Omega$ in order for Z_T to be real and the matching condition satisfied. We find that $Z_{nL} = 0.6 + j0.8$. In order to make Z_{n1} real, the shortest l from the Smith Chart is $\frac{\lambda}{8}$. Then $Z_{n1} = 3.0$, and $Z_1 = 150\Omega$. Since $Z_{in} = 50\Omega$, we need

$$Z_T = \sqrt{Z_{in} Z_1} = \sqrt{50 \times 150} = 86.6\Omega$$

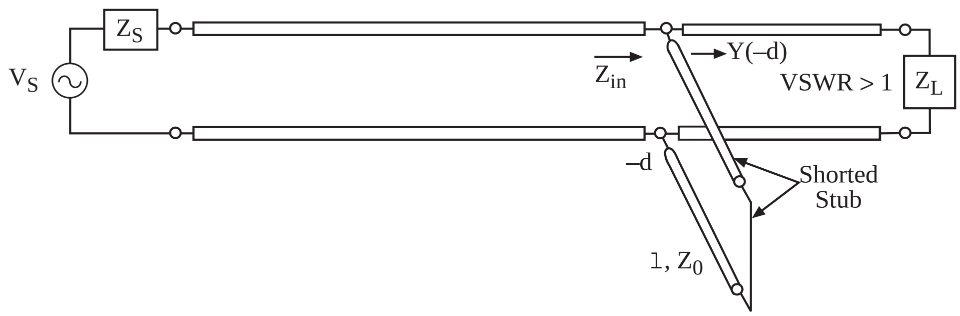
in order for matching condition to be satisfied.

Note that the quarter wave transformer only matches the circuit at one frequency. Often time, it has a small bandwidth of operation, i.e., it only works in the frequencies in a small neighborhood of the matching frequency. Sometimes, a cascade of two or more quarter-wave transformers are used in order to broaden the bandwidth of operation of the transformer.



(b) **Single Stub Tuning**

Another device for performing matching is a single stub (either shorted or opened at one end) which is shunted across the transmission line at $z = -d$ from the load.



The location d is chosen so that the admittance $Y(-d)$ looking toward the load is $Y_0 + jB$ ($Y_0 = \frac{1}{Z_0}$). The length l of the shorted stub is chosen so that its admittance is $-jB$. Hence, when the stub is connected in parallel to the transmission line at $z = -d$, the impedance $Z_{in} = Z_0$, so that matching condition is achieved.

A shorted stub has impedance and admittance given by

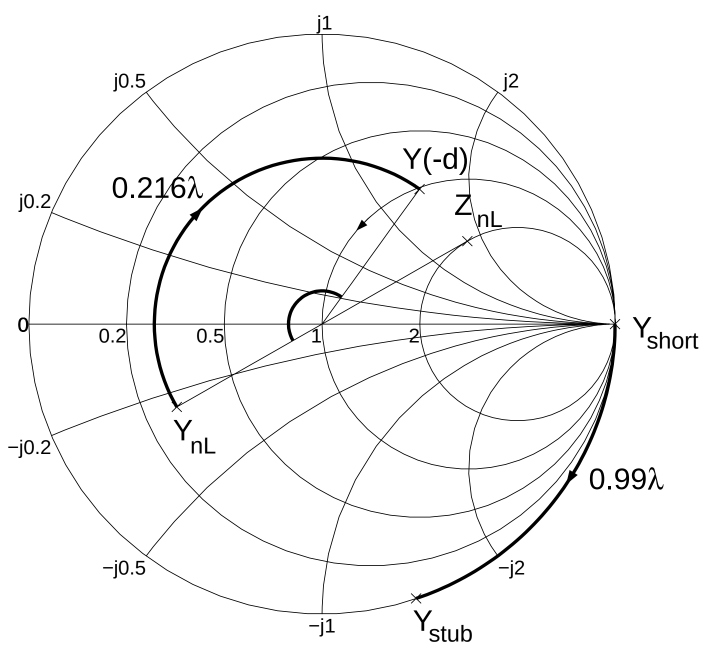
$$Z_s = jZ_0 \tan \beta l, \tag{3}$$

$$Y_s = -jY_0 \cot \beta l. \tag{4}$$

An open-circuited stub can also be used, and the impedance and admittance are given by

$$Z_{op} = -jZ_0 \cot \beta l, \tag{5}$$

$$Y_{op} = jY_0 \tan \beta l. \tag{6}$$



Example

Let $Z_L = (100+j85)\Omega$, find the minimum d and l that will reduce the VSWR of the main line to 1. Assume that $Z_0 = 50\Omega$.

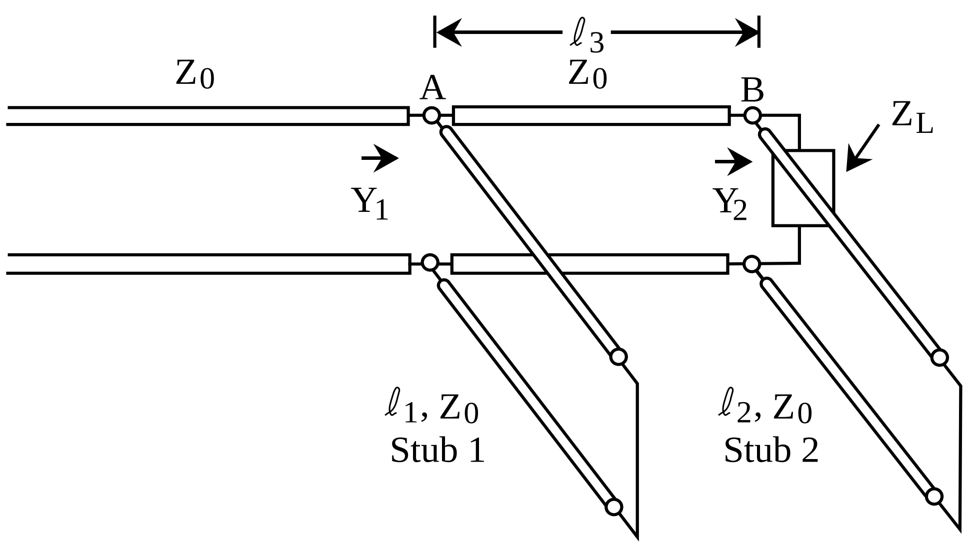
We find that the normalized load $Z_{nL} = 2 + 1.7j$ as shown on the Smith Chart. Since this problem involves parallel connections, it is more convenient to work with admittances. $Y_{nL} = \frac{1}{Z_{nL}}$ is as shown. When we move toward the generator, $Y_n(z)$ traces out a locus on the Smith Chart as shown. It intersects the $G = 1$ circle as shown, after moving through 0.216λ. Therefore, $d = 0.216\lambda$.

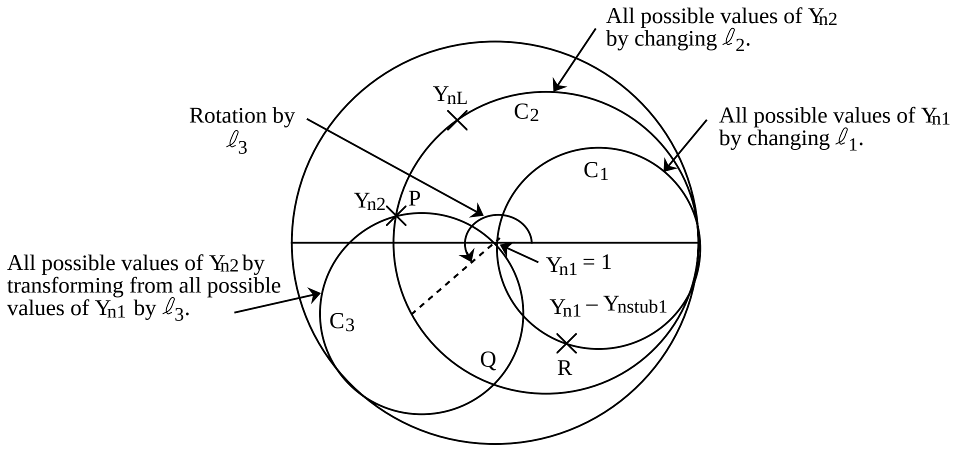
Now, $Y_n(-d) = 1 + j1.4$. Hence, $Y_{n\text{stub}} = -j1.4$. From the Smith Chart, we note that the admittance for a short is infinity, and is at the right end of the Smith Chart. To get a $Y_{n\text{stub}} = -j1.4$, we move toward the generator for 0.099λ. Hence, $l = 0.099\lambda$.

Often time, it is not easy to change d, but quite easy to change l. We note that both in the quarter wave transformer and the single stub tuner, we have to change 2 parameters for tuning. We can provide these 2 degrees of freedom by using two stubs, changing their length, but not their positions.

(c) **Double Stub Tuning (optional reading)**

Both single stub tuning and quarter wave transformer matching require changing the location of the stub or the transformer. In practice, this is difficult, and a double stub tuning removes the difficulty.





(1) In order to have a matched circuit, we should have $Y_1 = Y_0$ so that $Y_{n1} = 1$. However, if we change l_1, the possible values of Y_{n1} trace out a circle C_1 as shown.

(2) If Y_{nL} is as shown, by changing l_2, the possible values of Y_{n2} trace out a circle C_2 as shown.

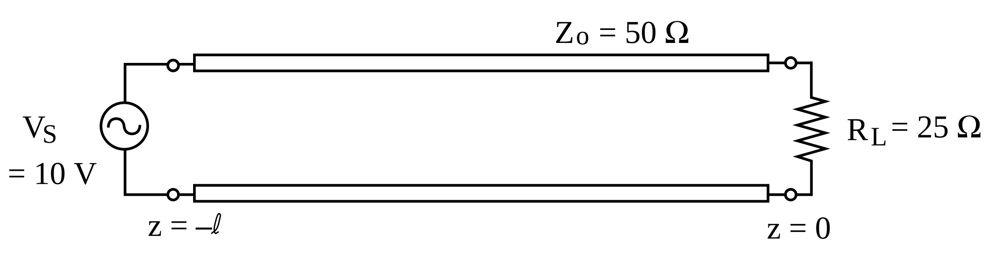
(3) When l_3 is added, all the possible values of Y_{n1} at A is transformed to B by a rotation according to the length of l_3. This constitute a circle C_3 which is all the possible values of Y_{n2} obtained from Y_{n1}. There are only two points, P and Q that the two circles C_2 and C_3 intersect. If we pick P, then this point should correspond to the value of Y_{n2}.

$$Y_{n2} = Y_{nl} + Y_{n\text{stub2}} \tag{25.1}$$

We can figure out $Y_{n\text{stub2}}$ and hence the length l_2.

(4) The length l_3 rotates the point P to the point R. Then R has the impedance $Y_{n1} - Y_{n\text{stub1}} = 1 - Y_{n\text{stub1}}$. We can figure out $Y_{n\text{stub1}}$ from the Smith Chart and hence the length l_1.

(d) **Ferranti Effect**

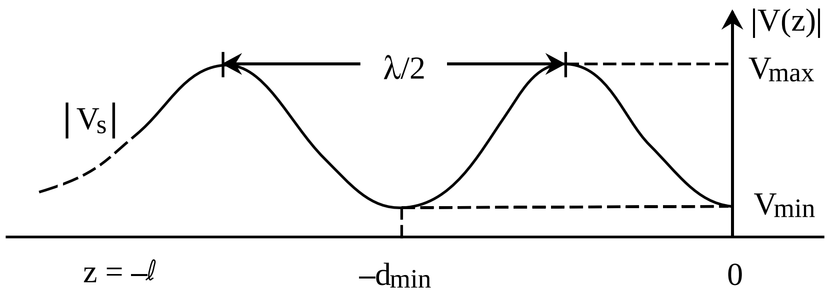


. Find VSWR on the line, and if l is allowed to vary arbitrarily, find the maximum voltage on the line.

We can find VSWR from the Smith Chart or by calculator.

$$P(0) = P_v = \frac{25 - 50}{25 + 50} = -\frac{1}{3},$$

$$\text{VSWR} = \frac{1 + |P_v|}{1 - |P_v|} = \frac{\frac{4}{3}}{\frac{2}{3}} = 2.$$



The voltage at $Z = -l$ is always fixed to be V_s. Hence, we can see that $|V(z)|$ on parts of the transmission line can be longer than $|V_s|$. If l is chosen so that V_s is at V_{min}, then

$$V_{max} = \text{VSWR} \times V_{min} = 10 \text{ volts} \times 2 = 20 \text{ volts}.$$

This amplification of voltage on a line is known as the Ferranti's effect. If the VSWR on the line is very high, V_{max} can be so large that it reaches the breakdown voltage of the line. This is something one should be cautious of in designing transmission line circuits.

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11. Lossy Transmission Lines.

When R and G are not zero, we have a lossy transmission line. In this case,

$$V(z) = V_0(e^{-\gamma z} + \rho_v e^{+\gamma z}) \tag{1}$$

where

$$\gamma = \sqrt{ZY} = \sqrt{(j\omega L + R)(j\omega C + G)} = \alpha + j\beta.$$

The current is derived using the telegrapher's equation to be

$$I(z) = \frac{V_0}{Z_0}(e^{-\gamma z} - \rho_v e^{\gamma z}), \tag{2}$$

where

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{j\omega L + R}{j\omega C + G}}.$$

When $\frac{R}{L} = \frac{G}{C}$, then Z_0 becomes frequency independent, and $Z_0 = \sqrt{\frac{L}{C}}$. Also,

$$\gamma = j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right)^{\frac{1}{2}}\left(1 + \frac{G}{j\omega L}\right)^{\frac{1}{2}} = j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right) \tag{3}$$

From (3), we see that $\alpha = R\sqrt{\frac{C}{L}} = \frac{R}{Z_0}$ while $\beta = \omega\sqrt{LC}$. Since α is frequency independent, and the $v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$ is also frequency independent, the transmission line is a distortionless line because any pulse that propagates on the line will not be distorted. This is because a pulse can be thought of as a superposition of Fourier harmonics. Each Fourier harmonic is a time harmonic signal. On a distortionless line, all the Fourier harmonics propagate at the same velocity and suffer the same attenuation. Hence the pulse is not distorted but only diminished in amplitude.

If we divide (1) by (2), we get

$$Z(z) = \frac{V(z)}{I(z)} = Z_0\frac{1 + \Gamma(z)}{1 - \Gamma(z)}, \tag{4}$$

where

$$\Gamma(z) = \rho_v e^{2\gamma z}. \tag{5}$$

Note that (4) also implies that

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} = \frac{Z_n(z) - 1}{Z_n(z) + 1}. \tag{6}$$

Equations (4) and (6) can be solved using a Smith Chart. However, now we have

$$|\Gamma(z)| = |\rho_v| \, e^{2\alpha z}. \tag{7}$$

The amplitude of $|\Gamma(z)|$ is diminishing when we move from the load to the source. From (5), we note that $\Gamma(z) \to 0$ when $z \to -\infty$, $Z(z) \to Z_0$ when $z \to -\infty$. Hence, a long lossy transmission line is always matched. The locus traced out by (7) is a spiral converging on the origin of the Smith Chart when we move from the load to the source.

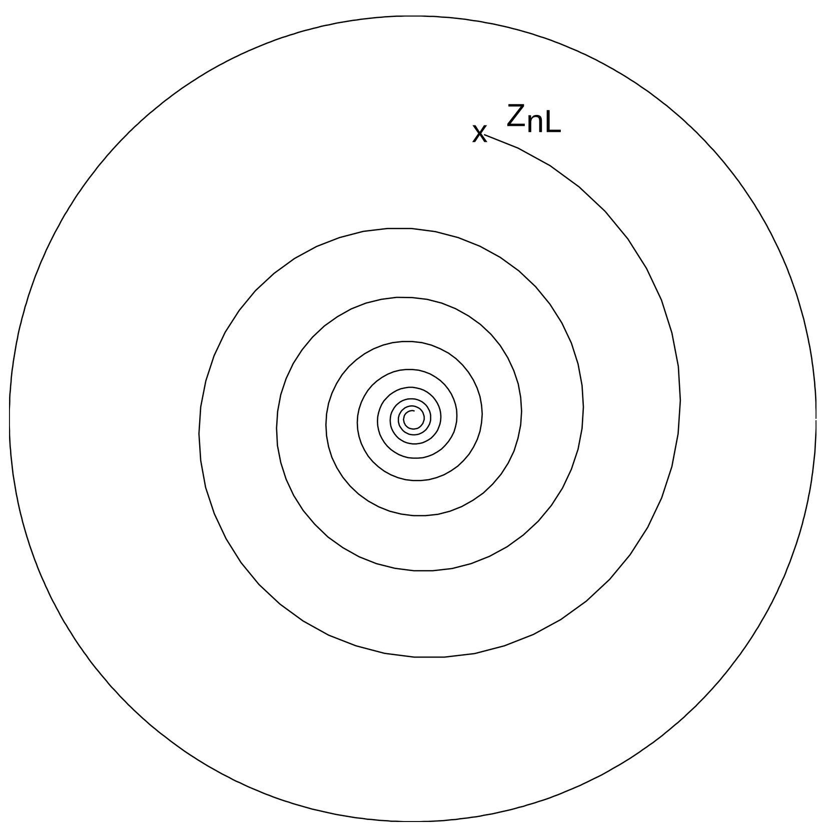
Also, the voltage standing wave pattern is given by

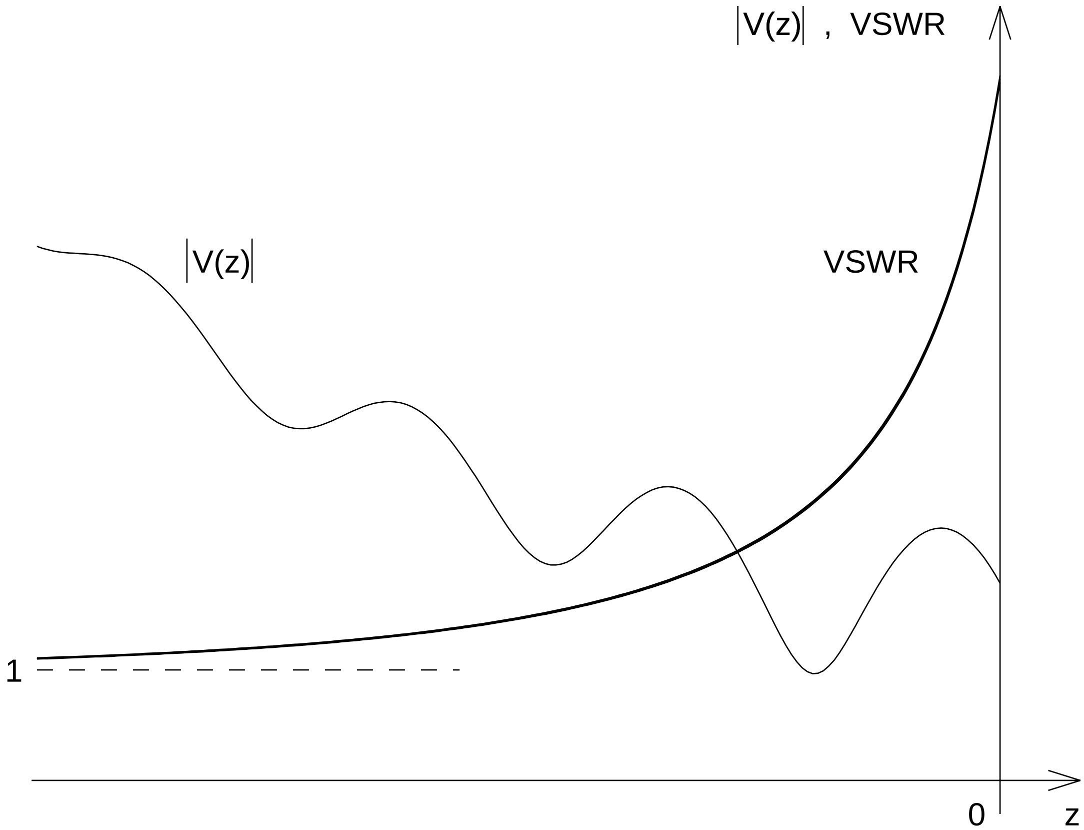
$$|V(z)| = |V_0| \, e^{-\alpha z} \, |1 + \Gamma(z)| \, . \tag{8}$$

A plot of $\Gamma(z)$ and $|V(z)|$ are as shown. Furthermore, we can define an ad hoc VSWR given to be

$$\text{VSWR} = \frac{1 + |\Gamma(z)|}{1 - |\Gamma(z)|} = \frac{1 + |\rho_v| \, e^{2\alpha z}}{1 - |\rho_v| \, e^{2\alpha z}}, \tag{9}$$

which is dependent on z.





Power on a Lossy Line

With $V(z)$ and $I(z)$ given by (1) and (2), one can define a complex power on a lossy line to be

$$\underline{P}(z) = V(z) I^\star(z), \tag{10}$$

where

$$V(z) = V_0 e^{-\gamma z}\big(1 + \Gamma(z)\big), \tag{11}$$

and

$$I(z) = \frac{V_0 e^{-\gamma z}}{Z_0}\big(1 - \Gamma(z)\big). \tag{12}$$

Hence,

$$\underline{P}(z) = \frac{|V_0|^2}{Z_0^\star} e^{-\gamma z - \gamma^\star z}\big(1 + \Gamma(z)\big)\big(1 - \Gamma^\star(z)\big), \tag{13}$$

which is equal to

$$\underline{P}(z) = \frac{|V_0|^2}{Z_0^\star} e^{-2\alpha\, z}\left[1 - |\Gamma(z)|^2 + 2j \Im m \Gamma(z)\right]. \tag{14}$$

Since $|\Gamma(z)| = |\rho_v|\, e^{2\alpha\, z}$, we have

$$\underline{P}(z) = \frac{|V_0|^2}{Z_0^\star} e^{-2\alpha\, z}\left[1 - |\rho_v|^2\, e^{4\alpha\, z} + 2j \Im m \Gamma(z)\right]. \tag{15}$$

We see that both the real part and the imaginary part of the complex power are functions of position. This is because real power is dissipated as the wave propagates. Hence, the real power at one point is not equal to the real power at another point.

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12. Transients on a Transmission Line.

When we do not have a time harmonic signal on a transmission line, we have to use transient analysis to understand the waves on a transmission line. A pulse waveform is an example of a transient waveform.

We have shown previously that if we have a forward going wave for a voltage on a transmission line, the voltage is

$$V(z,t) = f(z - vt). \tag{1}$$

The corresponding current can be derived via the telegrapher's equation

$$I(z,t) = \frac{1}{Z_0} f(z - vt). \tag{2}$$

If instead, we have a wave going in the negative direction,

$$V(z,t) = g(z + vt), \tag{3}$$

then the current from the telegrapher's equations, is

$$I(z,t) = -\frac{1}{Z_0} g(z + vt). \tag{4}$$

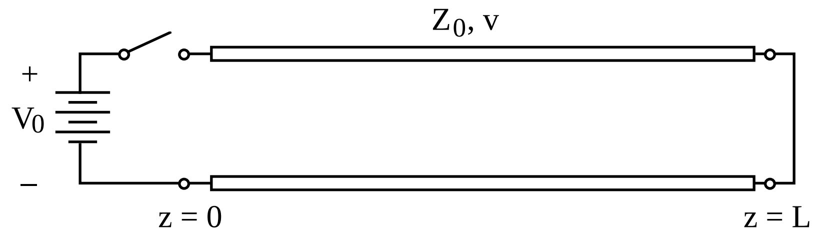
Hence, in general, if

$$V(z,t) = V_+(z,t) + V_-(z,t), \tag{5}$$

$$I(z,t) = Y_0 \big[V_+(z,t) - V_-(z,t) \big], \tag{6}$$

where $Y_0 = \frac{1}{Z_0}$, and the subscript $+$ indicates a positive going wave, while the subscript $-$ indicates a negative going wave.

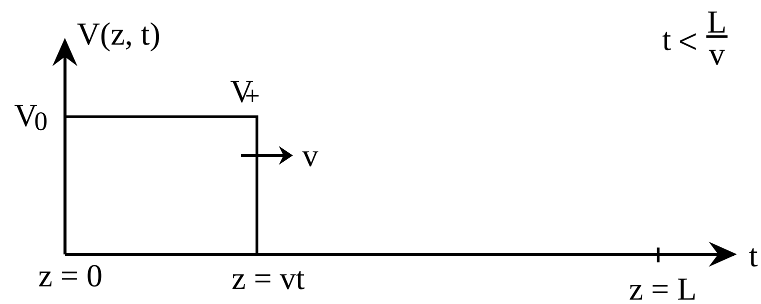
(a) **Reflection of a Transient Signal from a Shorted Termination**

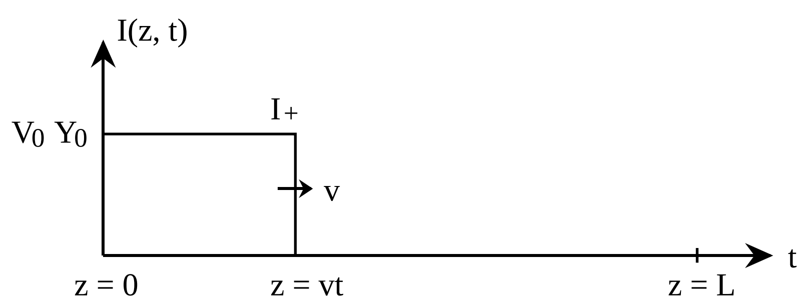


If we switch on the voltage of the above network at $t = 0$, the voltage at $z = 0$ has the form

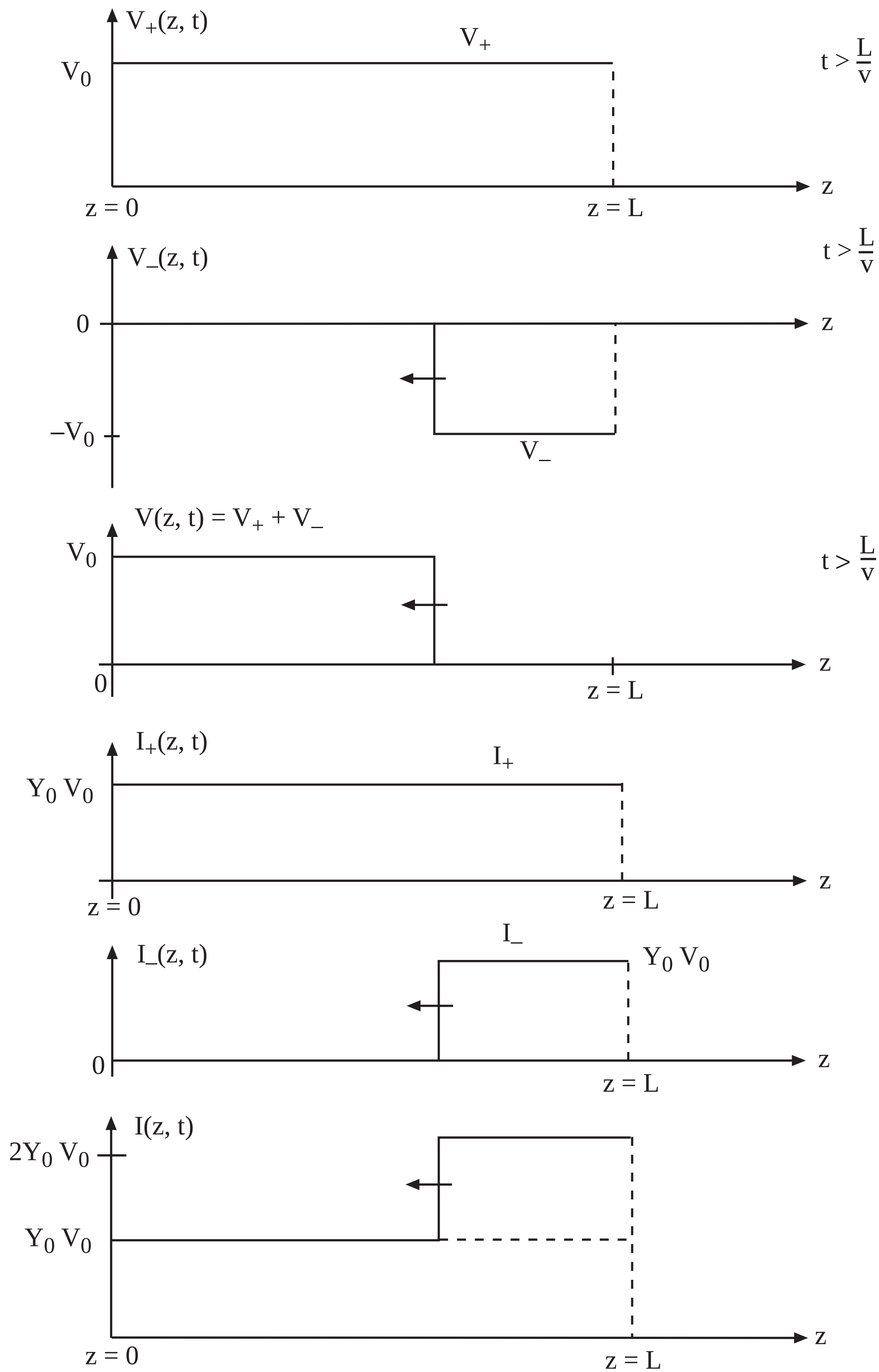
$$V(z = 0, t) = V_0 U(t). \tag{7}$$

The voltage on the transmission line is zero initially, the disturbance at $t = 0$ will create a wave front propagating to the right as t increases.

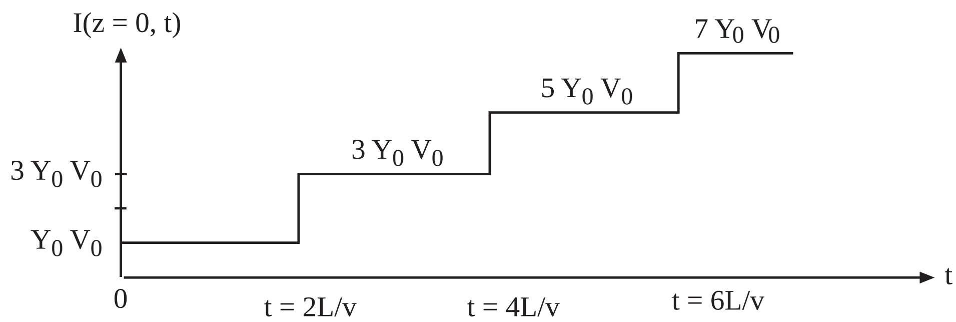




When the wave reaches the right end termination, the voltage and the current wave fronts will be reflected. However, the short at $z = L$ requires that $V(z = L, t) = 0$ always. Hence the reflected voltage wave, which is negative going, has an amplitude of $-V_0$. The corresponding current can be derived from (4) and is as shown.


V₊(z, t)
V₊
V₀
t > L/v
z
z = 0
z = L
V₋(z, t)
t > L/v
0
z
−V₀
V₋
V(z, t) = V₊ + V₋
V₀
t > L/v
z
0
z = L
I₊(z, t)
I₊
Y₀ V₀
z
z = 0
z = L
I₋
I₋(z, t)
Y₀ V₀
0
z
z = L
I(z, t)
2Y₀ V₀
Y₀ V₀
z
z = 0
z = L

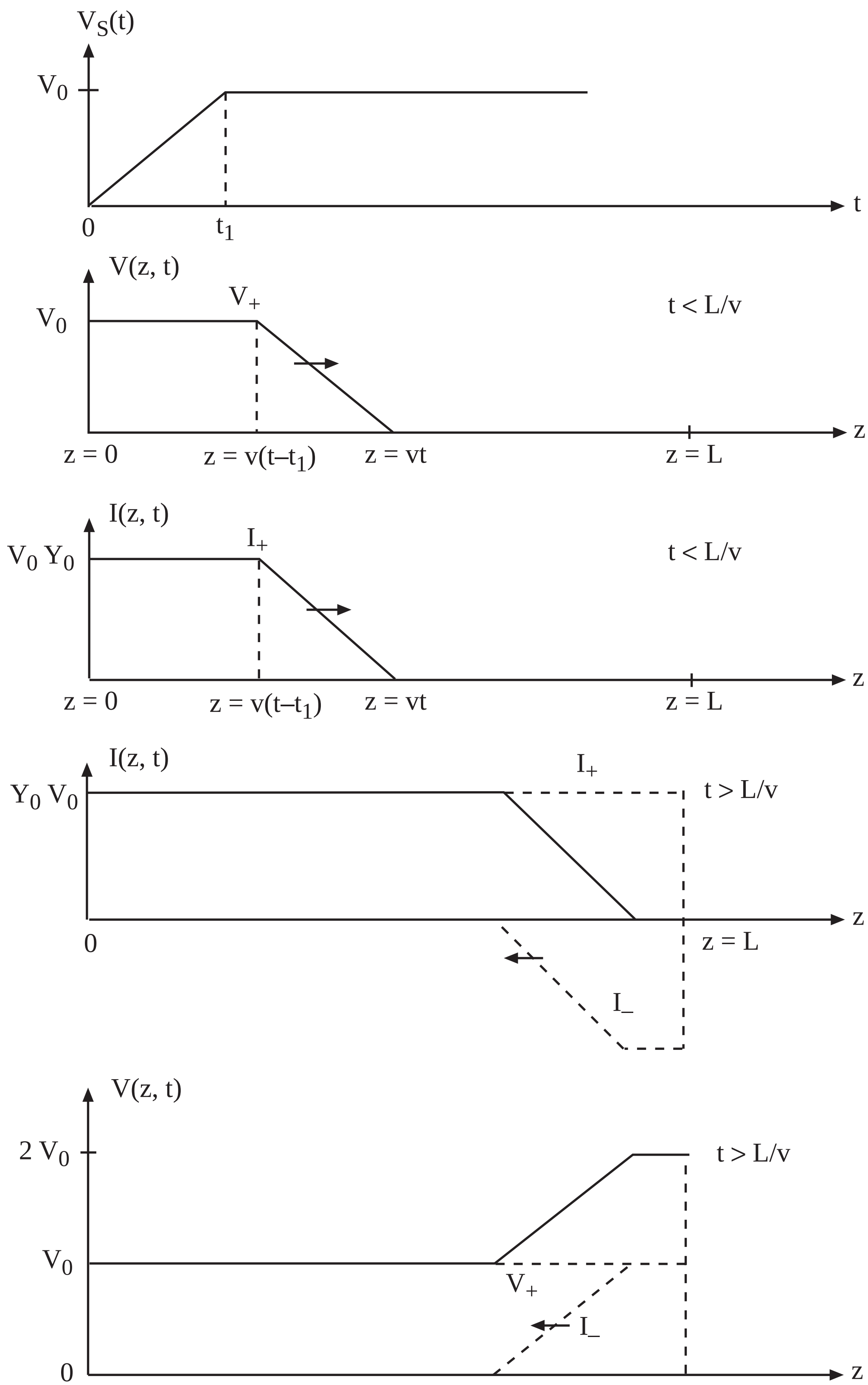
When the signal reaches the source end, it is being reflected again. A voltage source looks like a short circuit because the reflected voltage must cancel the incident voltage in order for the voltage across the voltage source remains unchanged. Hence the negative going voltage and current are again reflected like a short. Hence, if one is to measure the voltage at $z = 0$, it will always be V_0. However, the current at $z = 0$ will increase indefinitely with time as shown.



The current will eventually become infinitely large because the transmission line will become like a short circuit to the D.C. voltage source. Therefore, the current becomes infinite.

(b) **Open-Circuited Termination**

If we have an open-circuited termination at $z = L$, then the current has to be zero always. In this case, the reflected current is negative that of the incident current such that $I(z = L, t) = 0$ always. For example, if the source waveform looks like as shown below, the reflected waveform will behave as shown.



V_S(t)
V_0
t
0
t_1
V(z, t)
V_+
V_0
t < L/v
z
z = 0
z = v(t–t_1)
z = vt
z = L
I(z, t)
I_+
V_0 Y_0
t < L/v
z
z = 0
z = v(t–t_1)
z = vt
z = L
I(z, t)
I_+
Y_0 V_0
t > L/v
z
0
z = L
I_–
V(z, t)
2 V_0
t > L/v
V_0
V_+
I_–
0
z

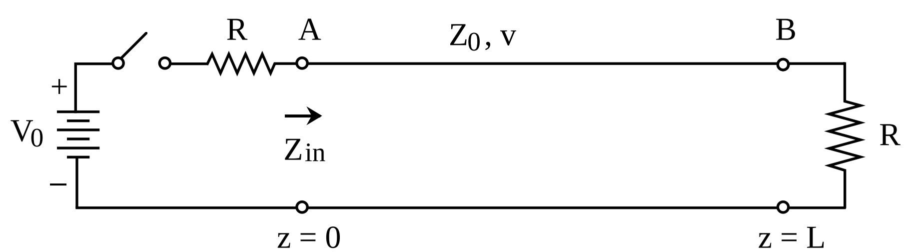
(c) **Resistive Termination**

We can think of transient signals as superpositions of time harmonic signals. This is a consequence of Fourier analysis. We see that the voltage reflection coefficient is -1 for a shorted termination for all frequencies. Hence, the voltage reflection coefficient is -1 for a transient signal. By a similar argument, the voltage reflection coefficient for an open-circuited termination is $+1$.

When the termination is resistive on a lossless transmission line, we recall that the voltage reflection coefficient is

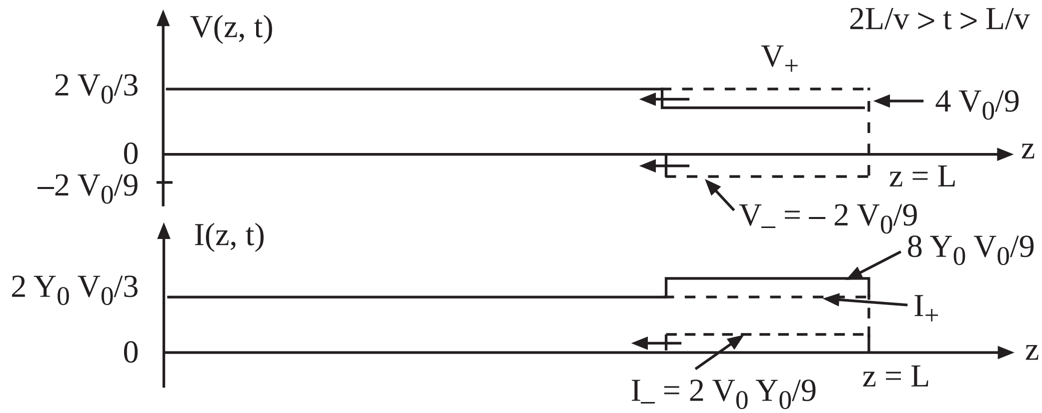
$$\rho_v = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{R_L - Z_0}{R_L + Z_0}. \tag{8}$$

Hence, the reflection coefficient is frequency independent. All frequency components in a transient signal will experience the same reflection. Hence, ρ_v is also the reflection coefficient for a voltage pulse.

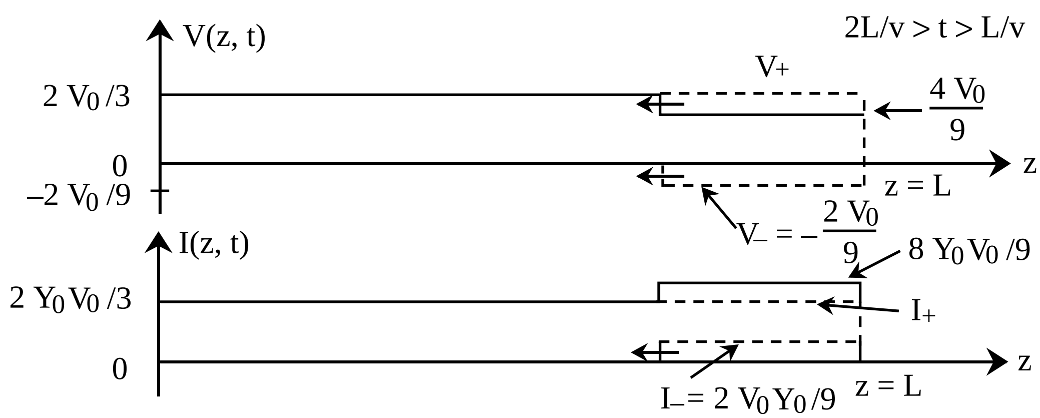


Consider, for example, a transmission line being driven via a source resistance R and a load termination R. If $R = \frac{1}{2}Z_0$, let us see what happens when we turn on the switch.

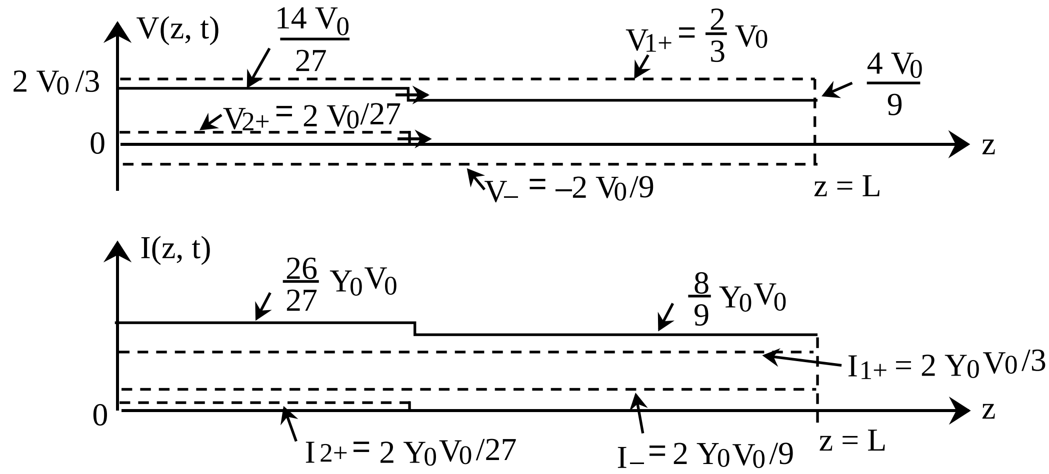
For $t < \frac{L}{V}$, the transmission line appears to be infinitely long to the source. Hence, Z_{in} looks like Z_0 to the source. Hence, $V_A = \frac{Z_0}{Z_0+R}V_0 = \frac{2}{3}V_0$ for $R = \frac{1}{2}Z_0$. Hence, we have a wavefront of $\frac{2}{3}V_0$ propagating to the right for $t < \frac{L}{V}$.



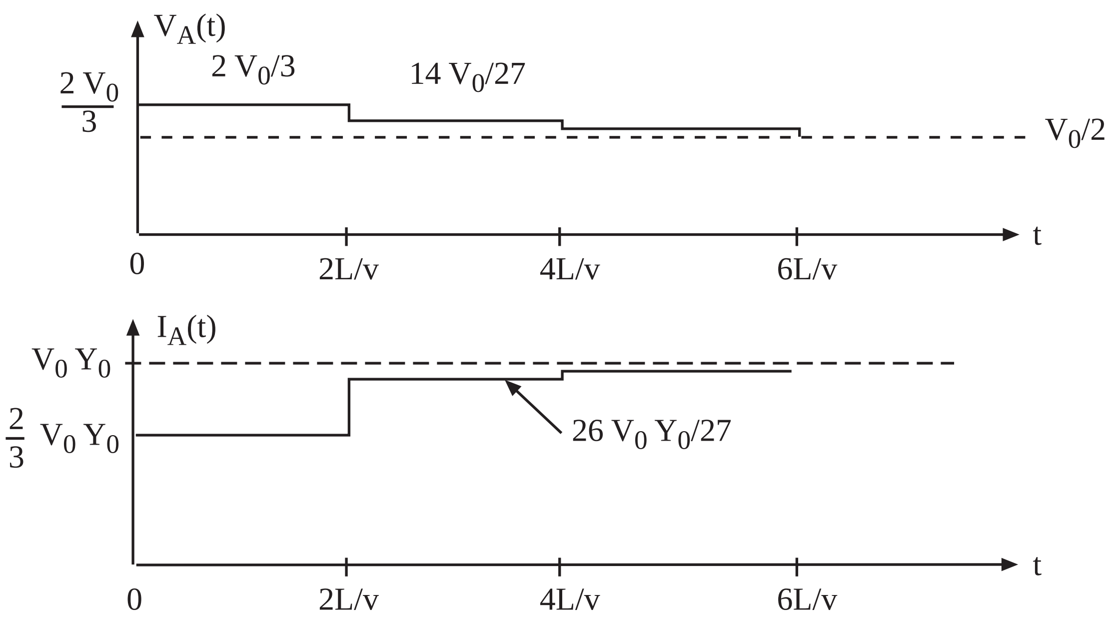
For $t > \frac{L}{V}$, a reflected voltage wave is generated at the termination and its amplitude is $\frac{2}{3}\rho_v V_0$. $\rho_v = -\frac{1}{3}$ for this termination.



For $t > 2\frac{L}{V}$, a voltage source looks like a short to the transient signal. The reflection from the left is again $-\frac{1}{3}$ for the voltage and $+\frac{1}{3}$ for the current.



When $t \to \infty$, the voltage and current on the line will settle down to a steady state. In that case, we have only DC signal on the line, and we need only to use DC circuit analysis to find the steady state solution. At DC, the transmission line becomes first two pieces of wires, $V_A = V_B = \frac{R}{2R}V_0 = \frac{1}{2}V_0$. The current through the circuit is $\frac{V_0}{Z_0}$. If one is to measure V_A as a function of time, it will look like



Transient analysis has important application to computer circuitry. We note that when we switch on a circuit with a delay line, we do not immediately arrive at the desired steady state value when we have a transmission line or a delay line. The settling time depends on the length of the line involved.

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13. Properties of Fields in a Transmission Line.

The field or wave in a transmission line is TEM (Transmission Electro-Magnetic) because both the **H**-field and the **E**-field are transverse to the direction of propagation. If the wave is propagating in the $\hat{z}$-direction, then both E_z and H_z are zero for such a wave. In such a case, the fields are

$$\mathbf{E} = \mathbf{E}_s, \mathbf{H} = \mathbf{H}_s, \tag{1}$$

where we have used the subscript s to denote fields transverse to the direction of propagation. We can also define a del operation such that

$$\nabla = \nabla_s + \hat{z}\frac{\partial}{\partial z}, \tag{2}$$

where ∇_s is transverse to the $\hat{z}$-direction, and in Cartesian coordinate, it is $\nabla_s = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$. From

$$\nabla \times \mathbf{H} = \epsilon\frac{\partial \mathbf{E}}{\partial t}, \tag{3}$$

or

$$\left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right) \times \mathbf{H}_s = \epsilon\frac{\partial \mathbf{E}}{\partial t}. \tag{4}$$

Since $\nabla_s \times \mathbf{H}_s$ points in the $\hat{z}$-direction, $\hat{z}\frac{\partial}{\partial z} \times \mathbf{H}_s$ is $\hat{z}$-directed, we have

$$\nabla_s \times \mathbf{H}_s = 0, \tag{5}$$

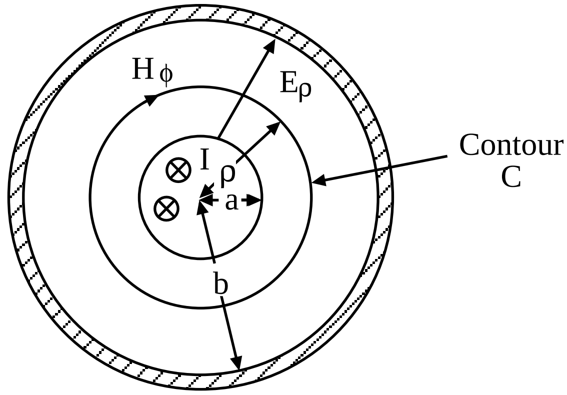
$$\frac{\partial}{\partial z}(\hat{z} \times \mathbf{H}_s) = \epsilon\frac{\partial \mathbf{E}_s}{\partial t}. \tag{6}$$

Similarly, from $\nabla_s \times \mathbf{E}_s = -\mu\frac{\partial \mathbf{H}_s}{\partial t}$, we can show that

$$\nabla_s \times \mathbf{E}_s = 0, \tag{7}$$

$$\frac{\partial}{\partial z}(\hat{z} \times \mathbf{E}_s) = -\mu\frac{\partial \mathbf{H}_s}{\partial t}. \tag{8}$$

Equations (5) and (7) shows that the transverse curl of the fields are zero. This implies that the fields in the transverse directions of a transmission line resembles that of the electrostatic fields. Furthermore, Equations (6) and (8) couple the $\mathbf{E}_s$ and $\mathbf{H}_s$ fields together. These two equations are the electromagnetic field analogues of the telegrapher's equations.



A current in a coaxial cable will produce a magnetic field polarized in the ϕ direction. From Ampere's Law, we have

$$\oint_C \mathbf{H}_s \cdot dl = \int_A \mathbf{J} \cdot ds = I, \tag{9}$$

or

$$\int_0^{2\pi} \rho \, d\phi H_\phi = I. \tag{10}$$

Hence,

$$H_\phi(\rho, z, t) = \frac{I(z,t)}{2\pi\rho}. \tag{11}$$

If we assume that the inner conductor in the coaxial line is charged up with the line charge Q in coulomb/m, then from $\oint \epsilon \mathbf{E} \cdot \hat{n} \, ds = Q$, we have

$$2\pi\rho\epsilon E_\rho = Q, \tag{12}$$

or

$$E_\rho = \frac{Q}{2\pi\rho\epsilon}. \tag{13}$$

Since the potential between a and b is $\int_a^b E_\rho \, d\rho$, we have

$$V = \int_a^b E_\rho \, d\rho = \frac{Q}{2\pi\epsilon} \ln\left(\frac{b}{a}\right). \tag{14}$$

Hence,

$$E_\rho(\rho, z, t) = \frac{V(z,t)}{\rho \ln(\frac{b}{a})} = \frac{Q(z,t)}{2\pi\epsilon\rho}. \tag{15}$$

The ratio $\frac{Q}{V}$ is the capacitance per unit length, and it is

$$C = \frac{2\pi\epsilon}{\ln(\frac{b}{a})}. \tag{16}$$

If $\mathbf{E}_s = \hat{\rho} E_\rho$, $\mathbf{H}_s = \hat{\phi} H_\phi$, equations (6) and (8) become

$$\frac{\partial}{\partial z} H_\phi = -\epsilon \frac{\partial E_\rho}{\partial t}, \tag{17}$$

$$\frac{\partial}{\partial z} E_\rho = -\mu \frac{\partial H_\phi}{\partial t}. \tag{18}$$

Substituting (11) for H_ϕ and (15) for E_ρ, we get

$$\frac{\partial}{\partial z} I(z,t) = -\frac{2\pi\epsilon}{\ln(\frac{b}{a})} \frac{\partial V}{\partial t}, \tag{19}$$

and

$$\frac{\partial}{\partial z} V(z,t) = -\frac{\mu \ln(\frac{b}{a})}{2\pi} \frac{\partial I}{\partial t}. \tag{20}$$

This is just the telegrapher's equations derived from Maxwell's equations. C is given by (16) while the inductance per unit length L is obtained by comparing (20) with the telegrapher's equations

$$L = \mu \frac{\ln(\frac{b}{a})}{2\pi}. \tag{21}$$

Note that the velocity of the wave on a transmission line is

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\epsilon}}, \tag{22}$$

which is independent of the dimensions of the line. This is because all TEM waves have velocity given by $\frac{1}{\sqrt{\mu\epsilon}}$.

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14. Skin Depth and Plane Wave in a Lossy Medium.

We learn earlier that in a lossy medium, $\mathbf{J} = \sigma \mathbf{E}$, and from

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} = \epsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E}. \tag{1}$$

Using phasor technique, we can convert the above to

$$\nabla \times \mathbf{H} = j\omega\epsilon \mathbf{E} + \sigma \mathbf{E} = j\omega \underline{\epsilon} \mathbf{E}, \tag{2}$$

where

$$\underline{\epsilon} = \epsilon - j\frac{\sigma}{\omega}, \tag{3}$$

is the ***complex permittivity.*** Furthermore, using that

$$\nabla \times \mathbf{E} = -j\omega\mu \mathbf{H}, \tag{4}$$

and that $\nabla \cdot \mathbf{H} = 0$, $\nabla \cdot \mathbf{E} = 0$, we can show that

$$\nabla^2 \mathbf{E} = -\omega^2 \mu \underline{\epsilon} \mathbf{E}, \tag{5}$$

$$\nabla^2 \mathbf{H} = -\omega^2 \mu \underline{\epsilon} \mathbf{H}. \tag{6}$$

[Refer to § 4 for details]. If we assume that $\mathbf{E} = \hat{x} E_x(z)$, then, we can show that

$$\frac{d^2}{dz^2} E_x(z) - \gamma^2 E_x(z) = 0, \tag{7}$$

where

$$\gamma = j\omega\sqrt{\mu\underline{\epsilon}} = \alpha + j\beta. \tag{7a}$$

The general solution to (7) is of the form

$$E_x(z) = c_1 e^{-\gamma z} + c_2 e^{\gamma z}. \tag{8}$$

If we assume that $c_2 = 0$, we have only

$$E_x(z) = c_1 e^{-\gamma z}. \tag{9}$$

We can convert the above into a real time quantity using phasor techniques, or

$$\begin{aligned} E_x(z,t) &= |c_1| \,\Re e[e^{-\alpha z - j\beta z + j\phi_1 + j\omega t}] \\ &= |c_1| \, e^{-\alpha z} \cos(\omega t - \beta z + \phi_1), \end{aligned} \tag{10}$$

where we have assumed that $c_1 = |c_1| e^{j\phi_1}$. Hence, we see that $E_x(z,t)$ is a wave that propagates to the right with velocity $v = \frac{\omega}{\beta}$ and attenuation constant α. We can find α from equation (7a), and

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\left(\epsilon - j\frac{\sigma}{\omega}\right)} = j\omega\sqrt{\mu\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)}. \tag{11}$$

The first term on the RHS of (1) is the displacement current term, while the second term is the conduction current term. From (2), we see that the ratio $\frac{\sigma}{\omega\epsilon}$ is the ratio of the conduction current to the displacement current in a lossy medium. $\frac{\sigma}{\omega\epsilon}$ is also known as the ***loss tangent*** of a lossy medium.

(i) When $\frac{\sigma}{\omega\epsilon} \ll 1$, the loss tangent is small, and the conduction current compared to the displacement current is small. The medium behaves more like a dielectric medium. In this case, we can use binomial expansions to approximate (11) to obtain

$$\gamma = j\omega\sqrt{\mu\epsilon}\left(1 - j\frac{1}{2}\frac{\sigma}{\omega\epsilon}\right) = \frac{1}{2}\sigma\sqrt{\frac{\mu}{\epsilon}} + j\omega\sqrt{\mu\epsilon}, \tag{12}$$

where

$$\alpha = \frac{1}{2}\sigma\sqrt{\frac{\mu}{\epsilon}}, \beta = \omega\sqrt{\mu\epsilon}. \tag{13}$$

(ii) When $\frac{\sigma}{\omega\epsilon} \gg 1$, the loss tangent is large because there is more conduction current than displacement current in the medium. In this case, the medium is conductive. According to equation (11), when $\frac{\sigma}{\omega\epsilon} \gg 1$, we have

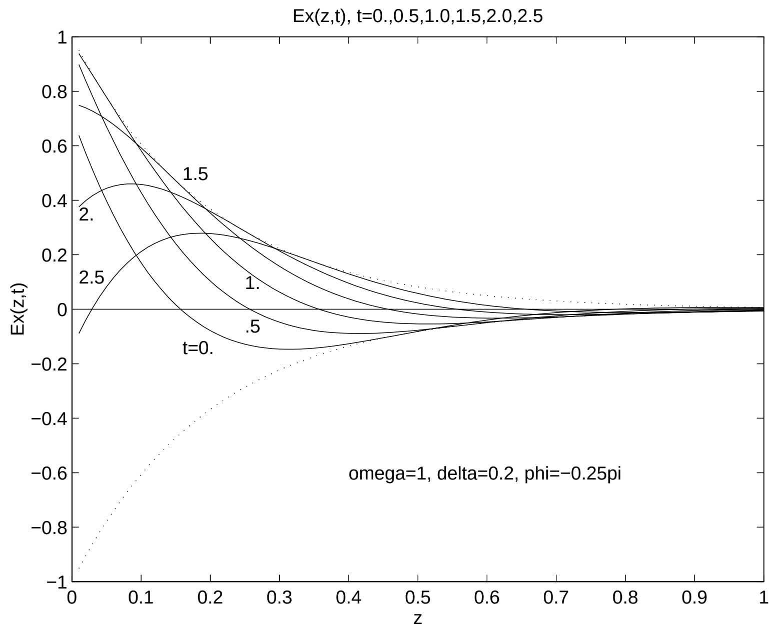
$$\gamma = j\omega\sqrt{-j\frac{\mu\sigma}{\omega}} = \sqrt{j\omega\mu\sigma} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}. \tag{14}$$

Hence

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \frac{1}{\delta}. \tag{15}$$

If we substitute $\alpha = \beta = \frac{1}{\delta}$ into (10), we have

$$E_x(z,t) = |c_1| e^{\frac{-z}{\delta}} \cos\left(\omega t - \frac{z}{\delta} + \phi_1\right). \tag{16}$$



This signal attenuates to e^{-1} of its original strength at $z = \delta$. Hence δ is also known as the ***penetration depth*** or the ***skin depth*** of a conductive medium. For other media, the penetration is $\frac{1}{\alpha}$, but for a conductive medium, it is

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f \mu\sigma}}. \tag{17}$$

This skin depth decreases with increasing frequencies and increasing conductivities.

(iii) When $\frac{\sigma}{\omega\epsilon} \approx 1$, it is a general lossy medium, and we have to resort to complex arithmatics to find α and β.
If we square (11), we have

$$\alpha^2 - \beta^2 + 2j\alpha\beta = -\omega^2\mu(\epsilon - j\frac{\sigma}{\omega}), \tag{18}$$

or

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon, \tag{19a}$$

$$2\alpha\beta = \omega\mu\sigma. \tag{19b}$$

Squaring (19a) and adding the square of (19b) to it, we have

$$(\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2 = (\alpha^2 + \beta^2)^2 = \omega^4\mu^2\epsilon^2 + \omega^2\mu^2\sigma^2, \tag{20}$$

or

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{\omega^2\epsilon^2 + \sigma^2}. \tag{21}$$

Combining with (19a), we deduce that

$$\alpha^2 = \frac{1}{2}(\omega\mu\sqrt{\omega^2\epsilon^2 + \sigma^2} - \omega^2\mu\epsilon), \tag{22a}$$

$$\beta^2 = \frac{1}{2}(\omega\mu\sqrt{\omega^2\epsilon^2 + \sigma^2} + \omega^2\mu\epsilon), \tag{22b}$$

Notice that when $\sigma = 0$, $\alpha = 0$.

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15. Group and Phase Velocities.

If we have two waves that are slightly different in frequency ω and phase constant β, a linear superposition of them is still a solution of the wave equation

$$E_x = E_0 \cos(\omega_1 t - \beta_1 z) + E_0 \cos(\omega_2 t - \beta_2 z). \tag{1}$$

If $\omega_1 = \omega - \Delta\omega$, $\beta_1 = \beta - \Delta\beta$, $\omega_2 = \omega + \Delta\omega$, $\beta_2 = \beta + \Delta\beta$, then

$$E_x = E_0 \cos[\omega t - \beta z - (\Delta\omega t - \Delta\beta z)] + E_0 \cos[\omega t - \beta z + (\Delta\omega t - \Delta\beta z)]. \tag{2}$$

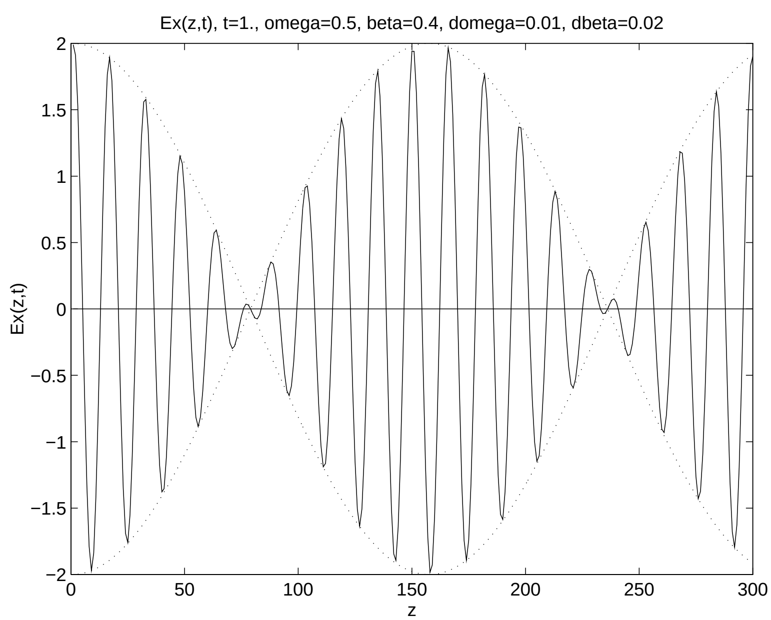
Using the fact that $\cos(A - B) + \cos(A + B) = 2 \cos A \cos B$, we have

$$E_x = 2E_0 \cos(\omega t - \beta z) \cos(\Delta\omega t - \Delta\beta z), \tag{3}$$

or

$$E_x(z, t) = 2E_0 \cos\left[\beta\left(\frac{\omega}{\beta}t - z\right)\right] \cos\left[\Delta\beta\left(\frac{\Delta\omega}{\Delta\beta}t - z\right)\right]. \tag{4}$$

At $t = 0$, we have $E_x = 2E_0 \cos\beta z \cos\Delta\beta z$ which is sketched below.



The first factor in (4) is rapidly varying while the second factor is slowly varying. The slowly varying term amplitude-modulates the rapidly varying term giving rise to the picture as shown.

We have learnt that a function of the form $f(vt - z)$ propagates in the positive z-direction with velocity v. From (10.5), we see that the rapidly varying term propagates with velocity $\frac{\omega}{\beta}$. Since this represents the propagation of the phases in the rapidly oscillating part in the figure, this is also known as phase velocity,

$$v_p = \frac{\omega}{\beta}. \tag{5}$$

The slowly varying part propagates with the velocity $\frac{\Delta\omega}{\Delta\beta}$, which is $\frac{d\omega}{d\beta}$ in the limit that $\Delta\omega$ and $\Delta\beta \to 0$. This represents the velocity on the envelope in the picture and hence, it is known as the group velocity,

$$v_g = \frac{d\omega}{d\beta} \text{ or } v_g^{-1} = \frac{d\beta}{d\omega}. \tag{6}$$

If $\beta = \omega\sqrt{\mu\epsilon}$, the phase velocity $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}$, the group velocity from (6) is also $\frac{1}{\sqrt{\mu\epsilon}}$. Hence, the group and the phase velocities are the same is β is a linear function of ω.

If β is not a linear function of ω, then, the phase velocity and the group velocities are functions of frequencies, and the medium is known to be ***dispersive***. In a dispersive medium, a pulse propagates with subsequent distortions because the different harmonics in the pulse propagate with different phase velocity. Example of a dispersive medium is a conductive medium where $\beta = \frac{1}{\delta} = \sqrt{\frac{\omega\mu\sigma}{2}}$, is not a linear function of ω.

In a distortionless line, the phase velocity is made to be frequency independent so that a pulse propagates without distortions.

Furthermore, a phase velocity can be larger than the velocity of light while the group velocity is always less than the speed of light. This is because energy propagates with the group velocity so that ***special relativity*** is not violated.

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16. Real Poynting Theorem.

Since $\mathbf{E} \times \mathbf{H}$ has the dimension of watts/m^2, we can study its divergence property and its conservative property. Using the vector identity in (1.26), we have,

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}. \tag{1}$$

From Maxwell's equations, we can replace $\nabla \times \mathbf{E}$ by $-\frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \times \mathbf{H}$ by $\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$. Hence,

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}) &= -\mathbf{H}\frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J} \\ &= -\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} - \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \mathbf{E} \cdot \mathbf{J}. \end{aligned} \tag{2}$$

We can show that

$$\frac{1}{2}\frac{\partial \left|\mathbf{H}\right|^2}{\partial t} = \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t}. \tag{3}$$

Hence,

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\mu \left|\mathbf{H}\right|^2 + \frac{1}{2}\epsilon \left|\mathbf{E}\right|^2\right) - \mathbf{E} \cdot \mathbf{J}. \tag{4}$$

We can define

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \text{ Poynting vector (Power Flow Density } watt\, m^{-2}), \tag{5}$$

$$U_H = \frac{1}{2}\mu \left|\mathbf{H}\right|^2 \text{ Magnetic Energy Density } (joule\, m^{-3}), \tag{6}$$

$$U_E = \frac{1}{2}\epsilon \left|\mathbf{E}\right|^2 \text{ Electric Energy Density} (joule\, m^{-3}), \tag{7}$$

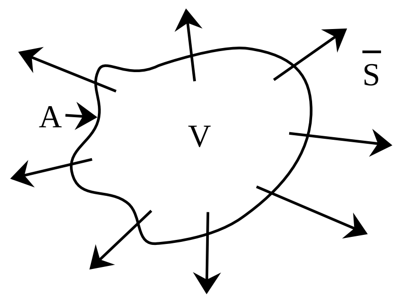
$$\mathbf{E} \cdot \mathbf{J} = \text{ Energy Dissipation Density} (watt\, m^{-3}). \tag{8}$$

U_H and U_E represent the energy stored in the magnetic field and electric field respectively. Alternatively, (4) becomes

$$\nabla \cdot \mathbf{S} = -\frac{\partial}{\partial t}(U_H + U_E) - \mathbf{E} \cdot \mathbf{J}. \tag{9}$$

Using the divergence theorem, (9) can be written in integral form,

$$\oint_A \mathbf{S} \cdot \hat{n}\, dA = -\frac{\partial}{\partial t} \int_V (U_H + U_E)\, dV - \int_V \mathbf{E} \cdot \mathbf{J}\, dV. \tag{10}$$



The equation says that the LHS will be positive only if there is a net outflow of the flux due to the vector field $\mathbf{S}$. If there is no current inside V so that $\mathbf{E} \cdot \mathbf{J} = 0$, then this is only possible if the stored energy $U_H + U_E$ inside V decreases with time.

If $\mathbf{J} = \sigma \mathbf{E}$, then the last term is $-\int \sigma \left|\mathbf{E}\right|^2 dV$ is always negative. Hence, the last term tends to make $\oint_S \mathbf{S} \cdot \hat{n}\, dA$ negative, because energy dissipation has to be compensated by power flux flowing into V. The Poynting theorems (9) and (10) are statements of energy conservation. For example, for a plane wave,

$$\mathbf{E} = \hat{x} f(z - vt), \quad \mathbf{H} = \hat{y}\sqrt{\frac{\epsilon}{\mu}} f(z - vt), \tag{11}$$

then

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \hat{z}\sqrt{\frac{\epsilon}{\mu}} f^2(z - vt). \tag{12}$$

Also,

$$U_E + U_H = \frac{1}{2}\epsilon f^2(z - vt) + \frac{1}{2}\epsilon f^2(z - vt) = \epsilon f^2(z - vt), \tag{13}$$

Therefore,

$$\mathbf{S} = \hat{z}\frac{1}{\sqrt{\mu\epsilon}}\epsilon f^2(z - vt) = \hat{z} v (U_E + U_H). \tag{14}$$

Hence, the velocity times the total energy density stored equals the power density flow in a plane wave.

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17. Complex Poynting Theorem.

The complex Poynting vector is defined to be

$$\underline{\mathbf{S}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}}^*. \tag{1}$$

It has the dimension of watt$/m^2$ and it denotes the flow of complex power. (We have used underbars to denote complex vectors).

Before we proceed further, let us look at Maxwell's equations for the phasor field. In phasor representation, Maxwell's equations become

$$\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}} + j\omega\epsilon\underline{\mathbf{E}}, \tag{2}$$
$$\nabla \times \underline{\mathbf{E}} = -j\omega\mu\underline{\mathbf{H}}. \tag{3}$$

First, we study the divergence property of (1),

$$\nabla \cdot (\underline{\mathbf{E}} \times \underline{\mathbf{H}}^*) = \underline{\mathbf{H}}^* \cdot \nabla \times \underline{\mathbf{E}} - \underline{\mathbf{E}} \cdot \nabla \times \underline{\mathbf{H}}^*. \tag{4}$$

Substituting (2) and (3) into (4), we have

$$\begin{aligned}\nabla \cdot (\underline{\mathbf{E}} \times \underline{\mathbf{H}}^*) &= -j\omega\mu\underline{\mathbf{H}} \cdot \underline{\mathbf{H}}^* + j\omega\epsilon\underline{\mathbf{E}} \cdot \underline{\mathbf{E}}^* - \underline{\mathbf{E}} \cdot \underline{\mathbf{J}}^* \\ &= -j\omega[\mu\,|\underline{\mathbf{H}}|^2 - \epsilon\,|\underline{\mathbf{E}}|^2] - \underline{\mathbf{E}} \cdot \underline{\mathbf{J}}^*. \end{aligned} \tag{5}$$

Comparing with (16.4), (5) involves the difference of the stored energy terms rather than the sum.

We have shown that for two quantities,

$$\mathbf{A}(z,t) = \Re e[\underline{\mathbf{A}}(z)e^{j\omega t}], \tag{6}$$
$$\mathbf{B}(z,t) = \Re e[\underline{\mathbf{B}}(z)e^{j\omega t}]. \tag{7}$$

The time average of $A(z,t)B(z,t)$, denoted by $\langle A, B\rangle$ is given by

$$\langle A, B\rangle = \frac{1}{2}\Re e[\underline{\mathbf{A}}(z)\underline{\mathbf{B}}^*(z)]. \tag{8}$$

Therefore,

$$\langle \mathbf{S}\rangle = \langle \mathbf{E} \times \mathbf{H}\rangle = \frac{1}{2}\Re e[\underline{\mathbf{E}} \times \underline{\mathbf{H}}^*] = \frac{1}{2}\Re e[\underline{\mathbf{S}}]. \tag{9}$$

The imaginary part of $\underline{\mathbf{S}}$ corresponds to instantaneous power that time averages to zero. It is also known as the reactive power. We can also convert (5) into integral from using the divergence theorem,

$$\oint_A (\underline{\mathbf{E}} \times \underline{\mathbf{H}}^*) \cdot \hat{n}\, dA = -j\omega \oint_V [\mu\, |\mathbf{H}|^2 - \epsilon\, |\mathbf{E}|^2]\, dV - \oint_V \sigma\, |\mathbf{E}|^2\, dV, \tag{10}$$

where we have assumed that $\mathbf{J} = \sigma\mathbf{E}$. If μ, ϵ, and σ are all real, then

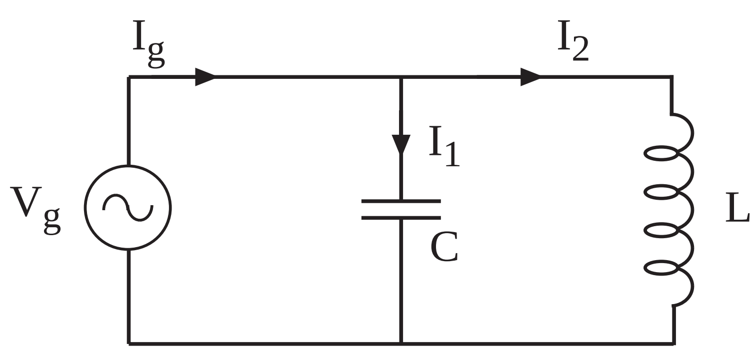
$$\oint_A \Re e(\mathbf{E} \times \mathbf{H}^*) \cdot \hat{n}\, dA = -\oint_V \sigma\, |\mathbf{E}|^2\, dV, \tag{11}$$

and

$$\oint_A \Im m(\mathbf{E} \times \mathbf{H}^*) \cdot \hat{n}\, dA = -\omega \oint_V [\mu\, |\mathbf{H}|^2 - \epsilon\, |\mathbf{E}|^2]\, dV. \tag{12}$$

We see that the real part of the power corresponds to power dissipated in V while the imaginary part of the power corresponds to difference in the magnetic energy stored and the electric energy stored. Hence, if a system has equal amount of magnetic and electric energy stored, it does not consume any reactive power.

Example of Reactive Power



We notice that in the complex Poynting theorem, the reactive power is proportional to $\omega(\mu|\mathbf{H}|^2 - \epsilon|\mathbf{E}|^2)$. It is zero when $\mu|\mathbf{H}|^2 = \epsilon|\mathbf{E}|^2$, or when the stored magnetic field energy equals the stored electric field energy. To comprehend this further, we look at a simple LC circuit driven by a time-harmonic voltage source.

At the resonant frequency of the tank circuit, $\omega = 1/\sqrt{LC}$, its input impedance is infinite, and hence $I_g = 0$. Therefore, there is no power delivered from the generator, be it real or reactive. However, $I_1 = -I_2 \neq 0$ at resonance, and as the tank circuit is resonating, the electric field energy stored in C is being converted into the magnetic field energy stored in L. Therefore,

$\frac{1}{2}L|I|^2 = \frac{1}{2}C|V|^2$ can be easily verified for a resonating tank circuit. This is precisely the case mentioned above.

Away from resonance,

$$I_g = V_g(j\omega C + \frac{1}{j\omega L}) = j\omega C V_g(1 - \frac{1}{\omega^2 LC}).$$

I_g is at 90° out-of-phase with V_g, and the complex power, $V_g I_g^*$ is purely imaginary. This implies that there is no time average power delivered by the source V_g, but it delivers nonzero reactive power. Away from resonance, the magnetic and electric stored energies are not in perfect balance with respect to each other, and we need to augment the system with external reactive power.

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18. Wave Polarization.

We learnt that

$$\mathbf{E} = \hat{x}E_x = \hat{x}E_1 \cos(\omega t - \beta z), \tag{1}$$

is a solution to the wave equation because $\nabla \cdot \mathbf{E} = 0$. Similarly,

$$\mathbf{E} = \hat{y}E_y = \hat{y}E_2 \cos(\omega t - \beta z + \phi), \tag{2}$$

is also a solution to the wave equation. Solutions (1) and (2) are known as ***linearly polarized waves***, because the electric field or the magnetic field are polarized in only one direction. However, a linear superposition of (1) and (2) are still a solution to Maxwell's equation

$$\mathbf{E} = \hat{x}E_x(z,t) + \hat{y}E_y(z,t). \tag{3}$$

If we observe this field at $z = 0$, it is

$$\mathbf{E} = \hat{x}E_1 \cos \omega t + \hat{y}E_2 \cos(\omega t + \phi). \tag{4}$$

When $\phi = 90^\circ$,

$$E_x = E_1 \cos \omega t \quad E_y = E_2 \cos(\omega t + 90^\circ), \tag{5}$$

$$\text{When } \omega t = 0^\circ, \quad E_x = E_1, \quad E_y = 0. \tag{6}$$

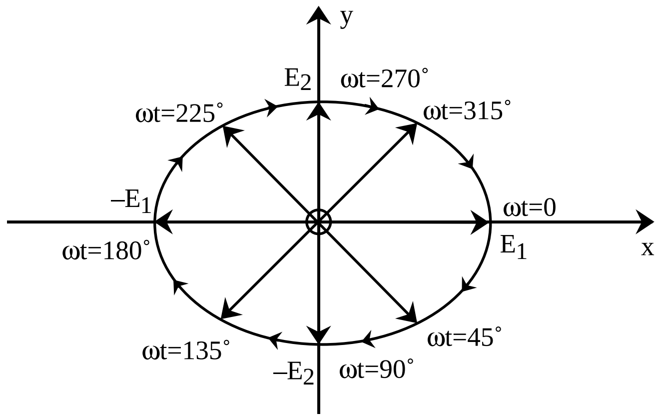
$$\text{When } \omega t = 45^\circ, \quad E_x = \frac{E_1}{\sqrt{2}}, \quad E_y = -\frac{E_2}{\sqrt{2}}. \tag{7}$$

$$\text{When } \omega t = 90^\circ, \quad E_x = 0, \quad E_y = -E_2. \tag{8}$$

$$\text{When } \omega t = 135^\circ, \quad E_x = -\frac{E_1}{\sqrt{2}}, \quad E_y = -\frac{E_2}{\sqrt{2}}. \tag{9}$$

$$\text{When } \omega t = 180^\circ, \quad E_x = -E_1, \quad E_y = 0. \tag{10}$$

If we continue further, we can sketch out the tip of the vector field $\mathbf{E}$. It traces out an ellipse as shown when $E_1 \neq E_2$. Such a wave is known as an ***elliptically polarized wave***.



When $E_1 = E_2$, the ellipse becomes a circle, and the wave is known as a ***circularly polarized wave***. When ϕ is $-90°$, the vector $\mathbf{E}$ rotates in the counter-clockwise direction.

A wave is classified as ***left hand elliptically (circularly) polarized*** when the wave is approaching the viewer. A counterclockwise rotation is classified as ***right hand elliptically (circularly) polarized***.

When $\phi \neq \pm 90°$, the tip of the vector $\mathbf{E}$ traces out a tilted ellipse. We can show this by expanding E_y in (5).

$$\begin{aligned} E_y &= E_2 \cos\omega t \cos\phi - E_2 \sin\omega t \sin\phi \\ &= \frac{E_2}{E_1} E_x \cos\phi - E_2 \left[1 - \left(\frac{E_x}{E_1}\right)^2\right]^{\frac{1}{2}} \sin\phi. \end{aligned} \tag{11}$$

Rearranging terms, we get

$$aE_x^2 - bE_xE_y + cE_y^2 = 1, \tag{12}$$

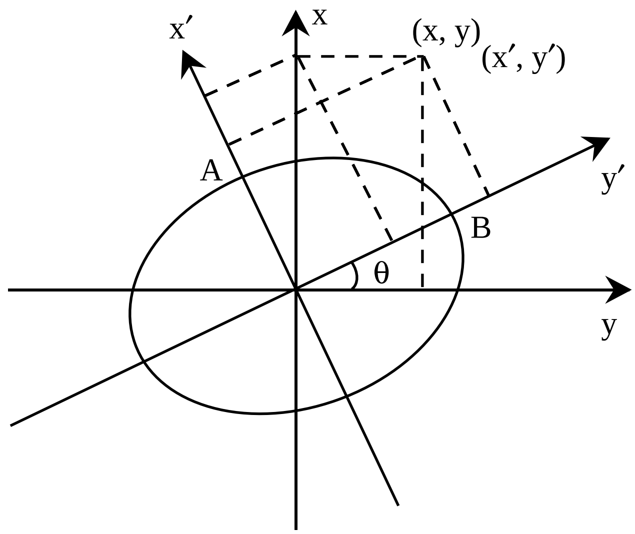
where

$$a = \frac{1}{E_1^2 \sin^2\phi}, \quad b = \frac{2\cos\phi}{E_1E_2\sin^2\phi}, \quad c = \frac{1}{E_2^2\sin^2\phi}. \tag{13}$$

Equation (12) is of the form

$$ax^2 - bxy + cy^2 = 1, \tag{14}$$

which is the equation of a tilted ellipse.



The equation of an ellipse in its self coordinate is

$$\left(\frac{x'}{A}\right)^2 + \left(\frac{y'}{B}\right)^2 = 1, \tag{15}$$

where A and B are the semi-axes of the ellipse. However,

$$x' = x\cos\theta - y\sin\theta, \tag{16}$$

$$y' = x\sin\theta + y\cos\theta, \tag{17}$$

we have

$$x^2\left(\frac{\cos^2\theta}{A^2} + \frac{\sin^2\theta}{B^2}\right) - xy\sin 2\theta\left(\frac{1}{A^2} - \frac{1}{B^2}\right) + y^2\left(\frac{\sin^2\theta}{A^2} + \frac{\cos^2\theta}{B^2}\right) = 1. \tag{18}$$

Equating (14) and (18), we can deduce that

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\cos\phi E_1 E_2}{E_2^2 - E_1^2}\right), \tag{19}$$

$$AR = \left(\frac{1+\Delta}{1-\Delta}\right)^{\frac{1}{2}}, \tag{20}$$

where

$$\Delta = \left[1 - \frac{4E_1^2E_2^2\sin^2\phi}{E_1^2 + E_2^2}\right]^{\frac{1}{2}}. \tag{21}$$

AR is the axial ratio which is the ratio of the two axes of the ellipse. It is defined to be larger than one always.

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19. Representation of a Plane Wave.

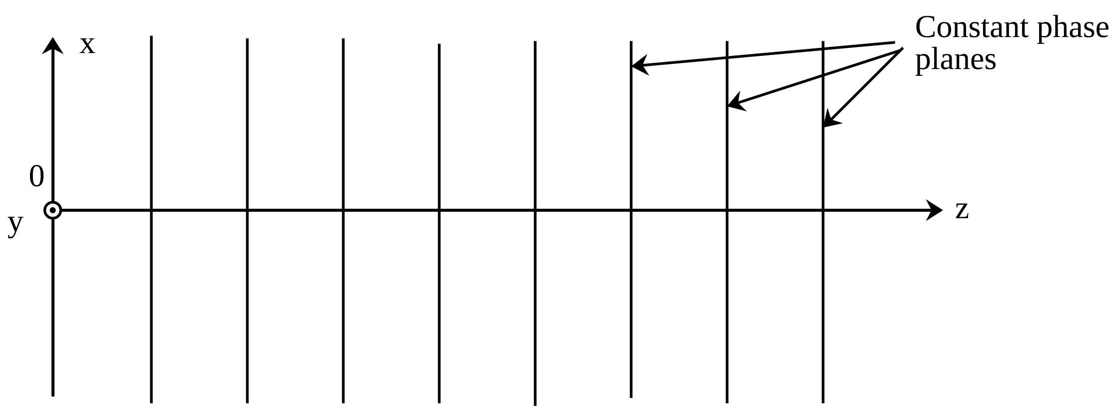
When $\nabla \cdot \mathbf{E} = 0$, the electric field satisfies the wave equation

$$\nabla^2 \mathbf{E} + \beta^2 \mathbf{E} = 0, \tag{1}$$

where $\beta^2 = \omega^2 \mu\epsilon$. We have learnt that one of the many possible solutions to the above equation is

$$\mathbf{E} = \hat{x} E_0 e^{-j\beta z}. \tag{2}$$

The expression $e^{-j\beta z}$, when viewed in three dimensions, has constant phase planes or wave fronts which are orthogonal to the z-axis.



To denote a plane wave propagating in other directions, we write it as

$$\mathbf{E} = \hat{a} E_0 e^{-j\beta_x x - j\beta_y y - j\beta_z z}, \tag{3}$$

where $\hat{a}$ is a constant unit vector, and E_0 a constant. If we substitute (3) into (1), we obtain

$$[-\beta_x^2 - \beta_y^2 - \beta_z^2 + \beta^2] E_0 = 0. \tag{4}$$

In order for (3) to satisfy (1) and that $E_0 \neq 0$, we require that

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2 \mu\epsilon. \tag{5}$$

If we define a vector $\boldsymbol{\beta} = \hat{x}\beta_x + \hat{y}\beta_y + \hat{z}\beta_z$, and $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$, then (3) can be written as

$$\mathbf{E} = \hat{a} E_0 e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}, \tag{6}$$

where the magnitude of $\boldsymbol{\beta}$ is

$$|\boldsymbol{\beta}| = [\beta_x^2 + \beta_y^2 + \beta_z^2]^{\frac{1}{2}} = \beta. \tag{7}$$

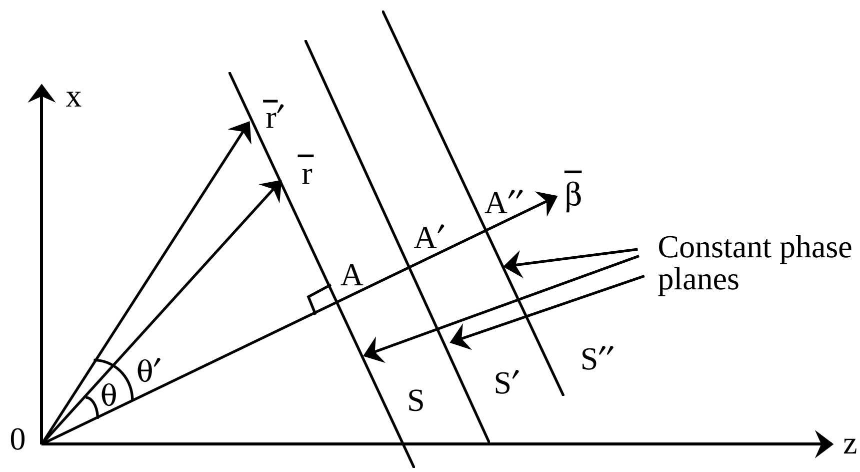
Equation (6) is a concise way to write a solution to (1). Since $\nabla \cdot \mathbf{E} = 0$ using (3), we note that

$$\nabla \cdot \mathbf{E} = -j[\hat{x}\beta_x + \hat{y}\beta_y + \hat{z}\beta_z] \cdot \hat{a}E_0 e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}. \tag{8}$$

Therefore, in order for $\nabla \cdot \mathbf{E} = 0$, we require that

$$\boldsymbol{\beta} \cdot \hat{a} = 0. \tag{9}$$

To explore further how the function $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$ look like, we assume $\boldsymbol{\beta}$ to be pointing in a direction as shown in the figure. The value of $\boldsymbol{\beta} \cdot \mathbf{r}$ is constant on a plane that is orthogonal to $\boldsymbol{\beta}$.



That is

$$\boldsymbol{\beta} \cdot \mathbf{r} = |\boldsymbol{\beta}|\,|\mathbf{r}| \cos\theta = \beta(OA), \tag{10}$$

for all $\mathbf{r}$ on the plane S that is orthogonal to $\boldsymbol{\beta}$. Hence, S is the constant phase plane of $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}} = e^{-j\beta(OA)}$. As one moves progressively in the $\boldsymbol{\beta}$ direction, the function $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$ has a phase that is linearly decreasing with distance. Therefore, $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$ denotes a plane wave that is propagating in the $\boldsymbol{\beta}$ direction. When $\boldsymbol{\beta}$ is pointing in the z-direction, such that $\boldsymbol{\beta} = \hat{z}\beta$, then $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}} = e^{-j\beta z}$, which is our familiar solution of a plane wave propagating in the z-direction.

An example of a plane wave electric field satisfying Maxwell's equations is

$$\mathbf{E} = \hat{y}E_0 e^{-j\beta_x x - j\beta_z z}, \tag{11}$$

where $\beta_x^2 + \beta_z^2 = \beta^2$. The corresponding magnetic field can be derived using Maxwell's equations.

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}. \tag{12}$$

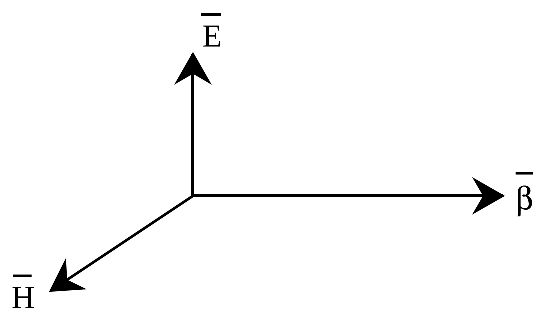
Hence,

$$\begin{aligned}\mathbf{H} &= \frac{-1}{j\omega\mu}\left(\hat{z}\frac{\partial}{\partial x}E_y - \hat{x}\frac{\partial}{\partial z}E_y\right) \\ &= (\hat{z}\beta_x - \hat{x}\beta_z)\,\frac{E_0}{\omega\mu}e^{-j\beta_x x - j\beta_z z}.\end{aligned} \tag{13}$$

In general, when ∇ operates on a plane wave phasor described by $e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$, it transforms into $-j\boldsymbol{\beta}$. This is obvious also from Equation (8). Therefore, from (12), we can express

$$\mathbf{H} = \frac{1}{\omega\mu}\boldsymbol{\beta} \times \mathbf{E}. \tag{14}$$

Therefore, $\mathbf{H}$ is orthogonal to both $\mathbf{E}$ and $\boldsymbol{\beta}$, or that $\mathbf{H} \cdot \mathbf{E} = 0$, and that $\mathbf{H}\cdot\boldsymbol{\beta} = 0$, in addition to $\mathbf{E}\cdot\boldsymbol{\beta} = 0$. Furthermore, $\mathbf{E}\times\mathbf{H}$ points in the direction of $\boldsymbol{\beta}$. Therefore, for a plane electromagnetic wave, $\mathbf{E}$, $\mathbf{H}$, and $\boldsymbol{\beta}$ form a right-handed orthogonal system. It is also a transverse electromagnetic (TEM) wave.



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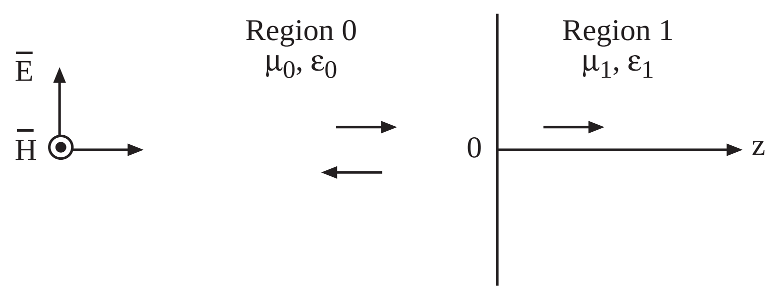
19a. Reflection and Transmission of a Simple Plane Wave Off an Interface.

We have learnt that in an infinite free space, a simple plane wave solution exists that is given by

$$\begin{aligned}\mathbf{E} &= \hat{x}E_x(z) = \hat{x}E_0 e^{-j\beta_0 z},\\ \mathbf{H} &= \hat{y}H_y(z) = \hat{y}H_0 e^{-j\beta_0 z} = \hat{y}\frac{E_0}{\eta_0}e^{-j\beta_0 z},\end{aligned} \tag{1}$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ is the intrinsic impedance, and $\beta_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the wavenumber. Also, $\beta_0 = 2\pi/\lambda_0$ where λ_0 is the free space wavelength.



When the simple plane wave is normally incident on a flat material interface, we expect to have a reflected wave in Region 0, and a transmitted wave in Region 1.

In Region 0, we can write the total fields as

$$\mathbf{E_0} = \hat{x}\left(E_0^+ e^{-j\beta_0 z} + E_0^- e^{+j\beta_0 z}\right), \tag{2}$$

$$\mathbf{H_0} = \hat{y}\left(\frac{E_0^+}{\eta_0}e^{-j\beta_0 z} - \frac{E_0^-}{\eta_0}e^{+j\beta_0 z}\right). \tag{3}$$

In Region 1, the total fields are

$$\mathbf{E_0} = \hat{x}E_1^+ e^{-j\beta_1 z}, \tag{4}$$

$$\mathbf{H_0} = \hat{x}\frac{E_1^+}{\eta_1}e^{-j\beta_1 z}, \tag{5}$$

where $\eta_1 = \sqrt{\mu_1/\epsilon_1}$ and $\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$. There are two unknowns in the above expressions, E_0^- and H_0^+. E_0^+ is known because it is the amplitude if the incident field. We can set up two equations to find two unknowns by matching boundary conditions at $z = 0$. The requisite boundary conditions are that the tangential components of the **E** field and **H** field should be continuous.

By imposing tangential **E** continuous, we arrive at

$$E_0^+ + E_0^- = E_1^+, \tag{6}$$

whereas imposing tangential **H** conditions yields

$$\frac{E_0^+}{\eta_0} - \frac{E_0^-}{\eta_0} = \frac{E_1^+}{\eta_1}. \tag{7}$$

Solving these two equations expresses E_0^- and E_1^+ in terms of E_0^+:

$$E_0^- = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} E_0^+, \tag{8}$$

$$E_1^- = \frac{2\eta_1}{\eta_1 + \eta_0} E_0^+. \tag{9}$$

We define the reflection coefficient to be

$$\Gamma = \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0}, \tag{10}$$

and the transmission coefficient to be

$$T = \frac{2\eta_1}{\eta_1 + \eta_0}. \tag{11}$$

Notice that $1 + \Gamma = T$.

When there is a mismatch at the interface, we expect most of the wave to be reflected. This occurs when $\eta_1 \ll \eta_0$. In this case, $\Gamma \simeq -1$, and $T \simeq 0$. It also occurs when $\eta_1 \gg \eta_0$, for which case, $\Gamma \simeq +1$, $T \simeq 2$.

The above derivation also holds true when Region 1 is a conductive lossy region. In this case, we replace ϵ_1 with a comlex permittivity $\tilde{\epsilon}_1$ which is given by

$$\tilde{\epsilon}_1 = \epsilon_1 - j\frac{\sigma_1}{\omega}. \tag{12}$$

Then $\eta_1 = \sqrt{\mu_1/\tilde{\epsilon}}$ where η_1 would be a complex number. Also, $j\beta_1$ becomes $\gamma_1 = j\omega\sqrt{\mu_1\tilde{\epsilon}_1} = \alpha_1 + j\beta_1$ which is a complex number also.

For a highly conductive medium like copper, $\sigma_1/\omega \gg \epsilon_1$, $\tilde{\epsilon}_1 \simeq -j\sigma_1/\omega$, and $\eta_1 = (1+j)\sqrt{\omega\mu_1/(2\sigma_1)}$. Consequently, $\eta_1 \ll \eta_0$ and $\Gamma \simeq -1$, $T \simeq= 0$.

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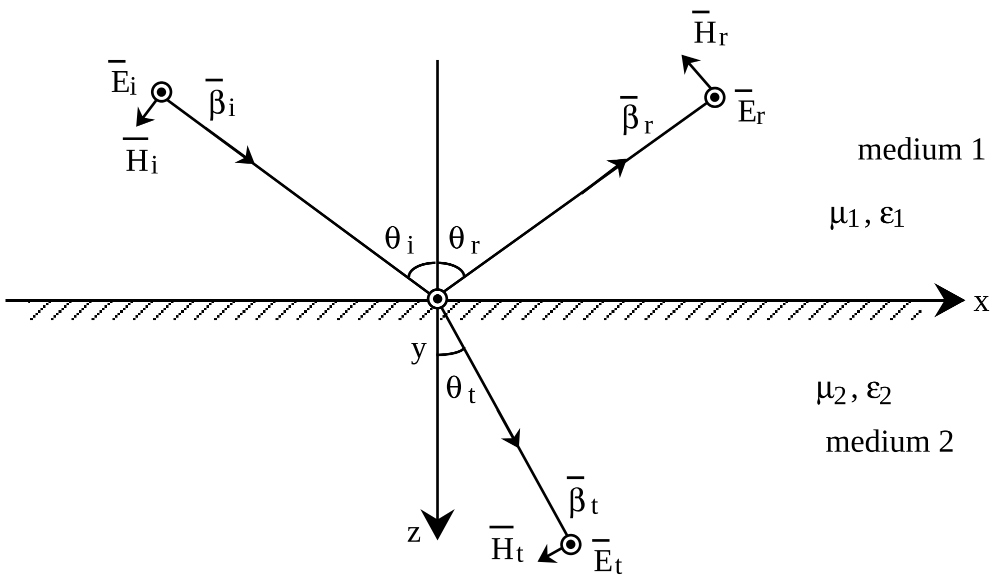
Date:November 7, 1997

20. Reflections and Refractions of Plane Waves.



Perpendicular Case (Transverse Electric or TE case)

When an incident wave impinges on a dielectric interface, a reflected wave as well as a transmitted wave is generated. We can express the three waves as

$$\mathbf{E}_i = \hat{y} E_0 e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}}, \tag{1}$$

$$\mathbf{E}_r = \hat{y} \rho_\perp E_0 e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}}, \tag{2}$$

$$\mathbf{E}_t = \hat{y} \tau_\perp E_0 e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}}. \tag{3}$$

The electric field is perpendicular to the xz plane, and $\boldsymbol{\beta}_i$, $\boldsymbol{\beta}_r$, and $\boldsymbol{\beta}_t$ are their respective directions of propagation. The $\boldsymbol{\beta}$'s are also known as **propagation vectors**. In particular,

$$\boldsymbol{\beta}_i = \hat{x}\beta_{ix} + \hat{z}\beta_{iz}, \tag{4}$$

$$\boldsymbol{\beta}_r = \hat{x}\beta_{rx} - \hat{z}\beta_{rz}, \tag{5}$$

$$\boldsymbol{\beta}_{tx} = \hat{x}\beta_{tx} + \hat{z}\beta_{tz}. \tag{6}$$

Since $\mathbf{E}_i$ and $\mathbf{E}_r$ are in medium 1, we have

$$\beta_{ix}^2 + \beta_{iz}^2 = \beta_1^2 = \omega^2 \mu_1 \epsilon_1, \tag{7}$$

$$\beta_{rx}^2 + \beta_{rz}^2 = \beta_1^2 = \omega^2 \mu_1 \epsilon_1, \tag{8}$$

and for $\mathbf{E}_t$ in medium 2, we have

$$\beta_{tx}^2 + \beta_{tz}^2 = \beta_2^2 = \omega^2 \mu_2 \epsilon_2. \tag{9}$$

(7), (8), and (9) are known as the **dispersion relations** for the components of the propagation vectors. From the figure, we note that

$$\beta_{ix} = \beta_1 \sin\theta_i, \quad \beta_{iz} = \beta_1 \cos\theta_i, \tag{10}$$
$$\beta_{rx} = \beta_1 \sin\theta_r, \quad \beta_{rz} = \beta_1 \cos\theta_r, \tag{11}$$
$$\beta_{tx} = \beta_2 \sin\theta_t, \quad \beta_{tz} = \beta_2 \cos\theta_t. \tag{12}$$

To find the unknown $\rho_\perp$ and $\tau_\perp$, we need to match boundary conditions for the fields at the dielectric interface. The boundary conditions are the equality of the tangential electric and magnetic fields on both sides of the interface. The magnetic fields can be derived via Maxwell's equations.

$$\mathbf{H}_i = \frac{\nabla \times \mathbf{E}_i}{-j\omega\mu_1} = \frac{\boldsymbol{\beta}_i \times \mathbf{E}_i}{\omega\mu_1} = (\hat{z}\beta_{ix} - \hat{x}\beta_{iz})\, \frac{E_0}{\omega\mu_1} e^{-j\boldsymbol{\beta}_i\cdot\mathbf{r}}. \tag{13}$$

Similarly,

$$\mathbf{H}_r = (\hat{z}\beta_{rx} + \hat{x}\beta_{rz})\frac{\rho_\perp E_0}{\omega\mu_1} e^{-j\boldsymbol{\beta}_r\cdot\mathbf{r}}, \tag{14}$$
$$\mathbf{H}_t = (\hat{z}\beta_{tx} - \hat{x}\beta_{tz})\frac{\tau_\perp E_0}{\omega\mu_2} e^{-j\boldsymbol{\beta}_t\cdot\mathbf{r}}. \tag{15}$$

Continuity of the tangential electric fields across the interface implies

$$E_0 e^{-j\beta_{ix}x} + \rho_\perp E_0 e^{-j\beta_{rx}x} = \tau_\perp E_0 e^{-j\beta_{tx}x}. \tag{16}$$

The above equation is to be satisfied for all x. This is only possible if

$$\beta_{ix} = \beta_{rx} = \beta_{tx} = \beta_x. \tag{17}$$

This condition is known as **phase matching**. From (10), (11), and (12), we know that (17) implies

$$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t. \tag{18}$$

The above implies that $\theta_r = \theta_i$. Furthermore,

$$\sqrt{\mu_1\epsilon_1}\sin\theta_i = \sqrt{\mu_2\epsilon_2}\sin\theta_t. \tag{19a}$$

If we define a refractive index $n_i = \sqrt{\frac{\mu_i\epsilon_i}{\mu_0\epsilon_0}}$, then (19a) becomes

$$n_1 \sin\theta_i = n_2 \sin\theta_t, \tag{19b}$$

which is the well known **Snell's Law**. Consequently, equation (16) becomes

$$1 + \rho_\perp = \tau_\perp. \tag{20}$$

From the continuity of the tangential magnetic fields, we have

$$-\beta_{iz}\frac{E_0}{\omega\mu_1}+\beta_{rz}\frac{\rho_\perp E_0}{\omega\mu_1}=-\beta_{tz}\frac{\tau_\perp E_0}{\omega\mu_2}. \tag{21}$$

Since $\theta_r=\theta_i$, we have $\beta_{iz}=\beta_{rz}$. Therefore, (21) becomes

$$1-\rho_\perp=\frac{\mu_1}{\mu_2}\frac{\beta_{tz}}{\beta_{iz}}\tau_\perp. \tag{22}$$

Solving (20) and (22), we have

$$\rho_\perp=\frac{\mu_2\beta_{iz}-\mu_1\beta_{tz}}{\mu_2\beta_{iz}+\mu_1\beta_{tz}}, \tag{23}$$

$$\tau_\perp=\frac{2\mu_2\beta_{iz}}{\mu_2\beta_{iz}+\mu_1\beta_{tz}}. \tag{24}$$

Using (10), (11), and (12), we can rewrite the above as

$$\rho_\perp=\frac{\eta_2\cos\theta_i-\eta_1\cos\theta_t}{\eta_2\cos\theta_i+\eta_1\cos\theta_t}, \tag{25}$$

$$\tau_\perp=\frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i+\eta_1\cos\theta_t}. \tag{26}$$

If the media are non-magnetic so that $\mu_1=\mu_2=\mu_0$, we can use (19) to rewrite (25) as

$$\rho_\perp=\frac{\eta_2\cos\theta_i-\eta_1\sqrt{1-\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i}}{\eta_2\cos\theta_i+\eta_1\sqrt{1-\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i}}. \tag{27}$$

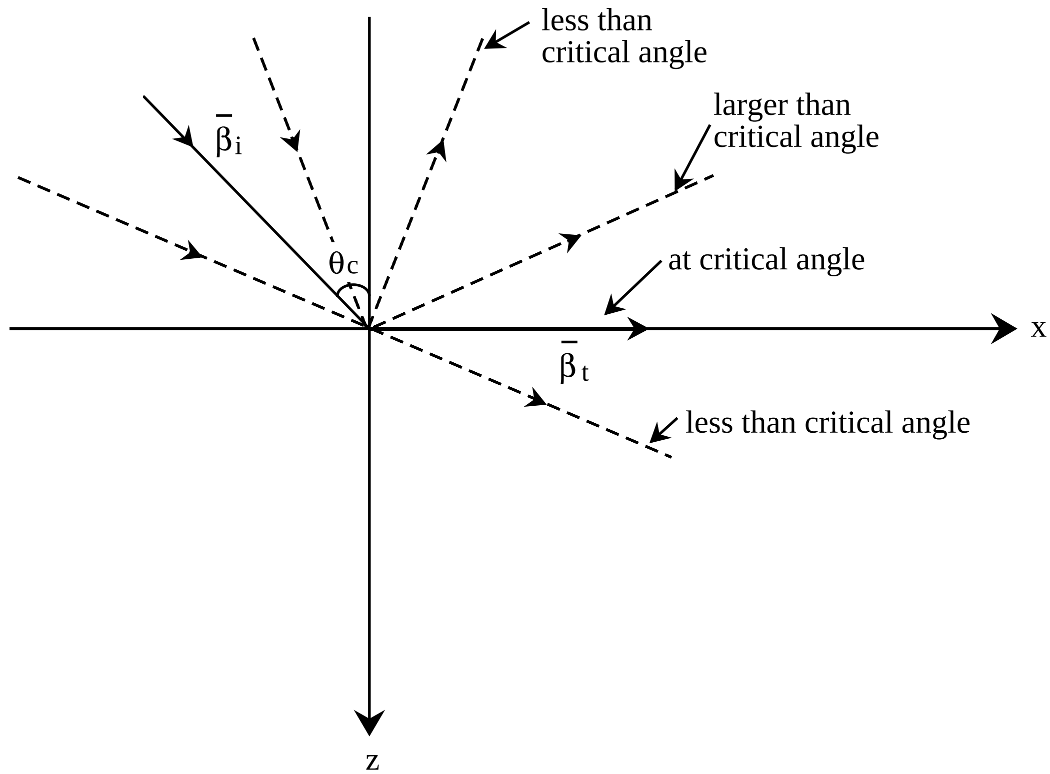
If $\sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i>1$, which is possible if $\frac{\epsilon_1}{\epsilon_2}>1$, when $\theta_i<\frac{\pi}{2}$, then $\rho_\perp$ is of the form

$$\rho_\perp=\frac{A-jB}{A+jB}, \tag{28}$$

which always has a magnitude of 1. In this case, all energy will be reflected. This is known as a **total internal reflection**. This occurs when $\theta_i>\theta_c$ where $\sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_c=1$. or

$$\theta_c=\sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}},\quad \epsilon_2<\epsilon_1. \tag{29}$$

When $\theta_i = \theta_c$, $\theta_t = 90^\circ$ from (19). The figure below denotes the phenomenon.



When $\theta_i > \theta_c$, $\beta_{tz} = \sqrt{\beta_2^2 - \beta_1^2 \sin^2 \theta_i}$, or

$$\beta_{tz} = \omega\sqrt{\mu_0 \epsilon_2}\left(1 - \frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i\right)^{\frac{1}{2}}. \tag{30}$$

The quantity in the parenthesis is purely negative, so that

$$\beta_{tz} = -j\alpha_{tz}, \tag{31}$$

a pure imaginary number. In this case, the electric field in medium 2 is

$$\mathbf{E}_t = \hat{y}\tau_\perp E_0 e^{-j\beta_x x - \alpha_{tz} z}. \tag{32}$$

The field is exponentially decaying in the positive z direction. We call such a wave an **evanescent wave**, or an **inhomogeneous wave** as opposed to **uniform plane wave**. The magnitude of a uniform plane wave is a **constant** of space while the magnitude of an evanescent wave or an inhomogeneous wave is **not a constant** of space. The corresponding magnetic field is

$$\mathbf{H}_t = (\hat{z}\beta_x + \hat{x}j\alpha_{tz})\frac{\tau_\perp E_0}{\omega\mu_2}e^{-j\beta_x x - \alpha_{tz} z}. \tag{33}$$

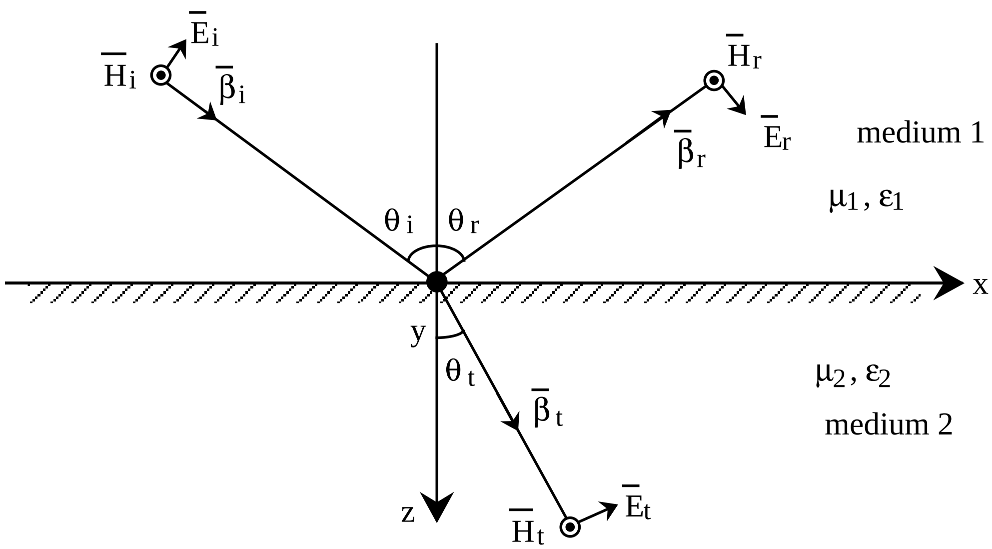
The complex power in the transmitted wave is

$$\underline{\mathbf{S}} = \mathbf{E}_t \times \mathbf{H}_t^* = (\hat{x}\beta_x + \hat{z}j\alpha_{tz})\frac{|\tau_\perp|^2 |E_0|^2}{\omega\mu_2} e^{-2\alpha_{tz}z}. \tag{34}$$

We note that $\underline{S}_x$ is pure real implying the presence of net time average power flowing in the $\hat{x}$-direction. However, $\underline{S}_z$ is pure imaginary implying that the power that is flowing in the $\hat{z}$-direction is purely reactive. Hence, no net time average power is flowing in the $\hat{z}$-direction.

Parallel case (Transverse Magnetic or TM case)

In this case, the electric field is parallel to the xz plane that contains the plane of incidence.



The magnetic field is polarized in the y direction, and they can be written as

$$\mathbf{H}_i = \hat{y}\frac{E_0}{\eta_1}e^{-j\boldsymbol{\beta}_i\cdot\mathbf{r}}, \tag{35}$$

$$\mathbf{H}_r = -\hat{y}\rho_\parallel\frac{E_0}{\eta_1}e^{-j\boldsymbol{\beta}_r\cdot\mathbf{r}}, \tag{36}$$

$$\mathbf{H}_t = \hat{y}\tau_\parallel\frac{E_0}{\eta_2}e^{-j\boldsymbol{\beta}_t\cdot\mathbf{r}}. \tag{37}$$

We put a negative sign in the definition for $\rho_\parallel$ to follow the convention of transmission line theory, where reflection coefficients are defined for voltages, and hence has a negative sign when used for currents. The magnetic field is the analogue of a current in transmission theory.

In this case, the electric field has to be orthogonal to $\boldsymbol{\beta}$ and $\hat{y}$, and they can be derived using

$$\mathbf{E}_i = -\frac{\boldsymbol{\beta}_i \times \mathbf{H}_i}{\omega \epsilon_1}$$

to be

$$\mathbf{E}_i = \frac{\hat{y} \times \boldsymbol{\beta}_i}{\beta} E_0 e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}} = (\hat{x}\beta_{iz} - \hat{z}\beta_{ix}) \frac{E_0}{\beta_1} e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}}, \tag{38}$$

$$\mathbf{E}_r = (\hat{x}\beta_{rz} + \hat{z}\beta_{rx}) \frac{\rho_\parallel E_0}{\beta_1} e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}}, \tag{39}$$

$$\mathbf{E}_t = (\hat{x}\beta_{tz} - \hat{z}\beta_{tx}) \frac{\tau_\parallel E_0}{\beta_2} e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}}. \tag{40}$$

Imposing the boundary conditions as before, we have

$$1 + \rho_\parallel = \frac{\beta_{tz}}{\beta_2} \frac{\beta_1}{\beta_{iz}} \tau_\parallel, \tag{41}$$

$$1 - \rho_\parallel = \frac{\eta_1}{\eta_2} \tau_\parallel. \tag{42}$$

The above can be solved to give

$$\rho_\parallel = \frac{\epsilon_1\beta_{tz} - \epsilon_2\beta_{iz}}{\epsilon_2\beta_{iz} + \epsilon_1\beta_{tz}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}, \tag{43}$$

and

$$\tau_\parallel = \frac{2\epsilon_2\beta_{iz}}{\epsilon_2\beta_{iz} + \epsilon_1\beta_{tz}} \frac{\eta_2}{\eta_1} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}. \tag{44}$$

In (43), $\rho_\parallel$ will be zero if

$$\eta_2^2 \cos^2\theta_t = \eta_1^2 \cos^2\theta_i. \tag{45}$$

Using Snell's Law, or (19), $\cos^2\theta_t = 1 - \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2} \sin^2\theta_i$, and (45) becomes

$$1 - \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2} \sin^2\theta_i = \frac{\mu_1\epsilon_2}{\mu_2\epsilon_1} \cos^2\theta_i. \tag{46}$$

Solving the above, we get

$$\sin\theta_i = \left(\frac{1 - \frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}}{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2} - \frac{\mu_1\epsilon_2}{\mu_2\epsilon_1}} \right)^{\frac{1}{2}}. \tag{47}$$

Most materials are non-magnetic in this world so that $\mu = \mu_0$, then

$$\sin\theta_i = \sqrt{\frac{\epsilon_2}{\epsilon_2 + \epsilon_1}}. \tag{48}$$

The angle for θ_i at which $\rho_\parallel = 0$ is known as the **Brewster angle**. It is given by

$$\theta_{ib} = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_2+\epsilon_1}} = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}. \tag{49}$$

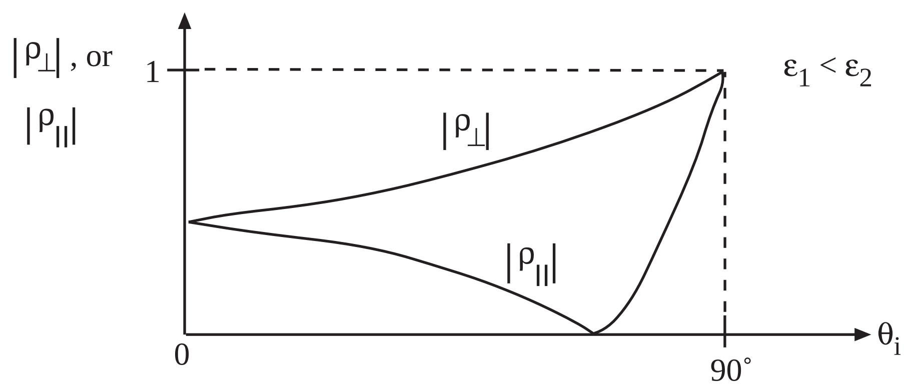
At this angle of incident, the wave will not be reflected but totally transmitted. Furthermore, we can show that

$$\sin^2\theta_{ib} + \sin^2\theta_{tb} = 1, \tag{50}$$

implying that

$$\theta_{ib} + \theta_{tb} = \frac{\pi}{2}. \tag{51}$$

On the contrary, $\rho_\perp$ can never be zero for $\mu = \mu_0$ or non-magnetic materials. Hence, a plot of $\left|\rho_\parallel\right|$ as a function of θ_i goes through a zero while the plot of $|\rho_\perp|$ is always larger than zero for non-magnetic materials.



At normal incidence, i.e., $\theta_i = 0, \rho_\perp = \rho_\parallel$ since we cannot distinguish between perpendicular and parallel polarizations. When $\theta_i = 90^\circ$, $|\rho_\perp| = \left|\rho_\parallel\right| = 1$. On the whole, $|\rho_\perp| \geq \left|\rho_\parallel\right|$ for non-magnetic materials.

The above equations are defined for lossless media. However, for lossy media, if we define a complex permittivity $\underline{\epsilon} = \epsilon - j\frac{\sigma}{\omega}$, Maxwell's equations remain unchanged. Hence, the expressions for $\rho_\perp$, $\tau_\perp$, $\rho_\parallel$, and $\tau_\parallel$ remain the same, except that we replace real permittivities with complex permittivities. For example, if medium 2 is metallic so that $\sigma \to \infty$, then, $\eta_2 = \sqrt{\frac{\mu_2}{\underline{\epsilon}_2}} \to 0$, and $\rho_\perp = -1$, and $\tau_\perp = 0$. Similarly, $\rho_\parallel = -1$ and $\tau_\parallel = 0$.

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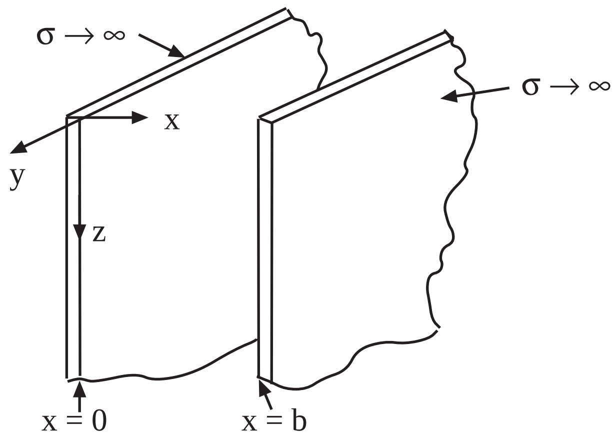
21. Infinite Parallel Plate Waveguide.



We have studied TEM (transverse electromagnetic) waves between two pieces of parallel conductors in the transmission line theory. We shall study other kinds of waves between two infinite parallel plates, or planes. We have learnt earlier that for a plane wave incident on a plane interface, the wave can be categorized into TE (transverse electric) with electric field polarized in the y-direction. Hence, between a parallel plate waveguide, we shall look for solutions of TE type with $\mathbf{E} = \hat{y}E_y$, or TM (transverse magnetic) type with $\mathbf{H} = \hat{y}H_y$. We shall assume that the field does not vary in the y-direction so that $\frac{\partial}{\partial y} = 0$.

We have shown earlier that if $\nabla \cdot \mathbf{E} = 0$, the equation for the $\mathbf{E}$ field in a source region is

$$(\nabla^2 + \omega^2 \mu\epsilon)\mathbf{E} = 0. \tag{1}$$

If $\nabla \cdot \mathbf{H} = 0$, the equation for the $\mathbf{H}$ field is

$$(\nabla^2 + \omega^2 \mu\epsilon)\mathbf{H} = 0. \tag{2}$$

Since $\frac{\partial}{\partial y} = 0$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ in these two equations.

I. TM Case, $\mathbf{H} = \hat{y}H_y$.

In this case,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu\epsilon\right) H_y = 0. \tag{3}$$

If we assume that

$$H_y = A(x)e^{-j\beta_z z}, \tag{4}$$

substituting (4) into (3), we have

$$\left[\frac{d^2}{dx^2} + \omega^2\mu\epsilon - \beta_z^2\right] A(x) = 0. \tag{5}$$

Letting $\beta_x^2 = \omega^2\mu\epsilon - \beta_z^2$, (5) becomes

$$\left[\frac{d^2}{dx^2} + \beta_x^2\right] A(x) = 0, \tag{6}$$

where the independent solutions are

$$A(x) = \left\{ \begin{array}{l} \cos\beta_x x \\ \sin\beta_x x \end{array} \right. . \tag{7}$$

Hence, H_y is of the form

$$H_y = H_0 \left\{ \begin{array}{c} \cos\beta_x x \\ \sin\beta_x x \end{array} \right\} e^{-j\beta_z z}, \tag{8}$$

where

$$\beta_x^2 + \beta_z^2 = \omega^2\mu\epsilon = \beta^2, \tag{9}$$

which are the **dispersion relation** for plane waves. We can also define $\beta_x = \beta\cos\theta$, $\beta_z = \beta\sin\theta$ so that (9) is automatically satisfied.

To decide a viable solution from (8), we look at the boundary conditions for the **E**-field at the metallic plates. From $\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$, we have

$$j\omega\epsilon E_x = \frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y, \tag{10}$$

(where $\frac{\partial}{\partial y}H_z = 0$ in the above equation) or

$$E_x = \frac{\beta_z}{\omega\epsilon}H_0 \left\{ \begin{array}{c} \cos\beta_x x \\ \sin\beta_x x \end{array} \right\} e^{-j\beta_z z}, \tag{11}$$

and

$$j\omega\epsilon E_z = \frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x, \tag{12}$$

(where $\frac{\partial}{\partial y}H_x = 0$ in the above equation) or

$$E_z = -\frac{\beta_x}{j\omega\epsilon}H_0 \left\{ \begin{array}{c} \sin\beta_x x \\ -\cos\beta_x x \end{array} \right\} e^{-j\beta_z z}. \tag{13}$$

The boundary conditions require that $E_z(x=0) = E_z(x=b) = 0$. Only the first solution gives $E_z(x=0) = 0$. Hence, we eliminate the second solution, or

$$E_z = -\frac{\beta_x}{j\omega\epsilon} H_0 \sin(\beta_x x) e^{-j\beta_z z}. \tag{14}$$

In order for $E_z(x=b) = 0$, we require that

$$\sin \beta_x b = 0, \tag{15}$$

or

$$\beta_x b = m\pi, \qquad m = 0, \pm 1, \pm 2, \pm 3, \ldots, \tag{16}$$

and consequently,

$$\beta_x = \frac{m\pi}{b}, \qquad m = 0, \pm 1, \pm 2, \pm 3, \ldots. \tag{17}$$

This is known as the **guidance condition** for the waveguide. Finally, we have

$$H_y = H_0 \cos\left(\frac{m\pi}{b} x\right) e^{-j\beta_z z}, \tag{18}$$

$$E_x = \frac{\beta_z}{\omega\epsilon} H_0 \cos\left(\frac{m\pi x}{b}\right) e^{-j\beta_z z}, \tag{19}$$

$$E_z = -\frac{m\pi}{j\omega\epsilon b} H_0 \sin\left(\frac{m\pi x}{b}\right) e^{-j\beta_z z}, \tag{20}$$

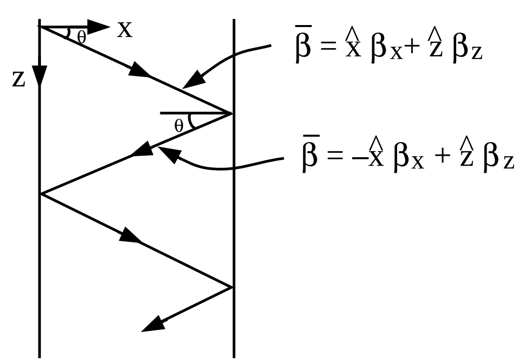
where

$$\beta_z = \left[\omega^2 \mu\epsilon - \left(\frac{m\pi}{b}\right)^2\right]^{\frac{1}{2}}, \tag{21}$$

which is the **dispersion relation** for the parallel plate waveguide. Equation (18) can be written as

$$H_y = \frac{H_0}{2}[e^{j\beta_x x} + e^{-j\beta_x x}]e^{-j\beta_z z} = \frac{H_0}{2}[e^{j\beta_x x - j\beta_z z} + e^{-j\beta_x x - j\beta_z z}]. \tag{22}$$

The first term in the above represents a plane wave propagating in the positive $\hat{z}$-direction and the negative $\hat{x}$-direction, while the second term corresponds to a wave propagating in the positive x and z directions. Hence, the field in between a parallel plate waveguide consists of a plane wave bouncing back and forth between the two plates, as shown.



Since we define $\beta_x = \beta \cos\theta$, $\beta_z = \beta \sin\theta$, the wave propagates in a direction making an angle θ with the $\hat{x}$-direction. Since the guidance condition requires that $\beta_x = \frac{m\pi}{b} = \beta \cos\theta$, the plane wave can be guided only for discrete values of θ.

From (21), we note that for different m's, β_z will assume different values. When $m = 0$, $\beta_z = \omega\sqrt{\mu\epsilon}$, $E_z = 0$, and we have a **TEM mode**. When $m > 0$, we have a **TM mode** of order m; we call it a TM_m mode. Hence, there are infinitely many solutions to Maxwell's equations between a parallel plate waveguide with the field given by (18), (19), (20), and the dispersion relation given by (21) where $m = 0, 1, 2, 3, \ldots$.

II. Cutoff Frequency

From (21), for a given TM_m mode, if $\omega\sqrt{\mu\epsilon} < \frac{m\pi}{b}$, then β_z is pure imaginary. In this case, the wave is purely decaying in the $\hat{z}$-direction, and it is **evanescent** and **non-propagating**. For a given TM_m mode, we can always lower the frequency so that this occurs. When this happens, we say that the mode is **cut off**. The **cutoff frequency** is the frequency for which a given TM_m mode becomes **cutoff** when the frequency of the TM_m mode is lower than this **cutoff frequency**. Hence,

$$\omega_{mc} = \frac{m\pi}{b\sqrt{\mu\epsilon}} \text{ or } f_{mc} = \frac{m}{2b\sqrt{\mu\epsilon}} = \frac{mv}{2b}. \tag{23}$$

When

$$\frac{(m+1)v}{2b} > f > \frac{mv}{2b} > \frac{(m-1)v}{2b} > \frac{(m-2)v}{2b} > \ldots > 0, \tag{24}$$

the TEM mode plus all the TM_n modes, where $0 < n \le m$ are **propagating** or **guided** while the TM_{m+1} and higher order modes are **evanescent** or **cutoff**. For the parallel plate waveguide, there is one mode with zero cutoff frequency and hence is guided for all frequencies. This is the **TEM mode** which is equivalent to the transmission line mode.

The wavelength that corresponds to the cutoff frequency is known as the **cutoff wavelength**, i.e.,

$$\lambda_{mc} = \frac{v}{f_{mc}} = \frac{2b}{m}. \tag{25}$$

When $\lambda < \lambda_{mc}$, the corresponding TM_m mode will be guided. You can think of λ as some kind of the "size" of the wave, and that only when the "size" of the wave is less than λ_{mc} can a wave "enter" the waveguide. Notice that λ_{mc} is proportional to the physical size of the waveguide.

IV. TE Case, $\mathbf{E} = \hat{y}E_y$.

The field for the TE case can be derived similarly to the TM case. The electric field is polarized in the $\hat{y}$-direction, and satisfies

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \mu\epsilon\right] E_y = 0. \tag{26}$$

The fields can be shown in a similar fashion to be

$$E_y = E_0 \sin(\beta_x x) e^{-j\beta_z z}, \tag{27}$$

$$H_x = -\frac{\beta_z}{\omega\mu} E_0 \sin(\beta_x x) e^{-j\beta_z z}, \tag{28}$$

$$H_z = -\frac{\beta_x}{j\omega\mu} E_0 \cos(\beta_x x) e^{-j\beta_z z}. \tag{29}$$

The boundary conditions are

$$E_y(x=0) = 0, \quad E_y(x=b) = 0. \tag{30}$$

This gives

$$\beta_x = \frac{m\pi}{b}, \tag{31}$$

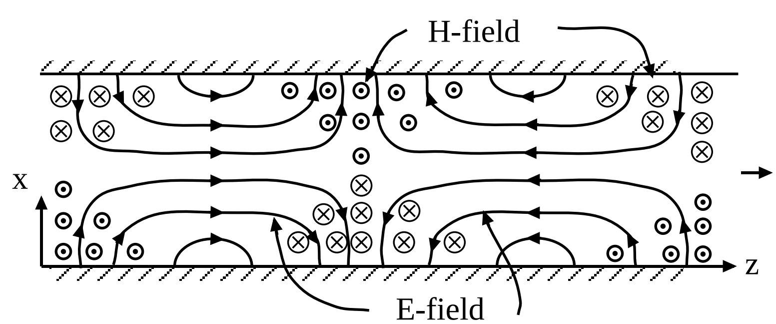
as before, where $\beta_x^2 + \beta_z^2 = \omega^2\mu\epsilon$. Hence, the TE$_m$ modes have the same dispersion relation and cut-off frequency as the TM$_m$ mode. However, when $m = 0$, $\beta_x = 0$, and (27)–(29) imply that we have zero field. Therefore, TE$_0$ mode does not exist. We say that TE$_m$ and TM$_m$ modes are **degenerate** when they have the same cutoff frequencies.

We can decompose (27) into plane waves, i.e.,

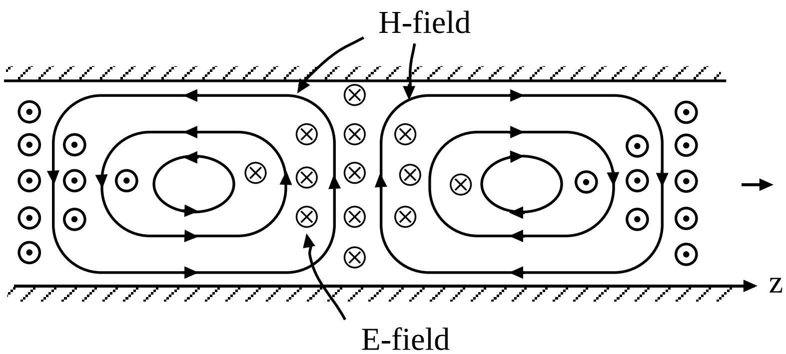
$$E_y = \frac{E_0}{2j}[e^{j\beta_x x - j\beta_z z} - e^{-j\beta_x x - j\beta_z z}], \tag{32}$$

and interpret the above as bouncing waves. Compared to (22), we see that the two bouncing waves in (32) are of the opposite signs whereas that in (22) are of the same sign. This is because the electric field has to vanish on the plates while the magnetic field need not.

TM$_1$ mode field



TE_1 mode field



The sketch of the fields for TM_1 and TE_1 modes are as shown above. For the TM mode, $H_z = 0$, and $E_z \neq 0$, while for the TE mode, $E_z = 0$, and $H_z \neq 0$. Tangential electric field is zero on the plates while tangential magnetic field is not zero on the plates. The above is the instantaneous field plots. $\mathbf{E} \times \mathbf{H}$ is in the direction of propagation of the waves.

III. Phase and Group Velocities.

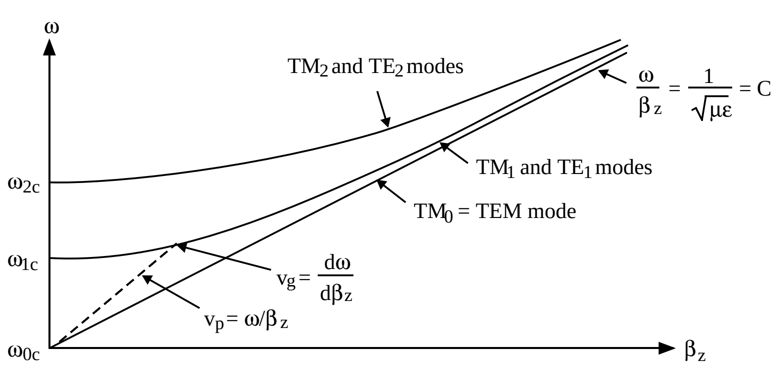
The phase velocity in the $\hat{z}$-direction of a wave in a waveguide is defined to be

$$v_p = \frac{\omega}{\beta_z} = \frac{\omega}{\left[\omega^2\mu\epsilon - \left(\frac{m\pi}{b}\right)^2\right]^{\frac{1}{2}}} = \frac{1}{\sqrt{\mu\epsilon}\left[1 - \left(\frac{f_{mc}}{f}\right)^2\right]^{\frac{1}{2}}}, \tag{33}$$

which is always **larger** than the speed of light for $f > f_{mc}$. The group velocity is

$$v_g = \frac{d\omega}{d\beta_z} = \left(\frac{d\beta_z}{d\omega}\right)^{-1} = \frac{\left[\omega^2\mu\epsilon - \left(\frac{m\pi}{b}\right)^2\right]^{\frac{1}{2}}}{\omega\mu\epsilon} = \frac{\left[1 - \left(\frac{f_{mc}}{f}\right)^2\right]^{\frac{1}{2}}}{\sqrt{\mu\epsilon}}, \tag{34}$$

which is always less than the speed of light.



Since $\beta_z = \frac{\omega}{c}\left[1 - \left(\frac{\omega_{mc}}{\omega}\right)^2\right]^{\frac{1}{2}}$, a plot of ω versus β_z is as shown. When $\beta_z \to 0$, the group velocity becomes zero while the phase velocity approaches infinity. When $\beta_z \to \infty$, or $\omega \to \infty$, the group and phase velocities both approach the velocity of light in free-space which is the TEM wave velocity.

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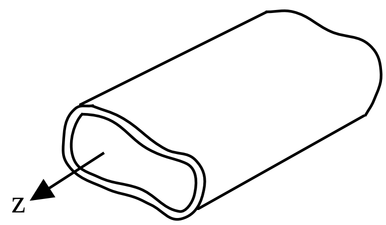
22. Hollow Waveguide.



A **hollow** cylindrical waveguide of uniform and arbitrary cross-section can guide waves. The fields inside a hollow waveguide can guide waves of both TE and TM types. When the field is of TE type, the electric field is purely transverse to the direction of wave propagation z; Hence $E_z = 0$. For TM fields, the magnetic field is purely transverse to the z-axis and hence, $H_z = 0$. Therefore, the field components of **TE fields** are

$$E_x, E_y, H_x, H_y, H_z,$$

and for **TM fields**, they are

$$H_x, H_y, E_x, E_y, E_z.$$

We can hence characterize **TE fields** as having $E_z = 0, H_z \neq 0$, and **TM fields** as $H_z = 0, E_z \neq 0$. Hence, the z-component of the **H** field can be used to characterize TE fields, while the z-component of the **E** field can be used to characterize TM fields in a hollow waveguide. Given E_z, and H_z, it will be desirable to derive the transverse components of the fields. We shall denote a vector transverse to $\hat{z}$ by a subscript s. In this notation, Maxwell's equations become

$$\left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right) \times (\mathbf{H}_s + \hat{z}H_z) = j\omega\epsilon(\mathbf{E}_s + \hat{z}E_z), \tag{1}$$

$$\left(\nabla_s + \hat{z}\frac{\partial}{\partial z}\right) \times (\mathbf{E}_s + \hat{z}E_z) = -j\omega\mu(\mathbf{H}_s + \hat{z}H_z), \tag{2}$$

where $\nabla_s = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$, and $\mathbf{E}_s$ and $\mathbf{H}_s$ are the electric field and the magnetic field, respectively, transverse to the z directon. Equating the transverse components in (1) and (2), we have

$$\nabla_s \times \hat{z}H_z + \frac{\partial}{\partial z}\hat{z} \times \mathbf{H}_s = j\omega\epsilon\mathbf{E}_s, \tag{3}$$

$$\nabla_s \times \hat{z}E_z + \frac{\partial}{\partial z}\hat{z} \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s. \tag{4}$$

Substituting (4) for $\mathbf{H}_s$ into (3), we have

$$\nabla_s \times \hat{z}H_z + \frac{\partial}{\partial z}\hat{z} \times \frac{j}{\omega\mu}\left(\nabla_s \times \hat{z}E_z + \frac{\partial}{\partial z}\hat{z} \times \mathbf{E}_s\right) = j\omega\epsilon\mathbf{E}_s. \tag{5}$$

Using the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}), \tag{6}$$

we can show that

$$\hat{z} \times \nabla_s \times \hat{z}E_z = \nabla_s(\hat{z} \cdot \hat{z}E_z) - \hat{z}E_z(\hat{z} \cdot \nabla_s) = \nabla_s E_z, \tag{7}$$

and

$$\hat{z} \times (\hat{z} \times \mathbf{E}_s) = \hat{z}(\hat{z} \cdot \mathbf{E}_s) - \mathbf{E}_s(\hat{z} \cdot \hat{z}) = -\mathbf{E}_s. \tag{8}$$

Hence, (5) becomes

$$\nabla_s \times \hat{z}H_z + \frac{j}{\omega\mu}\frac{\partial}{\partial z}\nabla_s E_z - \frac{j}{\omega\mu}\frac{\partial^2}{\partial z^2}\mathbf{E}_s = j\omega\epsilon\mathbf{E}_s. \tag{9}$$

If $\mathbf{E}$ is of the form $\mathbf{A}e^{-j\beta_z z} + \mathbf{B}e^{j\beta_z z}$, then $\frac{\partial^2}{\partial z^2} = -\beta_z^2$ and (9) becomes

$$\mathbf{E}_s = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\left[\frac{\partial}{\partial z}\nabla_s E_z - j\omega\mu\nabla_s \times \hat{z}H_z\right]. \tag{10}$$

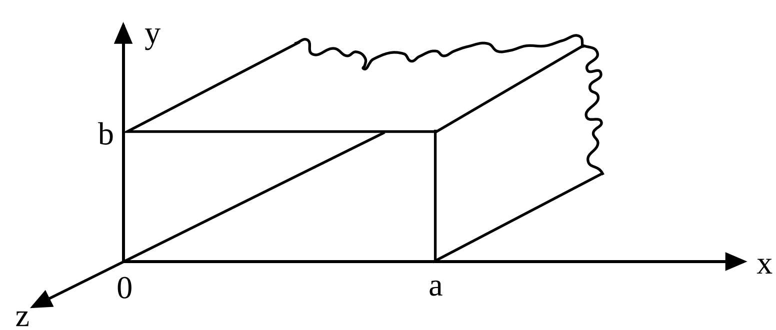
In a similar fashion, we obtain

$$\mathbf{H}_s = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\left[\frac{\partial}{\partial z}\nabla_s H_z + j\omega\epsilon\nabla_s \times \hat{z}E_z\right]. \tag{11}$$

The above equations can be used to derive the transverse components of the fields given the $\hat{z}$-components. Hence, in general, we only need to know the $\hat{z}$-components of the fields.

I. Rectangular Waveguides

Rectangular waveguides are a special case of cylindrical waveguides with uniform rectangular cross section. Hence, we can divide the waves inside the waveguide into TM and TE types.



TM Case, $H_z = 0, E_z \neq 0$

Inside the waveguide, we have a source free region, therefore

$$[\nabla^2 + \omega^2 \mu\epsilon]\mathbf{E} = 0, \tag{12}$$

or

$$[\nabla^2 + \omega^2 \mu\epsilon]E_z = 0. \tag{13}$$

Equation (13) admits solutions of the form

$$E_z = E_0 \begin{Bmatrix} \sin \beta_x x \\ \cos \beta_x x \end{Bmatrix} \begin{Bmatrix} \sin \beta_y y \\ \cos \beta_y y \end{Bmatrix} e^{-j\beta_z z}, \tag{14}$$

since

$$\frac{\partial^2}{\partial x^2} \begin{Bmatrix} \sin \beta_x x \\ \cos \beta_x x \end{Bmatrix} = \beta_x^2 \begin{Bmatrix} \sin \beta_x x \\ \cos \beta_x x \end{Bmatrix}, \tag{15}$$

$$\frac{\partial^2}{\partial y^2} \begin{Bmatrix} \sin \beta_y y \\ \cos \beta_y y \end{Bmatrix} = -\beta_y^2 \begin{Bmatrix} \sin \beta_y y \\ \cos \beta_y y \end{Bmatrix}, \quad \frac{\partial^2}{\partial z^2} e^{-j\beta_z z} = -\beta_z^2 e^{j\beta_z z}. \tag{16}$$

Therefore

$$(\nabla^2 + \omega^2 \mu\epsilon)E_z = (-\beta_x^2 - \beta_y^2 - \beta_z^2 + \omega^2 \mu\epsilon)E_z = 0. \tag{17}$$

This is only possible if

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = \omega^2 \mu\epsilon, \tag{18}$$

which is the **dispersion relation**. The boundary conditions require that

$$E_z(x = 0) = 0, \qquad E_z(y = 0) = 0. \tag{19}$$

Hence, the admissible solution is

$$E_z = E_0 \sin(\beta_x x) \sin(\beta_y y) e^{-j\beta_z z}. \tag{20}$$

Also, we require that

$$E_z(x = a) = 0, \qquad E_z(y = b) = 0. \tag{21}$$

This is only possible if $\sin(\beta_x a) = 0$ and $\sin(\beta_y b) = 0$, or

$$\beta_x a = m\pi, m = 0, 1, 2, \ldots, \qquad \beta_y b = n\pi, n = 0, 1, 2, 3, \ldots. \tag{22}$$

However, when m or $n = 0$, $E_z = 0$. Hence, we have

$$\beta_x = \frac{m\pi}{a}, \quad m \geq 1, \qquad \beta_y = \frac{n\pi}{b}, \quad n \geq 1, \tag{23}$$

which are the **guidance conditions**. To get the transverse **E** and **H** fields, we use (10) and (11)

$$E_x = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial z}\frac{\partial}{\partial x}E_z = \frac{-j\beta_x\beta_z}{\beta_x^2+\beta_y^2}E_0\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}, \tag{24}$$

$$E_y = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial z}\frac{\partial}{\partial y}E_z = \frac{-j\beta_x\beta_z}{\beta_x^2+\beta_y^2}E_0\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}, \tag{25}$$

$$H_x = \frac{j\omega\epsilon}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial y}E_z = \frac{j\omega\epsilon\beta_y}{\beta_x^2+\beta_y^2}E_0\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z}, \tag{26}$$

$$H_y = \frac{-j\omega\epsilon}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial x}E_z = \frac{-j\omega\epsilon\beta_x}{\beta_x^2+\beta_y^2}E_0\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z}. \tag{27}$$

We note that the electric fields satisfy their boundary conditions. From the dispersion relation (18), we have

$$\beta_z = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}. \tag{28}$$

The solution that corresponds to a particular choice of m and n in (23) is known as the **TM$_{mn}$ mode**. For a given TM$_{mn}$ mode, β_z will be pure imaginary if

$$\omega^2\mu\epsilon < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2, \tag{29}$$

or

$$\omega < \frac{1}{\sqrt{\mu\epsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}}. \tag{30}$$

In this case, the mode is cutoff, and the fields decay in the $\hat{z}$-direction and become purely **evanescent**. We define the cutoff frequency for the TM$_{mn}$ mode to be

$$\omega_{mnc} = \frac{1}{\sqrt{\mu\epsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}} = v\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}}. \tag{31}$$

The TM$_{mn}$ mode will not propagate if

$$\omega < \omega_{mnc} \text{ or } f < f_{mnc}, \tag{32}$$

where $f_{mnc} = \frac{\omega_{mnc}}{2\pi}$, $f = \frac{\omega}{2\pi}$. The corresponding cutoff wavelength is

$$\lambda_{mnc} = 2\pi\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{-\frac{1}{2}}. \tag{31a}$$

Only when the wavelength λ is smaller than this "size" can the wave "enter" the waveguide and be guided as the TM$_{mn}$ mode.

To find the power flowing in the waveguide, we use the Poynting theorem.

$$S_z = E_x H_y^* - E_y H_x^*, \tag{33}$$

$$= \frac{\omega\epsilon\beta_x^2\beta_z}{(\beta_x^2+\beta_y^2)^2} |E_0|^2 \cos^2(\beta_x x)\sin^2(\beta_y y) + \frac{\omega\epsilon\beta_y^2\beta_z}{(\beta_x^2+\beta_y^2)^2} |E_0|^2 \sin^2(\beta_x x)\cos^2(\beta_y y)$$

$$= \frac{\omega\epsilon\beta_z}{(\beta_x^2+\beta_y^2)^2} |E_0|^2 [\beta_x^2 \cos^2(\beta_x x)\sin^2(\beta_y y) + \beta_y^2 \sin^2(\beta_x x)\cos^2(\beta_y y)]. \tag{34}$$

The total power

$$P_z = \int_0^b dy \int_0^a dx S_z = \frac{\omega\epsilon\beta_z ab |E_0|^2}{4(\beta_x^2+\beta_y^2)^2}(\beta_x^2+\beta_y^2) = \frac{\omega\epsilon\beta_z ab |E_0|^2}{4(\beta_x^2+\beta_y^2)}. \tag{35}$$

When $f < f_{mnc}$, β_z is purely imaginary and the power becomes purely reactive. No real power or time average power flows down a waveguide when all the modes are cutoff.

TE Case, $E_z = 0, H_z \neq 0$.

In this case,

$$H_z = H_0 \cos(\beta_x x)\cos(\beta_y y) e^{-j\beta_z z}, \tag{36}$$

so that from equations (10) and (11), we have,

$$E_x = -\frac{j\omega\mu}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial y} H_z = \frac{j\omega\mu\beta_y}{\beta_x^2+\beta_y^2} H_0 \cos(\beta_x x)\sin(\beta_y y) e^{-j\beta_z z}, \tag{37}$$

$$E_y = \frac{j\omega\mu}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial x} H_z = \frac{-j\omega\mu\beta_x}{\beta_x^2+\beta_y^2} H_0 \sin(\beta_x x)\cos(\beta_y y) e^{-j\beta_z z}, \tag{38}$$

$$H_x = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial z}\frac{\partial}{\partial x} H_z = \frac{j\beta_x\beta_z}{\beta_x^2+\beta_y^2} H_0 \sin(\beta_x x)\cos(\beta_y y) e^{-j\beta_z z}, \tag{39}$$

$$H_y = \frac{1}{\omega^2\mu\epsilon - \beta_z^2}\frac{\partial}{\partial z}\frac{\partial}{\partial y} H_z = \frac{j\beta_y\beta_z}{\beta_x^2+\beta_y^2} H_0 \cos(\beta_x x)\sin(\beta_y y) e^{-j\beta_z z}, \tag{40}$$

where $\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2 = \omega^2\mu\epsilon$. Matching boundary conditions for the tangential electric field requires that

$$\beta_x = \frac{m\pi}{a}, m = 0, 1, 2, 3, \ldots, \quad \beta_y = \frac{n\pi}{b}, n = 0, 1, 2, 3, \ldots. \tag{41}$$

Unlike the TM case, the TE case can have either m or n equal to zero. Hence, TE_{m0} or TE_{0n} modes exist. However, when both m and n are zero, $H_z = H_0 e^{-j\beta_z z}$, $H_x = H_y = 0$, and $\nabla \cdot \mathbf{H} \neq 0$, therefore, TE_{00} mode cannot exist.

For the TE_{mn} modes, the subscript m is associated with the longer side of the rectangular waveguide, while n is associated with the shorter side. In the case of TE$_{m0}$ mode, $\beta_y = 0$, implying that $E_x = 0$, $E_y \neq 0$, $H_y = 0$, $H_x \neq 0$, $H_z \neq 0$. The fields resemble that of the TE$_m$ mode in a **parallel plate waveguide**. For the general TE$_{mn}$ mode, the dispersion relation is

$$\beta_z = \sqrt{\omega^2 \mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}. \tag{42}$$

Hence, the TE$_{mn}$ mode and the TM$_{mn}$ mode have the same cutoff frequency and they are **degenerate**.

Example: Designing a Waveguide to Propagate only the TE$_{10}$ mode

The cutoff frequency of a TM$_{mn}$ or a TE$_{mn}$ mode is given by

$$\omega_{mnc} = \frac{1}{\sqrt{\mu\epsilon}} \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}}. \tag{43}$$

Usually, a is assumed to be larger than b so that TE$_{10}$ mode has the lowest cutoff frequency, which is given by

$$f_{10c} = \frac{v}{2a} \text{ or } \lambda_{10c} = 2a, \tag{44}$$

where $v = \frac{1}{\sqrt{\mu\epsilon}}$, and $f_{10c} = \frac{\omega_{10c}}{2\pi}$. The next higher cutoff frequency is either f_{20c} or f_{01c} depending on the ratio of a to b.

$$f_{20c} = \frac{v}{a}, \quad f_{01c} = \frac{v}{2b}. \tag{45}$$

If $a > 2b$, $f_{20c} < f_{01c}$, and if $a < 2b$, $f_{20c} > f_{01c}$. $f_{20c} = f_{01c}$ if $a = 2b$. When $a = 2b$, and we want a waveguide to carry only the TE$_{10}$ mode between 10 GHz and 20 GHz. Therefore, we want $f_{10c} = 10$ GHz, and $f_{20c} = f_{01c} =$ 20GHz. If the waveguide is filled with air, then $v = 3 \times 10^8 \frac{m}{s}$, and we deduce that

$$a = \frac{v}{2f_{10c}} = 1.5\text{cm}, \quad b = \frac{v}{2f_{01c}} = 0.75. \tag{46}$$

In such a rectangular waveguide, only the TE$_{10}$ will propagate above 10 GHz and below 20 GHz. The other modes are all cutoff. Note that no mode could propagate below 10 GHz.

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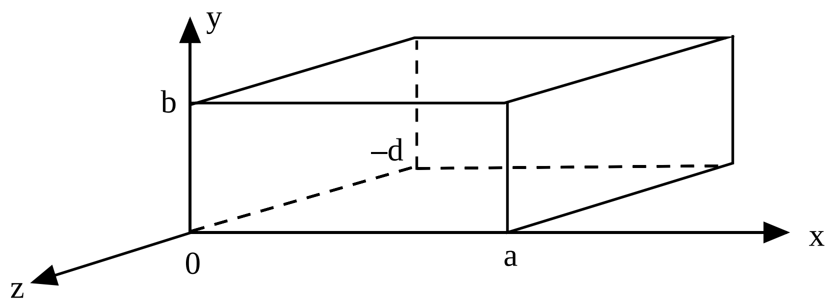
23. Cavity Resonator.



A cavity resonator is a useful microwave device. If we close off two ends of a rectangular waveguide with metallic walls, we have a rectangular cavity resonator. In this case, the wave propagating in the $\hat{z}$-direction will bounce off the two walls resulting in a standing wave in the $\hat{z}$-direction. For the **TM case**, we have

$$E_z = E_0 \sin(\beta_x x)\sin(\beta_y y)(e^{-j\beta_z z} + \rho e^{j\beta_z z}), \tag{1}$$
$$E_x = \frac{-j\beta_x\beta_z}{\beta_x^2+\beta_y^2} E_0 \cos(\beta_x x)\sin(\beta_y y)(e^{-j\beta_z z} - \rho e^{j\beta_z z}), \tag{2}$$
$$E_y = \frac{-j\beta_y\beta_z}{\beta_x^2+\beta_y^2} E_0 \sin(\beta_x x)\cos(\beta_y y)(e^{-j\beta_z z} - \rho e^{j\beta_z z}). \tag{3}$$

For the boundary conditions to be satisfied, we require that $E_x(z=0) = E_y(z=0) = 0$. Hence, $\rho = 1$, and

$$E_z = 2E_0 \sin(\beta_x x)\sin(\beta_y y)\cos(\beta_z z), \tag{4}$$
$$E_x = \frac{-2\beta_x\beta_z}{\beta_x^2+\beta_y^2} E_0 \cos(\beta_x x)\sin(\beta_y y)\sin(\beta_z z), \tag{5}$$
$$E_y = \frac{-2\beta_y\beta_z}{\beta_x^2+\beta_y^2} E_0 \sin(\beta_x x)\cos(\beta_y y)\sin(\beta_z z). \tag{6}$$

Furthermore, $E_x(z=-d) = E_y(z=-d) = 0$, implying that

$$\beta_z = \frac{p\pi}{d}, \qquad p = 0, 1, 2, 3, \ldots. \tag{7}$$

The guidance conditions for a waveguide demand that $\beta_x = \frac{m\pi}{a}$ and $\beta_y = \frac{n\pi}{b}$, where for TM case, neither m or n can be zero. Now that β_z has to satisfy (7), the TM mode in a cavity is classified as TM_{mnp} mode. We note from (4) that p can be zero while $E_z \neq 0$. Hence, the TM_{mn0} cavity mode can exist. In order for (4), (5), and (6) to be solutions to the wave equation, we require that

$$\omega^2\mu\epsilon = \beta_x^2 + \beta_y^2 + \beta_z^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2. \tag{8}$$

For a given choice of m, n, and p, only a single frequency can satisfy (8). This frequency is the **resonant frequency** of the cavity. It is only at this frequency that the cavity can sustain a free oscillation. At other frequencies, the fields interfere destructively and the free oscillation is not sustained. From (8), we gather that the resonant frequency for the TM_{mnp} mode is

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2\right]^{\frac{1}{2}}. \tag{9}$$

For the **TE case**, similar derivation shows that

$$H_z = H_0\cos(\beta_x x)\cos(\beta_y y)\sin(\beta_z z), \tag{10}$$

$$E_x = \frac{j\omega\mu\beta_y}{\beta_x^2+\beta_y^2}H_0\cos(\beta_x x)\sin(\beta_y y)\sin(\beta_z z), \tag{11}$$

$$E_y = -\frac{j\omega\mu\beta_x}{\beta_x^2+\beta_y^2}H_0\sin(\beta_x x)\cos(\beta_y y)\sin(\beta_z z). \tag{12}$$

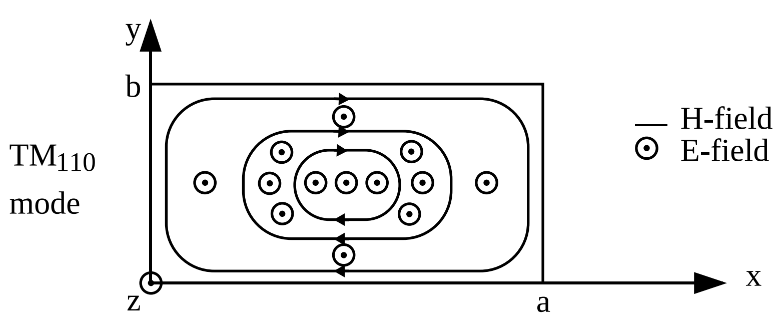
Similarly, the boundary conditions require that

$$\beta_x = \frac{m\pi}{a}, \beta_y = \frac{n\pi}{b}, \beta_z = \frac{p\pi}{d}. \tag{13}$$

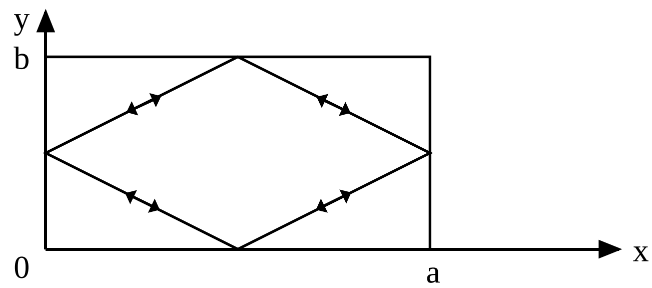
When $p = 0$, $H_z = 0$, hence TE_{mn0} mode does not exist. However, TE_{0np} or TE_{m0p} modes can exist. The resonant frequency formula is as given in (9). If $a > b > d$, the lowest resonant frequency is the TM_{110} mode. In this case,

$$\omega_{110} = \frac{1}{\sqrt{\mu\epsilon}}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]^{\frac{1}{2}}, \tag{14}$$

and $E_z \neq 0$, $H_x \neq 0$, $H_y \neq 0$, $E_x = E_y = 0$. A sketch of the field is as shown.



We can decompose the wave into plane waves bouncing off the four walls of the cavity.



As an example, for $a = 2$ cm, $b = 1$ cm, $d = 0.5$ cm, the resonant frequency of the TM_{110} mode is

$$2\pi f_{110} = 3 \times 10^8 \sqrt{\frac{5\pi^2}{4(10^{-2})^2}} = \frac{3 \times 10^8 \pi}{2 \times 10^{-2}} \sqrt{5}\text{Hz}, \tag{15}$$

or

$$f_{110} = \frac{3}{4} \times 10^{10} \times \sqrt{5}\text{Hz} = 1.68 \times 10^{10}\text{Hz} = 16.8\text{GHz}. \tag{16}$$

Cavity resonators are useful as filters and tuners in microwave circuits, as LC resonators are in RF circuits. Cavity resonators can also be used to measure the frequency of an electromagnetic signal.

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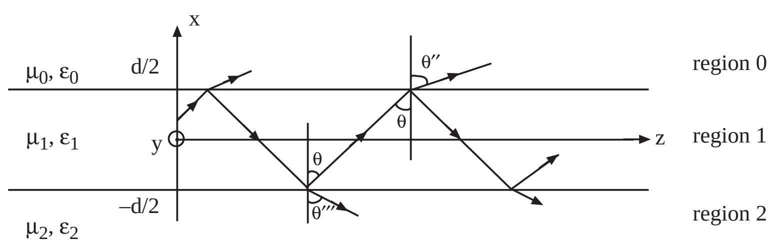
24. Dielectric Waveguides (Slab).

When a wave is incident from a medium with higher dielectric constant at an interface of two dielectric media, **total internal reflection** occurs when the angle of incident is larger than the **critical angle**. This fact can be used to make waves bouncing between two interfaces of a dielectric slab to be guided



Since total internal reflection occurs for both TE and TM waves, guidance is possible for both types of waves

I. TE Case $\mathbf{E} = \hat{y}E_y$

E_y is a solution to the wave equation in each region. In region 0, we assume a solution of the form

$$E_{0y} = E_0 e^{-j\beta_{0x}x - j\beta_z z}, \tag{1}$$

where

$$\beta_{0x}^2 + \beta_z^2 = \omega^2 \mu_0 \epsilon_0 = \beta_0^2. \tag{1a}$$

In region 1, we assume a solution of the form

$$E_{1y} = [A_1 e^{-j\beta_{1x}x} + B_1 e^{j\beta_{1x}x}]e^{-j\beta_z z}, \tag{2}$$

where

$$\beta_{1x}^2 + \beta_z^2 = \omega^2 \mu_1 \epsilon_1 = \beta_1^2. \tag{2a}$$

In region 2, the solution is of the form

$$E_{2y} = E_2 e^{j\beta_{2x}x - j\beta_z z}, \tag{3}$$

where

$$\beta_{2x}^2 + \beta_z^2 = \omega^2 \mu_2 \epsilon_2 = \beta_2^2. \tag{3a}$$

We assume that all the solutions in the three regions to have the same z-variation of $e^{-j\beta_z z}$ by the **phase matching** condition.

In region 1, we have an up-going wave as well as a down-going wave. The two waves have to be related by the reflection coefficient $\rho_\perp$ for the electric field at the boundaries. $\rho_\perp$ is derived earlier in the course. Therefore at $x = \frac{d}{2}$, we have

$$B_1 e^{j\beta_{1x}\frac{d}{2}} = \rho_{10\perp} A_1 e^{-j\beta_{1x}\frac{d}{2}}, \tag{4}$$

where $\rho_{10\perp}$ is the reflection coefficient at the regions 1 and 0 interface. At $x = -\frac{d}{2}$, we have

$$A_1 e^{j\beta_{1x}\frac{d}{2}} = \rho_{12\perp} B_1 e^{-j\beta_{1x}\frac{d}{2}}, \tag{5}$$

where $\rho_{12\perp}$ is the reflection coefficient at the regions 1 and 2 interface. Multiplying equations (4) and (5) together, we have,

$$A_1 B_1 e^{j\beta_{1x}d} = \rho_{12\perp}\rho_{10\perp} A_1 B_1 e^{-j\beta_{1x}d}. \tag{6}$$

A_1 and B_1 are non-zero only if

$$1 = \rho_{12\perp}\rho_{10\perp} e^{-2j\beta_{1x}d}. \tag{7}$$

The above is known as the **guidance condition** of a dielectric slab waveguide. If medium 3 is equal to medium 1, then $\rho_{12\perp} = \rho_{10\perp}$, and the guidance condition becomes

$$1 = \rho_{10\perp}^2 e^{-2j\beta_{1x}d}. \tag{8}$$

From before, for a wave incident at an angle θ,

$$\rho_{10\perp} = \frac{\eta_0 \cos\theta - \eta_1 \cos\theta''}{\eta_0 \cos\theta + \eta_1 \cos\theta''}. \tag{9}$$

Since $\beta_{1x} = \beta_1 \cos\theta$, $\beta_{0x} = \beta_0 \cos\theta''$, (9) could be written as

$$\rho_{10\perp} = \frac{\frac{\eta_0}{\beta_1}\beta_{1x} - \frac{\eta_1}{\beta_0}\beta_{0x}}{\frac{\eta_0}{\beta_1}\beta_{1x} + \frac{\eta_1}{\beta_0}\beta_{0x}} = \frac{\mu_0\beta_{1x} - \mu_1\beta_{0x}}{\mu_0\beta_{1x} + \mu_1\beta_{0x}}. \tag{10}$$

Taking the square root of (8), we have

$$\rho_{10\perp} e^{-j\beta_{1x}d} = \pm 1. \tag{11}$$

When we choose the plus sign, $B_1 = A_1$ from (4), and from (2)

$$E_{1y} = 2A_1 \cos(\beta_{1x}x) e^{-j\beta_z z} \qquad \Rightarrow \text{even in x.} \tag{12}$$

When we choose the minus sign in (11) we have $B_1 = -A_1$, and

$$E_{1y} = -2jA_1 \sin(\beta_{1x}x) e^{-j\beta_z z} \qquad \Rightarrow \text{odd in x.} \tag{13}$$

Multiplying (11) by $e^{j\beta_{1x}\frac{d}{2}}$ and manipulating, we have

$$\frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\tan\left(\beta_{1x}\frac{d}{2}\right) = j\beta_{0x}\frac{d}{2} \qquad \text{even solutions,} \tag{14}$$

$$\frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\cot\left(\beta_{1x}\frac{d}{2}\right) = j\beta_{0x}\frac{d}{2} \qquad \text{odd solutions.} \tag{15}$$

Subtracting (1a) from (2a) and solving for β_{0x}, we have

$$\beta_{0x} = [\omega^2(\mu_0\epsilon_0 - \mu_1\epsilon_1) + \beta_{1x}^2]^{\frac{1}{2}}. \tag{16}$$

In order for (14) and (15) to be satisfied, β_{0x} has to be pure imaginary. In other words, the waves in region 0 and 3 have to be evanescent and decay exponentially away from the slab. Hence

$$\beta_{0x} = -j\alpha_{0x} = -j[\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0) - \beta_{1x}^2]^{\frac{1}{2}}, \tag{17}$$

and (14) and (15) become

$$\frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\tan\beta_{1x}\frac{d}{2} = \alpha_{0x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2} \ \text{even solutions,} \tag{18}$$

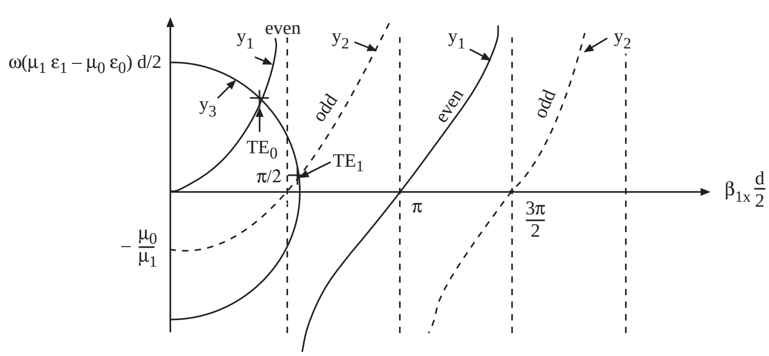
$$-\frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\cot\beta_{1x}\frac{d}{2} = \alpha_{0x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2} \ \text{odd solutions.} \tag{19}$$

We can solve the above graphically by plotting

$$y_1 = \frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\tan\left(\beta_{1x}\frac{d}{2}\right) \qquad \text{even solutions,} \tag{20}$$

$$y_2 = -\frac{\mu_0}{\mu_1}\beta_{1x}\frac{d}{2}\cot\left(\beta_{1x}\frac{d}{2}\right) \qquad \text{odd solutions,} \tag{21}$$

$$y_3 = \left[\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2\right]^{\frac{1}{2}} = \alpha_{0x}\frac{d}{2}. \tag{22}$$



y_3 is the equation of a circle; the radius of the circle is given by

$$\omega(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}\frac{d}{2}. \tag{23}$$

The solutions to (18) and (19) are given by the intersections of y_3 with y_1 and y_2. We note from (23) that the radius of the circle can be increased in three ways; (i) by increasing the frequency, (ii) by increasing the contrast $\frac{\mu_1\epsilon_1}{\mu_0\epsilon_0}$, and (iii) by increasing the thickness d of the slab.

When $\beta_{0x} = -j\alpha_{0x}$, the reflection coefficient is

$$\rho_{10\perp} = \frac{\mu_0\beta_{1x} + j\mu_1\alpha_{0x}}{\mu_0\beta_{1x} - j\mu_1\alpha_{0x}} = \exp\left[+2j\tan^{-1}\left(\frac{\mu_1\alpha_{0x}}{\mu_0\beta_{1x}}\right)\right], \tag{24}$$

and $|\rho_{10\perp}| = 1$. Hence there is total internal reflections and the wave is guided by total internal reflections. **Cut-off occurs** when the total internal reflection ceases to occur, i.e. when the frequency decreases such that $\alpha_{0x} = 0$. From the diagram, we see that $\alpha_{0x} = 0$ when

$$\omega(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}\frac{d}{2} = \frac{m\pi}{2}, \qquad m = 0, 1, 2, 3, \ldots, \tag{25}$$

or

$$\omega_{mc} = \frac{m\pi}{d(\mu_1\epsilon_1 - \mu_0\epsilon_0)^{\frac{1}{2}}}, \qquad m = 0, 1, 2, 3, \ldots. \tag{26}$$

The mode that corresponds to the m-th cut-off frequency above is labeled the TE$_m$ mode. TE$_0$ mode is the mode that has no cut-off or propagates at all frequencies.

At cut-off, $\alpha_{0x} = 0$, and from (1a),

$$\beta_z = \omega\sqrt{\mu_0\epsilon_0}, \tag{27}$$

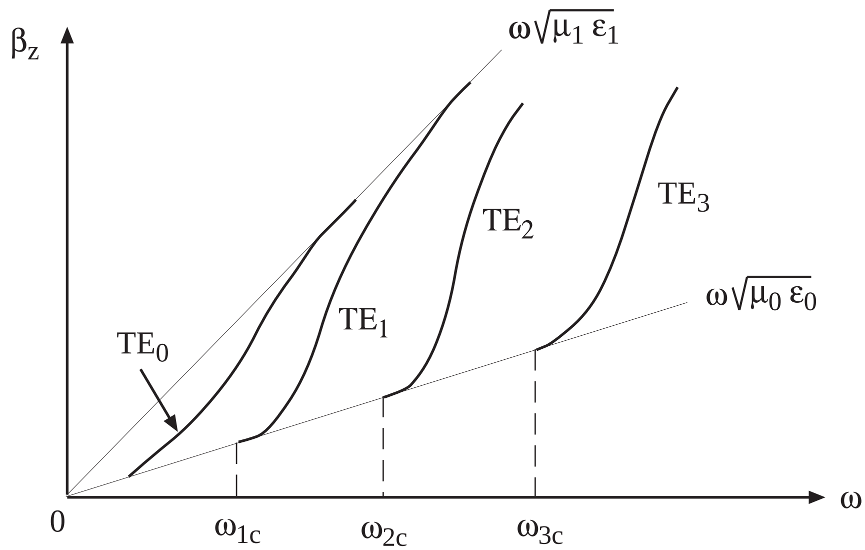
for all the modes. Hence, both the group and the phase velocities are that of the outer region. This is because when $\alpha_{0x} = 0$, the wave is not evanescent outside, and most of the energy of the mode is carried by the exterior field.

When $\omega \to \infty$, $\beta_{1x} \to \frac{n\pi}{d}$ from the diagram for all the modes. From (2a),

$$\beta_z = \sqrt{\omega^2\mu_1\epsilon_1 - \beta_{1x}^2} \approx \omega\sqrt{\mu_1\epsilon_1}, \qquad \omega \to \infty. \tag{28}$$

Hence the group and phase velocities approach that of the dielectric slab. This is because when $\omega \to \infty$, $\alpha_{0x} \to \infty$, and all the fields are trapped in the slab and propagating within it.

Because of this, the dispersion diagram of the different modes appear as below.



II. TM Case $\mathbf{H} = \hat{y}H_y$

For the TM case, a similar guidance condition analogous to (27) can be derived

$$1 = \rho_{12\|}\rho_{10\|}e^{-2j\beta_{1x}d}, \tag{29}$$

where ρ is the reflection coefficient for the TM field. Similar derivations show that the above guidance condition, for $\epsilon_2 = \epsilon_0$, $\mu_2 = \mu_0$, reduces to

$$\frac{\epsilon_0}{\epsilon_1}\beta_{1x}\frac{d}{2}\tan\beta_{1x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2} \qquad \text{even solution,} \tag{30}$$

$$-\frac{\epsilon_0}{\epsilon_1}\beta_{1x}\frac{d}{2}\cot\beta_{1x}\frac{d}{2} = \sqrt{\omega^2(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{d^2}{4} - \left(\beta_{1x}\frac{d}{2}\right)^2} \qquad \text{odd solution.} \tag{31}$$

Note that for equations (7) and (29), when we have two parallel metallic plates, $\rho_{\|} = 1$, and $\rho_{\perp} = \pm 1$, and the guidance condition becomes

$$1 = e^{-2j\beta_{1x}d} \quad \Rightarrow \beta_{1x} = \frac{m\pi}{d}, m = 0, 1, 2, \ldots, \tag{32}$$

which is what we have observed before.

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25. Vector Potential - Introduction to Antennas & Radiations

Maxwell's equations are

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}, \tag{1}$$
$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{J}, \tag{2}$$
$$\nabla \cdot \mu\mathbf{H} = 0, \tag{3}$$
$$\nabla \cdot \epsilon\mathbf{E} = \rho. \tag{4}$$

Since $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, we can let

$$\mu\mathbf{H} = \nabla \times \mathbf{A}, \tag{5}$$

so that equation (3) is automatically satisfied. Substituting (5) into (1), we have

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0. \tag{6}$$

Since $\nabla \times \nabla\phi = 0$, we have

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\phi. \tag{7}$$

Hence, knowing $\mathbf{A}$ and ϕ uniquely determines $\mathbf{E}$ and $\mathbf{H}$. We shall relate $\mathbf{A}$ and ϕ to the **sources** $\mathbf{J}$ and ρ of Maxwell's equations. Substituting (5) and (7) into (2), we have

$$\nabla \times \nabla \times \mathbf{A} = j\omega\mu\epsilon[-j\omega\mathbf{A} - \nabla\phi] + \mu\mathbf{J}, \tag{8}$$

or

$$\nabla^2\mathbf{A} + \omega^2\mu\epsilon\mathbf{A} = -\mu\mathbf{J} + j\omega\mu\epsilon\nabla\phi + \nabla\nabla \cdot \mathbf{A}. \tag{9}$$

Using (7) in (4), we have

$$\nabla \cdot (j\omega\mathbf{A} + \nabla\phi) = -\frac{\rho}{\epsilon}. \tag{10}$$

The above could be simplified for the following observation. Equations (5) and (7) give the same $\mathbf{E}$ and $\mathbf{H}$ fields under the transformation

$$\mathbf{A}' = \mathbf{A} + \nabla\psi, \tag{11}$$
$$\phi' = \phi - j\omega\psi. \tag{12}$$

The above are known as the **Gauge Transformation**. With the new $\mathbf{A}'$ and ϕ', we can substitute into (5) and (7) and they give the same $\mathbf{E}$ and $\mathbf{H}$ fields, i.e.

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times \nabla\psi = \nabla \times \mathbf{A} = \mu\mathbf{H}, \tag{13}$$
$$-j\omega\mathbf{A}' - \nabla\phi' = -j\omega\mathbf{A} - j\omega\nabla\psi - \nabla\phi + j\omega\nabla\psi = \mathbf{E}. \tag{14}$$

It implies that $\mathbf{A}$ and ϕ are not unique. The vector field $\mathbf{A}$ is not unique unless we specify both its curl and its divergence. Hence, in order to make $\mathbf{A}$ unique, we have to specify its divergence. If we specify the divergence of $\mathbf{A}$ such that

$$\nabla \cdot \mathbf{A} = -j\omega\mu\epsilon\phi, \tag{15}$$

then (9) and (10) become

$$\nabla^2\mathbf{A} + \omega^2\mu\epsilon = -\mu\mathbf{J}, \tag{16}$$

$$\nabla^2\phi + \omega^2\mu\epsilon\phi = -\frac{\rho}{\epsilon}. \tag{17}$$

The condition in (15) is also known as the **Lorentz gauge**. Equations (16) and (17) represent a set of four inhomogeneous wave equations driven by the sources of Maxwell's equations. Hence given the sources ρ and $\mathbf{J}$, we may find $\mathbf{A}$ and ϕ. $\mathbf{E}$ and $\mathbf{H}$ may in turn be found using (5) and (7). However, as a consequence of the Lorentz gauge, we need only to find $\mathbf{A}$; ϕ follows directly from equation (15).

Let us consider the relation due to an elemental current that can be described by

$$\mathbf{J} = \hat{z}Il\delta(\mathbf{r}) \quad A/m^2, \tag{18}$$

where Il denotes the strength of this current, and $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$. Equation (16) becomes

$$\nabla^2 A_z + \omega^2\mu\epsilon A_z = -\mu Il\delta(\mathbf{r}). \tag{19}$$

Taking advantage of the spherical symmetry of the problem, ∇^2 has only r dependence in spherical coordinates, we have

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}A_z + \beta^2 A_z = -\mu Il\delta(\mathbf{r}), \tag{20}$$

where $\beta^2 = \omega^2\mu\epsilon$. Equations (19) and (20) are similar in form to Poisson's equation with a point charge Q at the origin,

$$\nabla^2\phi = -\frac{Q}{\epsilon}\delta(\mathbf{r}). \tag{21}$$

We know that (21) has the solution of the form

$$\phi = \frac{Q}{4\pi\epsilon r}. \tag{22}$$

Hence, we guess that the solution to (20) is of the form

$$A_z = \frac{\mu Il}{4\pi r}C(r). \tag{23}$$

It can be shown that

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}f(r) = \frac{1}{r}\frac{d^2}{dr^2}rf(r). \tag{24}$$

Outside the origin, the RHS of (20) is zero, and after using (23) and (24) in (20), we have

$$\frac{d^2}{dr^2}C(r) + \beta^2 C(r) = 0. \tag{25}$$

This gives

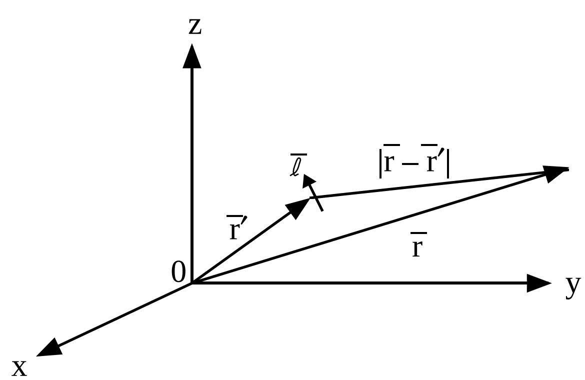
$$C(r) = e^{\pm j\beta r}. \tag{26}$$

Since we are looking for a solution that radiates energy to infinity, we choose an outgoing solution in (26). Hence,

$$A_z(r) = \frac{\mu I l}{4\pi r} e^{-j\beta r}, \tag{27}$$

for a source directed at a $\hat{z}$-direction. From (16), we note that $\mathbf{A}$ and $\mathbf{J}$ always point in the same direction. Therefore, for a point source directed at $\mathbf{l}$ and located at $\mathbf{r}'$ instead of the origin, the vector potential $\mathbf{A}$ is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu I \mathbf{l}}{4\pi |\mathbf{r} - \mathbf{r}'|} e^{-j\beta|\mathbf{r}-\mathbf{r}'|}. \tag{28}$$



By linear superposition, the vector potential due to an arbitrary source $\mathbf{J}$ is

$$\mathbf{A} = \frac{\mu}{4\pi} \iiint d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} e^{-j\beta|\mathbf{r}-\mathbf{r}'|}. \tag{29}$$

Similarly, we can show that

$$\phi = \frac{1}{4\pi\epsilon} \iiint d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} e^{-j\beta|\mathbf{r}-\mathbf{r}'|}. \tag{30}$$

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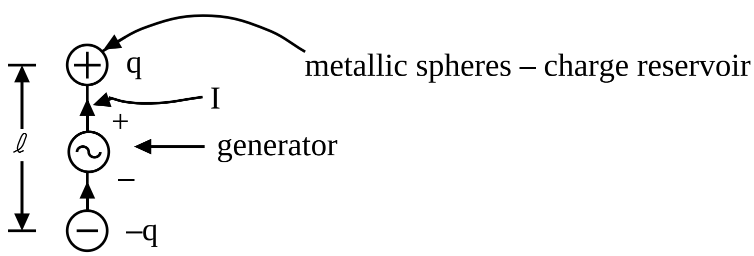
26. The Fields of a Hertzian Dipole

A Hertzian dipole is a dipole which is much smaller than the wavelength under construction so that we can approximate it by a point current distribution,

$$\mathbf{J}(\mathbf{r}) = \hat{z} I l \delta(\mathbf{r}). \tag{1}$$

The dipole may look like the following



l is the effective length of the dipole so that the dipole moment $p = ql$. The charge q is varying time harmonically because it is driven by the generator. Since $\frac{dq}{dt} = I$, we have

$$Il = \frac{dq}{dt} l = j\omega q l = j\omega p, \tag{2}$$

for a Hertzian dipole. We already know that the corresponding vector potential is given by

$$\mathbf{A}(\mathbf{r}) = \hat{z} \frac{\mu I l}{4\pi r} e^{-j\beta r}. \tag{3}$$

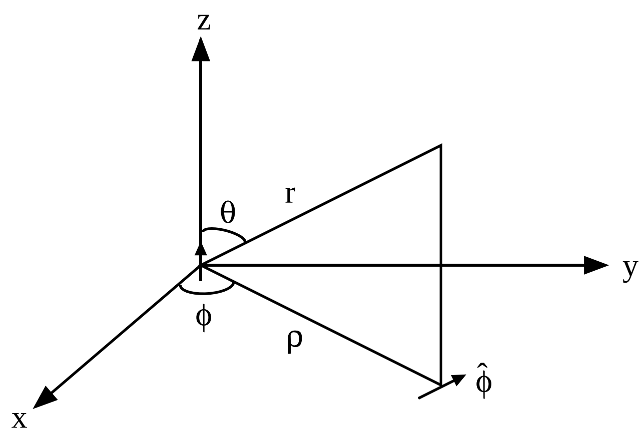
The magnetic field is obtained, using cylindrical coordinates, as

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} = \frac{1}{\mu} \left(\hat{\rho} \frac{1}{\rho} \frac{\partial}{\partial \phi} A_z - \hat{\phi} \frac{\partial}{\partial \rho} A_z \right), \tag{4}$$

where $\frac{\partial}{\partial \phi} = 0$, $r = \sqrt{\rho^2 + z^2}$. In the above, $\frac{\partial}{\partial \rho} = \frac{\partial r}{\partial \rho} \frac{\partial}{\partial r} = \frac{\rho}{\sqrt{\rho^2+z^2}} \frac{\partial}{\partial r} = \frac{\rho}{r} \frac{\partial}{\partial r}$.

Hence,

$$\mathbf{H} = -\hat{\phi} \frac{\rho}{r} \frac{Il}{4\pi} \left(-\frac{1}{r^2} - j\beta \frac{1}{r} \right) e^{-j\beta r}. \tag{5}$$



In spherical coordinates, $\frac{\rho}{r} = \sin\theta$, and (5) becomes

$$\mathbf{H} = \hat{\phi}\frac{Il}{4\pi r^2}(1 + j\beta r)e^{-j\beta r}\sin\theta. \tag{6}$$

The electric field can be derived using Maxwell's equations.

$$\begin{aligned}\mathbf{E} &= \frac{1}{j\omega\epsilon}\nabla\times\mathbf{H} = \frac{1}{j\omega\epsilon}\left(\hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta H_\phi - \hat{\phi}\frac{1}{r}\frac{\partial}{\partial r}rH_\phi\right)\\ &= \frac{Ile^{-j\beta r}}{j\omega\epsilon 4\pi r^3}\left[\hat{r}2\cos\theta(1 + j\beta r) + \hat{\theta}\sin\theta(1 + j\beta r - \beta^2 r^2)\right]. \end{aligned}\tag{7}$$

Case I. Near Field, $\beta r \ll 1$

$$\mathbf{E} \cong \frac{\rho}{4\pi\epsilon r^3}(\hat{r}2\cos\theta + \hat{\theta}\sin\theta), \qquad \beta r \ll 1, \tag{8}$$

$$\mathbf{H} \ll \mathbf{E}, \qquad \text{when } \beta r \ll 1. \tag{9}$$

βr could be made very small by making $\frac{r}{\lambda}$ small or by making $\omega \to 0$. The above is like the static field of a dipole.

Case II. Far Field (Radiation Field), $\beta r \gg 1$

In this case,

$$\mathbf{E} \cong \hat{\theta}j\omega\mu\frac{Il}{4\pi r}e^{-j\beta r}\sin\theta, \tag{10}$$

and

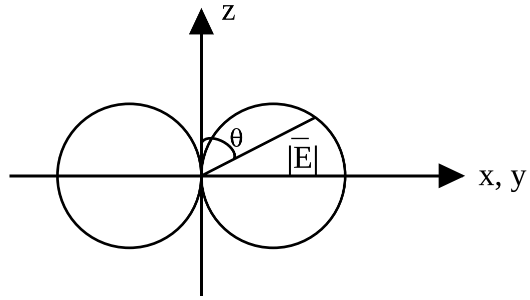
$$\mathbf{H} \cong \hat{\phi}j\beta\frac{Il}{4\pi r}e^{-j\beta r}\sin\theta. \tag{11}$$

Note that $\frac{E_\theta}{H_\phi} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\epsilon}} = \eta_0$. **E** and **H** are orthogonal to each other and are both orthogonal to the direction of propagation, i.e. as in the case of a plane wave. A spherical wave resembles a plane wave in the far field approximation.

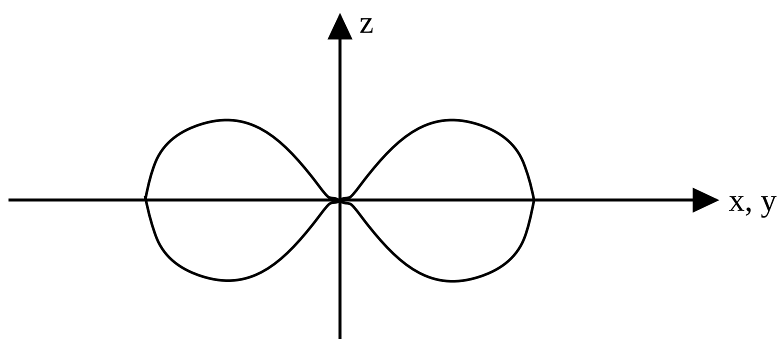
The time average power flow is given by

$$\langle \mathbf{S} \rangle = \frac{1}{2} \Re e[\mathbf{E} \times \mathbf{H}^*] = \hat{r} \frac{1}{2} \eta_0 \left| H_\phi \right|^2 = \hat{r} \frac{\eta_0}{2} \left(\frac{\beta I l}{4\pi r} \right)^2 \sin^2 \theta. \tag{12}$$

The **radiation field pattern** of a Hertzian dipole is the plot of $|\mathbf{E}|$ as a function of θ at a constant $\mathbf{r}$.



The **radiation power pattern** is the plot of $\langle S_r \rangle$ at a constant r.



The total power radiated by a Hertzian dipole is given by

$$P = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta r^2 \sin\theta \langle S_r \rangle = 2\pi \int_0^{\pi} d\theta \frac{\eta_0}{2} \left(\frac{\beta I l}{4\pi} \right)^2 \sin^3 \theta. \tag{13}$$

Since

$$\int_0^{\pi} d\theta \sin^3 \theta = -\int_1^{-1} (d \cos\theta)[1 - \cos^2 \theta] = \int_{-1}^{1} dx (1 - x^2) = \frac{4}{3}, \tag{14}$$

then

$$P = \frac{4}{3} \pi \eta_0 \left(\frac{\beta I l}{4\pi} \right)^2. \tag{15}$$

The **directive gain** of an antenna, $D(\theta, \phi)$, is defined as

$$D(\theta, \phi) = \frac{\langle S_r \rangle}{\frac{P}{4\pi r^2}}, \tag{16}$$

where $\frac{P}{4\pi r^2}$ is the power density if the power P were uniformly distributed over a sphere. Substituting (12) and (15) into the above, we have

$$D(\theta, \phi) = \frac{\frac{\eta_0}{2}\left(\frac{\beta I l}{4\pi r}\right)^2 \sin^2\theta}{\frac{1}{4\pi r^2}\frac{4}{3}\eta_0 \pi \left(\frac{\beta I l}{4\pi}\right)^2} = \frac{3}{2}\sin^2\theta. \tag{17}$$

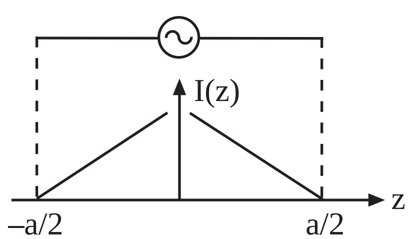
The peak of $D(\theta, \phi)$ is known as the **directivity** of an antenna. It is 1.5 in this case. If an antenna is radiating isotropically, its directivity is 1. Therefore, the lowest possible values for the directivity of an antenna is 1, whereas it can be over 100 for some antennas like reflector antennas. A **directive gain pattern** is a plot of the above function $D(\theta, \phi)$ and it resembles the radiation power pattern.

If the total power fed into the antenna instead of the total radiated power is used in the denominator of (16), the ratio is known as the **power gain** or just bf gain. The total power fed into the antenna is not equal to the total radiated power because there could be some loss in the antenna system like metallic loss.

Defining a **radiation resistance** R_r by $P = \frac{1}{2}I^2 R_r$, we have

$$R_r = \frac{2P}{I^2} = \eta_0 \left(\frac{\beta l}{6\pi}\right)^2, \qquad \text{where } \eta_0 = 377\Omega. \tag{18}$$

For example, for a Hertzian dipole with $l = 0.1\lambda$, $R_r \approx 8\Omega$. For a small dipole with no charge reservoir at the two ends, the currents have to vanish at the tip of the dipole.



The effective length of the dipole is **half** of its actual length due to the manner the currents are distributed. For example, for a half-wave dipole, $a = \frac{\lambda}{2}$, and if we use $l_{\text{eff}} = \frac{\lambda}{4}$ in (18), we have

$$R_r \approx 50\Omega. \tag{19}$$

However, a half-wave dipole is not much smaller than a wavelength and does not qualify to be a Hertzian dipole. Furthermore, the current distribution on the half-wave dipole is not triangular in shape as above. A more precise calculation shows that $R_r = 73\Omega$ for a half-wave dipole.

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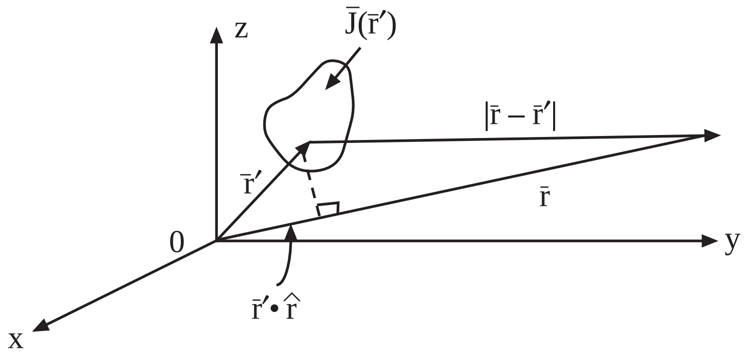
27. Radiation Field Approximations

The vector potential due to a source $\mathbf{J}(\mathbf{r})$, can be calculated from the equation

$$\mathbf{A}(\mathbf{r}) = \iiint_V d\mathbf{r}' \frac{\mu \mathbf{J}(\mathbf{r}')}{4\pi |\mathbf{r} - \mathbf{r}'|} e^{-j\beta|\mathbf{r}-\mathbf{r}'|}, \tag{1}$$

where V is the volume occupied by $\mathbf{J}(\mathbf{r})$.



When $|\mathbf{r}| \gg |\mathbf{r}'|$, then $|\mathbf{r} - \mathbf{r}'| = r - \mathbf{r}' \cdot \hat{r}$. Equation (1) becomes

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &\cong \iiint_V d\mathbf{r}' \frac{\mu \mathbf{J}(\mathbf{r}')}{r - \mathbf{r}' \cdot \hat{r}} e^{-j\beta r} e^{j\beta \mathbf{r}' \cdot \hat{r}} \\ &= \frac{\mu e^{-j\beta r}}{4\pi r} \iiint_V d\mathbf{r}' \mathbf{J}(\mathbf{r}') e^{j\beta \mathbf{r}' \cdot \hat{r}} \\ &= e^{-j\beta r} \frac{\mathbf{f}(\theta, \phi)}{r} = \hat{\theta} A_\theta + \hat{\phi} A_\phi + \hat{r} A_r. \end{aligned} \tag{2}$$

In the above we have assumed that $|\mathbf{r}' \cdot \hat{r}| \ll r$ but $\beta \mathbf{r}' \cdot \hat{r}$ is not small, since β can be large. When βr is large, $\frac{f(\theta,\phi)}{r}$ is a slowly varying function compared to $e^{-j\beta r}$. Hence, we can regard $\frac{\mathbf{f}(\theta,\phi)}{r}$ almost to be a constant compared to $e^{-j\beta r}$. The magnetic field can be derived to be

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \approx -\frac{1}{\mu} \left[\hat{\theta} \frac{\partial}{\partial r} A_\phi - \hat{\phi} \frac{\partial}{\partial r} A_\theta \right]. \tag{3}$$

However, $\frac{\partial}{\partial r} \sim -j\beta$ when βr is large. Hence,

$$\mathbf{H} = \frac{j\beta}{\mu} (\hat{\theta} A_\phi - \hat{\phi} A_\theta), \qquad \text{when } \beta r \to \infty. \tag{4}$$

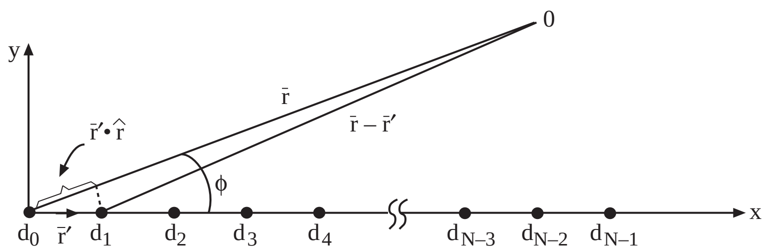
Similarly,

$$\mathbf{E} = \frac{1}{j\omega\epsilon}\nabla \times \mathbf{H} \cong -j\omega[\hat{\theta}A_\theta + \hat{\phi}A_\phi]. \tag{5}$$

Linear Array of Dipole Antennas

If $\mathbf{J}(\mathbf{r}')$ is of the form

$$\begin{aligned}\mathbf{J}(\mathbf{r}') = \hat{z}Il[&A_0\delta(x') + A_1\delta(x'-d_1) + A_2\delta(x'-d_2) \\ &+ \cdots + A_{N-1}\delta(x'-d_{N-1})]\delta(y')\delta(z'),\end{aligned} \tag{6}$$



the vector potential on the xy-plane can be derived to be

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \hat{z}\frac{\mu Il}{4\pi r}e^{j\beta r}\iiint d\mathbf{r}'[A_0\delta(x') + A_1\delta(x'-d_1) + \cdots]\delta(y')\delta(z')e^{+j\beta\mathbf{r}'\cdot\hat{r}} \\ &= \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}\left[A_0 + A_1 e^{+j\beta d_1\cos\phi} + A_2 e^{j\beta d_2\cos\phi} + \cdots + A_{N-1}e^{j\beta d_{N-1}\cos\phi}\right].\end{aligned} \tag{7}$$

If $d_n = nd$, and $A_n = e^{jn\psi}$, then (7) becomes

$$\mathbf{A}(\mathbf{r}) = \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}\left[1 + e^{j(\beta d\cos\phi+\psi)} + e^{2j(\beta d\cos\phi+\psi)} + \cdots + e^{j(N-1)(\beta d\cos\phi+\psi)}\right], \tag{8}$$

which is of the form

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}. \tag{9}$$

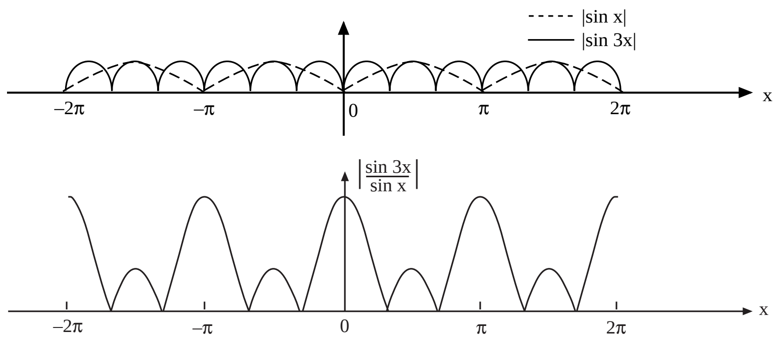
Therefore,

$$\mathbf{A}(\mathbf{r}) = \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}\frac{1-e^{jN(\beta d\cos\phi+\psi)}}{1-e^{j(\beta d\cos\phi+\psi)}}. \tag{10}$$

The electric field on the xy-plane is $E_\theta = -j\omega A_\theta = +j\omega A_z$. Hence, $|E_\theta|$ is of the form

$$\begin{aligned}|E_\theta| &= |E_0|\left|\frac{1-e^{jN(\beta d\cos\phi+\psi)}}{1-e^{j(\beta d\cos\phi+\psi)}}\right| \\ &= |E_0|\left|\frac{\sin\frac{N}{2}(\beta d\cos\phi+\psi)}{\sin\frac{1}{2}(\beta d\cos\phi+\psi)}\right|.\end{aligned} \tag{11}$$

Equation (11) is of the form $\frac{|\sin Nx|}{|\sin x|}$. Plots of $|\sin 3x|$ and $|\sin x|$ are shown as an example.



In equation (11), $\lambda = \frac{1}{2}(\beta d \cos\phi + \psi)$. We notice that the **maximum** in (11) would occur if $\lambda = n\pi$, or if

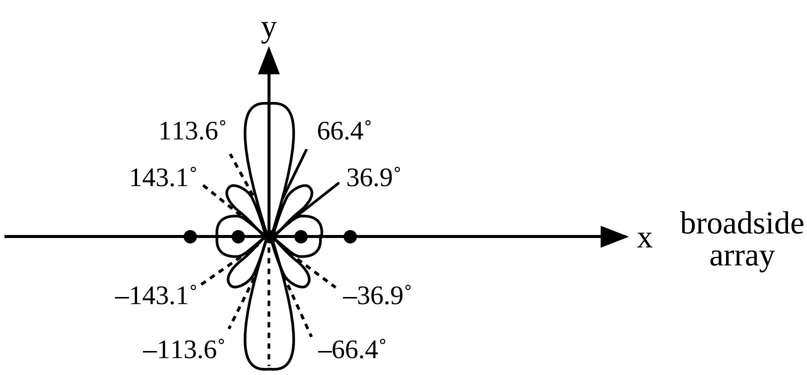
$$\beta d \cos\phi + \psi = 2n\pi, \qquad n = 0, \pm 1, \pm 2, \pm 3, \cdots. \tag{12}$$

The **zeros** or **nulls** will occur at $Nx = n\pi$, or

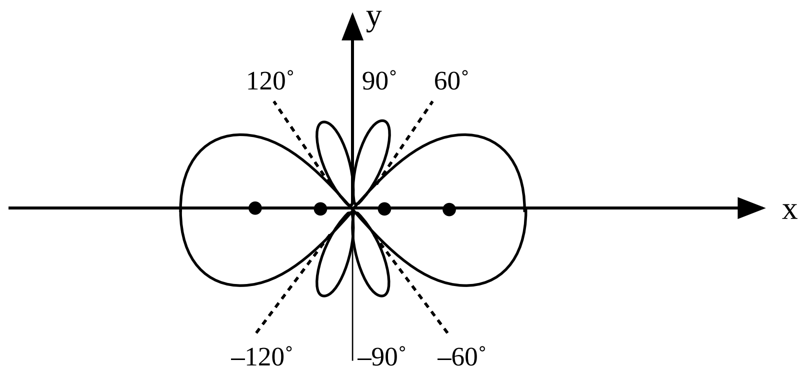
$$\beta d \cos\phi + \psi = \frac{2n\pi}{N}, \qquad n = \pm 1, \pm 2, \pm 3, \cdots, \quad n \neq mN. \tag{13}$$

For example,

Case I. $\psi = 0, \beta d = \pi$, principal maximum is at $\phi = \pm\frac{\pi}{2}$ if $N = 5$, nulls are at $\phi = \pm\cos^{-1}\left(\frac{2n}{5}\right)$, or $\phi = \pm 66.4^\circ, \pm 36.9^\circ, \pm 113.6^\circ, \pm 143.1^\circ$.



Case II. $\psi = \pi, \beta d = \pi$, principal maximum is at $\phi = 0, \pi$, if $N = 4$, nulls are at $\phi = \pm\cos^{-1}\left(\frac{n}{2} - 1\right)$, or $\phi = \pm 120^\circ, \pm 90^\circ, \pm 60^\circ$.



The interference effects between the different antenna elements of a linear array focus the power in a given direction. We can use linear array to increase the directivity of antennas.

Note that equation (7) can also be derived by other means. We know that the vector potential due to one dipole is

$$\mathbf{A} = \hat{z}\frac{\mu I l}{4\pi}\frac{e^{-j\beta|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}, \tag{14}$$

when the dipole is located at $\mathbf{r}'$ and pointing in the $\hat{z}$-direction. Hence for an array of dipoles of different phases and amplitudes, located at $x = \hat{x}d_0, \hat{x}d_1, \hat{x}d_2, \cdots, \hat{x}d_{N-1}$, the vector potential by linear superposition is

$$\mathbf{A}(\mathbf{r}) = \hat{z}\frac{\mu I l}{4\pi}\left[\frac{e^{-j\beta|\mathbf{r}-\hat{x}d_0|}}{|\mathbf{r}-\hat{x}d_0|}A_0 + \frac{e^{-j\beta|\mathbf{r}-\hat{x}d_1|}}{|\mathbf{r}-\hat{x}d_1|}A_1 + \cdots + \frac{e^{-j\beta|\mathbf{r}-\hat{x}d_{N-1}|}}{|\mathbf{r}-\hat{x}d_{N-1}|}A_{N-1}\right]. \tag{15}$$

If we approximate $|\mathbf{r}-\hat{x}d_n|$ by $r - \hat{r}\cdot\hat{x}d_N = r - d_N\cos\phi$, in the phase, and by r in the denominator, then (15) becomes

$$\mathbf{A}(\mathbf{r}) = \hat{z}\frac{\mu I l}{4\pi r}e^{-j\beta r}\left[A_0 + A_1 e^{+j\beta d_1\cos\phi} + A_2 e^{j\beta d_2\cos\phi} + \cdots + A_{N-1}e^{j\beta d_{N-1}\cos\phi}\right], \tag{16}$$

which is the same as equation (7). The interference between the terms in (16) can be used to generate different radiation patterns for different communication applications.

Let $c = a + jb$, and $h = f + jg$, then

$$c + h = (a + f) + j(b + g), \tag{4}$$

and

$$c - h = (a - f) + j(b - g), \tag{5}$$

Multiplication and Division

$$ch = (a + jb)(f + jg) = (af - bg) + j(bf + ag), \tag{6}$$

$$\frac{c}{h} = \frac{a + jb}{f + jg} = \frac{(a + jb)(f - jg)}{(f + jg)(f - jg)} = \frac{af + bg}{f^2 + g^2} + j\frac{bf - ag}{f^2 + g^2}. \tag{7}$$

Multiplication and division are more conveniently carried out in a polar form. Let

$$c = |c|\, e^{j\phi_1}, \quad h = |h|\, e^{j\phi_2}, \tag{8}$$

then

$$ch = |c|\, |h|\, e^{j(\phi_1 + \phi_2)}, \tag{9}$$

$$\frac{c}{h} = \frac{|c|}{|h|} e^{j(\phi_1 - \phi_2)}. \tag{10}$$

Square Root of a Complex Number

It is most convenient to take the square root of a complex number in ***polar form*** or by converting it to ***polar form***.

$$c = |c|\, e^{j\phi_1} = \sqrt{a^2 + b^2} e^{j \tan^{-1} \frac{b}{a}}, \tag{11}$$

$$\sqrt{c} = |c|^{\frac{1}{2}}\, e^{j\frac{\phi_1}{2}} = (a^2 + b^2)^{\frac{1}{4}} e^{j\frac{1}{2}\tan^{-1}\frac{b}{a}}. \tag{12}$$

In fact

$$c^{\frac{1}{m}} = |c|^{\frac{1}{m}}\, e^{j\frac{\phi_1}{m}} = (a^2 + b^2)^{\frac{1}{2m}} e^{j\frac{1}{m}\tan^{-1}\frac{b}{a}}. \tag{13}$$

Phasor Representation of a Time-Harmonic Scalar

If $V(t)$ is a time-harmonic signal such that

$$V(t) = V_0 \cos(\omega t + \phi), \tag{14}$$

it could also be written as

$$V(t) = \Re e\{V_0 e^{j\phi} e^{j\omega t}\}. \tag{15}$$

The term $\tilde{V} = V_0 e^{j\phi}$ is known as the phasor representation of $V(t)$.

If $U(t) = U_0 \cos(\omega t + \phi_1)$, or the phasor representation of $U(t)$ is

$$\tilde{U} = U_0 e^{j\phi_1}. \tag{16}$$

It can be shown easily that

$$V(t) + U(t) = \Re e\{[\underbrace{V_0 e^{j\phi}}_{\tilde{V}} + \underbrace{U_0 e^{j\phi_1}}_{\tilde{U}}] e^{j\omega t}\}. \tag{17}$$

Hence $\tilde{V} + \tilde{U}$ is a phasor representation of $V(t) + U(t)$.
Also

$$\frac{\partial V(t)}{\partial t} = \frac{\partial}{\partial t} \Re e\{V_0 e^{j\phi} e^{j\omega t}\} = \Re e\{j\omega \underbrace{V_0 e^{j\phi}}_{\tilde{V}} e^{j\omega t}\}. \tag{18}$$

Therefore $j\omega\tilde{V}$ is a phasor representation of $\frac{\partial}{\partial t} V(t)$. However, as a word of caution, $\tilde{V}\tilde{U}$ is not a phasor representation of $V(t)U(t)$. You can convince yourself of this.

Exercise

1) Show that,
   (a) $c + c^*$ is always real,
   (b) $c - c^*$ is always imaginary,
   (c) c/c^* has magnitude equal to 1.

2) Consider $z^2 = 1 + 2j$. It is a second order polynomial with two roots. Find the two roots.

3) Obtain the phasor representation of the following
   (a) $V(t) = 10 \cos(\omega t + \frac{\pi}{3})$,
   (b) $I(t) = -8 \sin(\omega t + \frac{\pi}{3})$,
   (c) $A(t) = 3 \sin \omega t - 2 \cos \omega t$,
   (d) $C(t) = 3 \cos(\omega t + \frac{\pi}{4}) + 4 \sin(\omega t + \frac{\pi}{3})$.

4) Obtain $C(t)$ in terms of ω from the following phasors:
   (a) $c = 1 + j$,

(b) $c = 4\exp(j0.8)$,
(c) $c = 3e^{j\frac{\pi}{2}} + 4e^{j0.8}$,
(d) $c = j\sin 3z$.

5) (a) Using binomial theorem, show that

$$\sqrt{1+ja} \simeq \pm\left(1 + j\frac{a}{2}\right), \qquad \text{if } |a| \ll 1.$$

(b) Show that

$$\sqrt{1+ja} \simeq \pm(1+j)\left(\frac{a}{2}\right)^{\frac{1}{2}}, \qquad \text{if } |a| \gg 1.$$