# Lecture 3

# Constitutive Relations, Wave Equation, Electrostatics, and Static Green's Function

Constitutive relations are important for defining the electromagnetic material properties of the media involved. Also, wave phenomenon is a major triumph of Maxwell's equations. Hence, we will study that derivation of this phenomenon here. To make matter simple, we will reduce the problem to electrostatics to simplify the math to introduce the concept of the Green's function.

As mentioned previously, for time-varying problems, only the first two of the four Maxwell's equations suffice. The latter two are derivable from the first two. But the first two equations have four unknowns **E**, **H**, **D**, and **B**. Hence, two more equations are needed to solve for these unknowns. These equations come from the constitutive relations.

## 3.1 Simple Constitutive Relations

The constitution relation between electric flux **D** and the electric field **E** in free space (or vacuum) is

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \tag{3.1.1}$$

When material medium is present, one has to add the contribution to **D** by the polarization density **P** which is a dipole density.[1] Then [31, 32, 41]

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{3.1.2}$$

[1]Note that a dipole moment is given by $Q\ell$ where $Q$ is its charge in coulomb and $\ell$ is its length in m. Hence, dipole density, or polarization density as dimension of coulomb/m$^2$, which is the same as that of electric flux **D**.

The second term $\mathbf{P}$ above is due to material property, and the contribution to the electric flux due to the polarization density of the medium. It is due to the little dipole contribution due to the polar nature of the atoms or molecules that make up a medium.

By the same token, the first term $\varepsilon_0\mathbf{E}$ can be thought of as the polarization density contribution of vacuum. Vacuum, though represents nothingness, has electrons and positrons, or electron-positron pairs lurking in it [42]. Electron is matter, whereas positron is anti-matter. In the quiescent state, they represent nothingness, but they can be polarized by an electric field $\mathbf{E}$. That also explains why electromagnetic wave can propagate through vacuum.

For many media, they are approximately linear media. Then $\mathbf{P}$ is linearly proportional to $\mathbf{E}$, or $\mathbf{P} = \varepsilon_0\chi_0\mathbf{E}$, or

$$\begin{aligned}\mathbf{D} &= \varepsilon_0\mathbf{E} + \varepsilon_0\chi_0\mathbf{E}\\ &= \varepsilon_0(1+\chi_0)\mathbf{E} = \varepsilon\mathbf{E}, \quad \varepsilon = \varepsilon_0(1+\chi_0) = \varepsilon_0\varepsilon_r\end{aligned} \tag{3.1.3}$$

where $\chi_0$ is the electric susceptibility . In other words, for linear material media, one can replace the vacuum permittivity $\varepsilon_0$ with an effective permittivity $\varepsilon = \varepsilon_0\varepsilon_r$ where $\varepsilon_r$ is the relative permittivity. Thus, $\mathbf{D}$ is linearly proportional to $\mathbf{E}$. In free space,[2]

$$\varepsilon = \varepsilon_0 = 8.854 \times 10^{-12} \approx \frac{10^{-8}}{36\pi} \text{ F/m} \tag{3.1.4}$$

The constitutive relation between magnetic flux $\mathbf{B}$ and magnetic field $\mathbf{H}$ is given as

$$\mathbf{B} = \mu\mathbf{H}, \qquad \mu = \text{permeability H/m} \tag{3.1.5}$$

In free space or vacuum,

$$\mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \tag{3.1.6}$$

As shall be explained later, this is an assigned value giving it a precise value as shown above. In other materials, the permeability can be written as

$$\mu = \mu_0\mu_r \tag{3.1.7}$$

The above can be derived using similar argument for permittivity, where the different permeability is due to the presence of magnetic dipole density in a material medium. In the above, $\mu_r$ is termed the relative permeability.

## 3.2 Emergence of Wave Phenomenon, Triumph of Maxwell's Equations

One of the major triumphs of Maxwell's equations is the prediction of the wave phenomenon. This was experimentally verified by Heinrich Hertz in 1888 [18], some 23 years after the

[2]It is to be noted that we will use MKS unit in this course. Another possible unit is the CGS unit used in many physics texts [43]

completion of Maxwell's theory in 1865 [17]. To see this, we consider the first two Maxwell's equations for time-varying fields in vacuum or a source-free medium.[3] They are

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{3.2.1}$$

$$\nabla \times \mathbf{H} = -\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{3.2.2}$$

Taking the curl of (3.2.1), we have

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \frac{\partial}{\partial t} \nabla \times \mathbf{H} \tag{3.2.3}$$

It is understood that in the above, the double curl operator implies $\nabla \times (\nabla \times \mathbf{E})$. Substituting (3.2.2) into (3.2.3), we have

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E} \tag{3.2.4}$$

In the above, the left-hand side can be simplified by using the identity that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$,[4] but be mindful that the operator $\nabla$ has to operate on a function to its right. Therefore, we arrive at the identity that

$$\nabla \times \nabla \times \mathbf{E} = \nabla \nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} \tag{3.2.5}$$

Since $\nabla \cdot \mathbf{E} = 0$ in a source-free medium, we have

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E} = 0 \tag{3.2.6}$$

where

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The above is known as the Laplacian operator. Here, (3.2.6) is the wave equation in three space dimensions [32, 44].

To see the simplest form of wave emerging in the above, we can let $\mathbf{E} = \hat{x} E_x(z, t)$ so that $\nabla \cdot \mathbf{E} = 0$ satisfying the source-free condition. Then (3.2.6) becomes

$$\frac{\partial^2}{\partial z^2} E_x(z, t) - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} E_x(z, t) = 0 \tag{3.2.7}$$

Eq. (3.2.7) is known mathematically as the wave equation in one space dimension. It can also be written as

$$\frac{\partial^2}{\partial z^2} f(z, t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} f(z, t) = 0 \tag{3.2.8}$$

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[3]Since the third and the fourth Maxwell's equations are derivable from the first two when $\partial/\partial_t \neq 0$.

[4]For mnemonics, this formula is also known as the "back-of-the-cab" formula.

where $c_0^2 = (\mu_0\varepsilon_0)^{-1}$. Eq. (3.2.8) can also be factorized as

$$\left(\frac{\partial}{\partial z} - \frac{1}{c_0}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} + \frac{1}{c_0}\frac{\partial}{\partial t}\right) f(z,t) = 0 \tag{3.2.9}$$

or

$$\left(\frac{\partial}{\partial z} + \frac{1}{c_0}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} - \frac{1}{c_0}\frac{\partial}{\partial t}\right) f(z,t) = 0 \tag{3.2.10}$$

The above can be verified easily by direct expansion, and using the fact that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}\frac{\partial}{\partial t} \tag{3.2.11}$$

The above implies that we have either

$$\left(\frac{\partial}{\partial z} + \frac{1}{c_0}\frac{\partial}{\partial t}\right) f_+(z,t) = 0 \tag{3.2.12}$$

or

$$\left(\frac{\partial}{\partial z} - \frac{1}{c_0}\frac{\partial}{\partial t}\right) f_-(z,t) = 0 \tag{3.2.13}$$

Equation (3.2.12) and (3.2.13) are known as the one-way wave equations or the advective equations [45]. From the above factorization, it is seen that the solutions of these one-way wave equations are also the solutions of the original wave equation given by (3.2.8). Their general solutions are then

$$f_+(z,t) = F_+(z - c_0 t) \tag{3.2.14}$$

$$f_-(z,t) = F_-(z + c_0 t) \tag{3.2.15}$$

We can verify the above by back substitution into (3.2.12) and (3.2.13). Eq. (3.2.14) constitutes a right-traveling wave function of any shape while (3.2.15) constitutes a left-traveling wave function of any shape. Since Eqs. (3.2.14) and (3.2.15) are also solutions to (3.2.8), we can write the general solution to the wave equation as

$$f(z,t) = F_+(z - c_0 t) + F_-(z + c_0 t) \tag{3.2.16}$$

This is a wonderful result since $F_+$ and $F_-$ are arbitrary functions of any shape (see Figure 3.1); they can be used to encode information for communication!

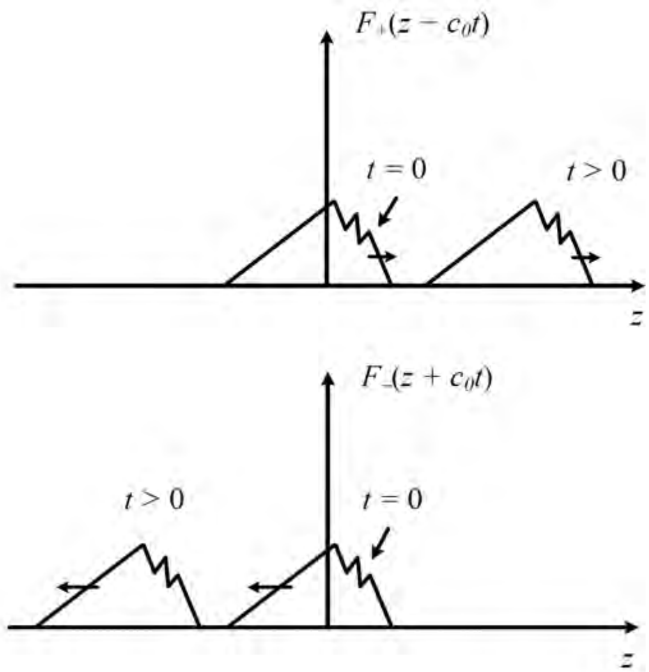


Figure 3.1: Solutions of the wave equation can be a single-valued function of any shape. In the above, $F_+$ travels in the positive $z$ direction, while $F_-$ travels in the negative $z$ direction as $t$ increases.

Furthermore, one can calculate the velocity of this wave to be

$$c_0 = 299,792,458\text{m/s} \simeq 3 \times 10^8\text{m/s} \tag{3.2.17}$$

where $c_0 = \sqrt{1/\mu_0\varepsilon_0}$.

Maxwell's equations (3.2.1) implies that $\mathbf{E}$ and $\mathbf{H}$ are linearly proportional to each other. Thus, there is only one independent constant in the wave equation, and the value of $\mu_0$ is defined neatly to be $4\pi \times 10^{-7}$ henry m$^{-1}$, while the value of $\varepsilon_0$ has been measured to be about $8.854 \times 10^{-12}$ farad m$^{-1}$. Now it has been decided that the velocity of light is used as a standard and is defined to be the integer given in (3.2.17). A meter is defined to be the distance traveled by light in 1/(299792458) seconds. Hence, the more accurate that unit of time or second can be calibrated, the more accurate can we calibrate the unit of length or meter. Therefore, the design of an accurate clock like an atomic clock is an important research problem.

The value of $\varepsilon_0$ was measured in the laboratory quite early. Then it was realized that electromagnetic wave propagates at a tremendous velocity which is the velocity of light.[5] This

[5]The velocity of light was known in astronomy by (Roemer, 1676) [19].

was also the defining moment which revealed that the field of electricity and magnetism and the field of optics were both described by Maxwell's equations or electromagnetic theory.

## 3.3 Static Electromagnetics–Revisted

We have seen static electromagnetics previously in integral form. Now we look at them in differential operator form. When the fields and sources are not time varying, namely that $\partial/\partial t = 0$, we arrive at the static Maxwell's equations for electrostatics and magnetostatics, namely [31, 32, 46]

$$\nabla \times \mathbf{E} = 0 \tag{3.3.1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{3.3.2}$$

$$\nabla \cdot \mathbf{D} = \varrho \tag{3.3.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.3.4}$$

Notice the the electrostatic field system is decoupled from the magnetostatic field system. However, in a resistive system where

$$\mathbf{J} = \sigma \mathbf{E} \tag{3.3.5}$$

the two systems are coupled again. This is known as resistive coupling between them. But if $\sigma \to \infty$, in the case of a perfect conductor, or superconductor, then for a finite $\mathbf{J}$, $\mathbf{E}$ has to be zero. The two systems are decoupled again.

Also, one can arrive at the equations above by letting $\mu_0 \to 0$ and $\epsilon_0 \to 0$. In this case, the velocity of light becomes infinite, or retardation effect is negligible. In other words, there is no time delay for signal propagation through the system in the static approximation.

Finally, it is important to note that in statics, the latter two Maxwell's equations are not derivable from the first two. Hence, all four equations have to be considered when one seeks the solution in the static regime or the long-wavelength regime.

### 3.3.1 Electrostatics

We see that Faraday's law in the static limit is

$$\nabla \times \mathbf{E} = 0 \tag{3.3.6}$$

One way to satisfy the above is to let $\mathbf{E} = -\nabla\Phi$ because of the identity $\nabla \times \nabla = 0$.[6] Alternatively, one can assume that $\mathbf{E}$ is a constant. But we usually are interested in solutions that vanish at infinity, and hence, the latter is not a viable solution. Therefore, we let

$$\mathbf{E} = -\nabla\Phi \tag{3.3.7}$$

[6] One an easily go through the algebra in cartesian coordinates to convince oneself of this.

### 3.3.2 Poisson's Equation

As a consequence of the above,

$$\nabla \cdot \mathbf{D} = \varrho \Rightarrow \nabla \cdot \varepsilon \mathbf{E} = \varrho \Rightarrow -\nabla \cdot \varepsilon \nabla \Phi = \varrho \tag{3.3.8}$$

In the last equation above, if $\varepsilon$ is a constant of space, or independent of $\mathbf{r}$, then one arrives at the simple Poisson's equation, which is a partial differential equation

$$\nabla^2 \Phi = -\frac{\varrho}{\varepsilon} \tag{3.3.9}$$

Here, the Laplacian operator

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

For a point source, we know from Coulomb's law that

$$\mathbf{E} = \frac{q}{4\pi\varepsilon r^2}\hat{r} = -\nabla\Phi \tag{3.3.10}$$

From the above, we deduce that[7]

$$\Phi = \frac{q}{4\pi\varepsilon r} \tag{3.3.11}$$

Therefore, we know the solution to Poisson's equation (3.3.9) when the source $\varrho$ represents a point source. Since this is a **linear equation**, we can use the principle of linear superposition to find the solution when the charge density $\varrho(\mathbf{r})$ is arbitrary.

A point source located at $\mathbf{r}'$ is described by a charge density as

$$\varrho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}') \tag{3.3.12}$$

where $\delta(\mathbf{r}-\mathbf{r}')$ is a short-hand notation for $\delta(x-x')\delta(y-y')\delta(z-z')$. Therefore, from (3.3.9), the corresponding partial differential equation for a point source is

$$\nabla^2\Phi(\mathbf{r}) = -\frac{q\delta(\mathbf{r} - \mathbf{r}')}{\varepsilon} \tag{3.3.13}$$

The solution to the above equation, from Coulomb's law in accordance to (3.3.11), has to be

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon|\mathbf{r} - \mathbf{r}'|} \tag{3.3.14}$$

whereas (3.3.11) is for a point source at the origin, (3.3.14) is for a point source located and translated to $\mathbf{r}'$. The above is a coordinate independent form of the solution. Here, $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$ and $\mathbf{r}' = \hat{x}x' + \hat{y}y' + \hat{z}z'$, and $|\mathbf{r} - \mathbf{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$.

[7] One can always take the gradient or $\nabla$ of $\Phi$ to verify this. Mind you, this is best done in spherical coordinates.

### 3.3.3 Static Green's Function

Let us define a partial differential equation given by

$$\nabla^2 g(\mathbf{r}-\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{3.3.15}$$

The above is similar to Poisson's equation with a point source on the right-hand side as in (3.3.13). Such a solution, a response to a point source, is called the Green's function.[8] By comparing equations (3.3.13) and (3.3.15), then making use of (3.3.14), we deduced that the static Green's function is

$$g(\mathbf{r}-\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{3.3.16}$$

An arbitrary source can be expressed as

$$\varrho(\mathbf{r}) = \iiint_V dV' \varrho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') \tag{3.3.17}$$

The above is just the statement that an arbitrary charge distribution $\varrho(\mathbf{r})$ can be expressed as a linear superposition of point sources $\delta(\mathbf{r}-\mathbf{r}')$. Using the above in (3.3.9), we have

$$\nabla^2 \Phi(\mathbf{r}) = -\frac{1}{\varepsilon}\iiint_V dV' \varrho(r')\delta(\mathbf{r}-\mathbf{r}') \tag{3.3.18}$$

We can let

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon}\iiint_V dV' g(\mathbf{r}-\mathbf{r}')\varrho(\mathbf{r}') \tag{3.3.19}$$

By substituting the above into the left-hand side of (3.3.18), exchanging order of integration and differentiation, and then making use of equation (3.3.15), it can be shown that (3.3.19) indeed satisfies (3.3.9). The above is just a convolutional integral. Hence, the potential $\Phi(\mathbf{r})$ due to an arbitrary source distribution $\varrho(\mathbf{r})$ can be found by using convolution, namely,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon}\iiint_V \frac{\varrho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dV' \tag{3.3.20}$$

In a nutshell, the solution of Poisson's equation when it is driven by an arbitrary source $\varrho$, is the convolution of the source with the static Green's function, a point source response.

### 3.3.4 Laplace's Equation

If $\varrho = 0$, or if we are in a source-free region, then

$$\nabla^2 \Phi = 0 \tag{3.3.21}$$

which is the Laplace's equation. Laplace's equation is usually solved as a boundary value problem. In such a problem, the potential $\Phi$ is stipulated on the boundary of a region with a certain boundary condition, and then the solution is sought in the region so as to match the boundary condition.

Examples of such boundary value problems are given at the end of the lecture.

[8]George Green (1793-1841), the son of a Nottingham miller, was self-taught, but his work has a profound impact in our world.

## 3.4 Homework Examples

**Example 1**

Fields of a sphere of radius $a$ with uniform charge density $\rho$:

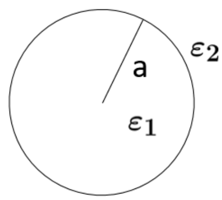


Figure 3.2: Figure of a sphere with uniform charge density for the example above.

Assuming that $\Phi|_{r=\infty} = 0$, what is $\Phi$ at $r \leq a$? And $\Phi$ at $r > a$.

**Example 2**

A capacitor has two parallel plates attached to a battery, what is $\mathbf{E}$ field inside the capacitor?

First, one guess the electric field between the two parallel plates. Then one arrive at a potential $\Phi$ in between the plates so as to produce the field. Then the potential is found so as to match the boundary conditions of $\Phi = V$ in the upper plate, and $\Phi = 0$ in the lower plate. What is the $\Phi$ that will satisfy the requisite boundary condition?

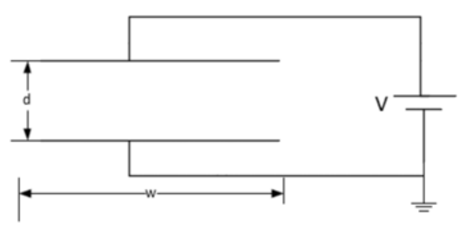


Figure 3.3: Figure of a parallel plate capacitor. The field in between can be found by solving Laplace's equation as a boundary value problem [31].

**Example 3**

A coaxial cable has two conductors. The outer conductor is grounded and hence is at zero potential. The inner conductor is at voltage $V$. What is the solution?

For this, one will have to write the Laplace's equation in cylindrical coordinates, namely,

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{3.4.1}$$

In the above, we assume that the potential is constant in the $z$ direction, and hence, $\partial/\partial z = 0$, and $\rho, \phi, z$ are the cylindrical coordinates. By assuming axi-symmetry, we can let $\partial/\partial\phi = 0$ and the above becomes

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) = 0 \tag{3.4.2}$$

Show that $\Phi = A\ln\rho + B$ is a general solution to Laplace's equation in cylindrical coordinates inside a coax. What is the $\Phi$ that will satisfy the requisite boundary condition?

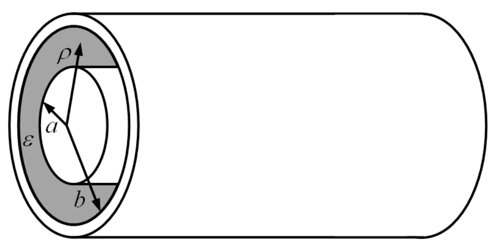


Figure 3.4: The field in between a coaxial line can also be obtained by solving Laplace's equation as a boundary value problem (courtesy of Ramo, Whinnery, and Van Duzer [31]).