Lecture 14

Reflection, Transmission, and Interesting Physical Phenomena

We have seen the derivation of a reflection coefficient in a transmission line that relates the amplitude of the reflected wave to that of the incident wave. By doing so, we have used a simplified form of Maxwell's equations, the telegrapher's equations which are equations in one dimension. Here, we will solve Maxwell's equations in its full glory, but in order to do so, we will look at a very simple problem of plane wave reflection and transmission at a single plane interface.

This will give rise to the Fresnel reflection and transmission coefficients, and embedded in them are interesting physical phenomena. We will study these interesting phenomena as well.

(Much of the contents of this lecture can be found in Kong, and also the ECE 350X lecture notes. They can be found in many textbooks, even though the notations can be slightly different [31, 32, 44, 50, 54, 65, 79, 83, 85, 86].)

14.1 Reflection and Transmission—Single Interface Case

We will derive the plane-wave reflection coefficients for the single interface case between two dielectric media. These reflection coefficients are also called the Fresnel reflection coefficients because they were first derived by Austin-Jean Fresnel (1788-1827). Note that he lived before the completion of Maxwell's equations in 1865. But when Fresnel derived the reflection coefficients in 1823, they were based on the elastic theory of light; and hence, the formulas are not exactly the same as what we are going to derive (see Born and Wolf, Principles of Optics, p. 40 [58]).

The single plane interface, plane wave reflection and transmission problem, with complicated mathematics, is homomorphic to the transmission line problem. The complexity comes because we have to keep track of the 3D polarizations of the electromagnetic fields in this case. We shall learn later that the mathematical homomorphism can be used to exploit the simplicity of transmission line theory in seeking the solutions to the multiple dielectric interface problems.

14.1.1 TE Polarization (Perpendicular or E Polarization)[1]

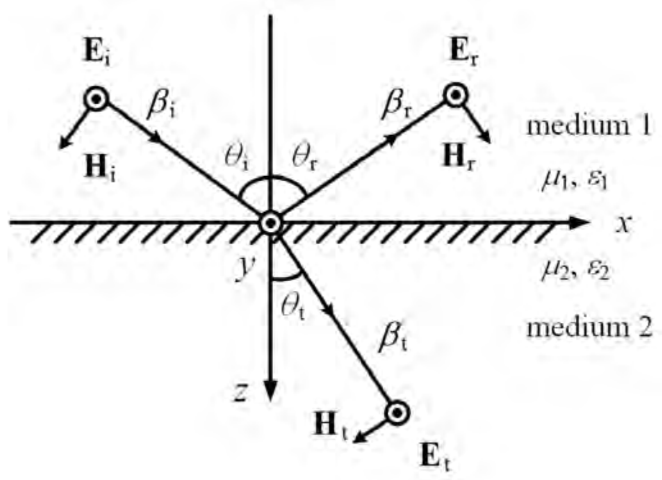

Figure 14.1: A schematic showing the reflection of the TE polarization wave impinging on a dielectric interface.

To set up the above problem, the wave in Region 1 can be written as the superposition or sum of the incident plane wave and reflected plane wave. Here, $\mathbf{E}_i(\mathbf{r})$ and $\mathbf{E}_r(\mathbf{r})$ are the incident and reflected plane waves, respectively. The total field is $\mathbf{E}_i(\mathbf{r}) + \mathbf{E}_r(\mathbf{r})$ which are the phasor representations of the fields. We assume plane wave polarized in the y direction where the wave vectors are $\boldsymbol{\beta}_i = \hat{x}\beta_{ix} + \hat{z}\beta_{iz}$, $\boldsymbol{\beta}_r = \hat{x}\beta_{rx} - \hat{z}\beta_{rz}$, $\boldsymbol{\beta}_t = \hat{x}\beta_{tx} + \hat{z}\beta_{tz}$, respectively for the incident, reflected, and transmitted waves. Then, from Section 7.3, we have

$$\mathbf{E}_i = \hat{y}E_0 e^{-j\boldsymbol{\beta}_i\cdot\mathbf{r}} = \hat{y}E_0 e^{-j\beta_{ix}x - j\beta_{iz}z} \tag{14.1.1}$$

which represents a uniform incident plane wave, and

$$\mathbf{E}_r = \hat{y}R^{TE}E_0 e^{-j\boldsymbol{\beta}_r\cdot\mathbf{r}} = \hat{y}R^{TE}E_0 e^{-j\beta_{rx}x + j\beta_{rz}z} \tag{14.1.2}$$

which is a uniform reflected wave. In Region 2, we only have transmitted plane wave; hence

$$\mathbf{E}_t = \hat{y}T^{TE}E_0 e^{-j\boldsymbol{\beta}_t\cdot\mathbf{r}} = \hat{y}T^{TE}E_0 e^{-j\beta_{tx}x - j\beta_{tz}z} \tag{14.1.3}$$

[1]These polarizations are also variously know as TE_z, or the s and p polarizations, a descendent from the notations for acoustic waves where s and p stand for shear and pressure waves respectively.

In the above, the incident wave is known and hence, E_0 is known. From (14.1.2) and (14.1.3), R^{TE} and T^{TE} are unknowns yet to be sought. To find them, we need two boundary conditions to yield two equations.[2] These boundary conditions are tangential **E** field continuous and tangential **H** field continuous, which are $\hat{n} \times \mathbf{E}$ continuous and $\hat{n} \times \mathbf{H}$ continuous conditions at the interface.

Imposing $\hat{n} \times \mathbf{E}$ continuous at $z = 0$, we get

$$E_0 e^{-j\beta_{ix}x} + R^{TE} E_0 e^{-j\beta_{rx}x} = T^{TE} E_0 e^{-j\beta_{tx}x}, \quad \forall x \tag{14.1.4}$$

where $\forall$ implies "for all". In order for the above to be valid for all x, it is necessary that $\beta_{ix} = \beta_{rx} = \beta_{tx}$, which is also known as the phase matching condition.[3] From the above, by letting $\beta_{ix} = \beta_{rx} = \beta_1 \sin\theta_i = \beta_1 \sin\theta_r$, we obtain that $\theta_r = \theta_i$ or that the law of reflection that the angle of reflection is equal to the angle of incidence. By letting $\beta_{ix} = \beta_1 \sin\theta_i = \beta_{tx} = \beta_2 \sin\theta_t$, we obtain Snell's law that $\beta_1 \sin\theta_i = \beta_2 \sin\theta_t$. (This law of refraction that was also known in the Islamic world in the 900 AD. [94]).

The exponential terms or the phase terms on both sides of (14.1.4) are the same. Now, canceling common terms on both sides of the equation (14.1.4), the above simplifies to

$$1 + R^{TE} = T^{TE} \tag{14.1.5}$$

To impose $\hat{n} \times \mathbf{H}$ continuous, one needs to find the **H** field using $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$. By letting ∇ with $-j\beta$, then we have $\mathbf{H} = -j\boldsymbol{\beta} \times \mathbf{E}/(-j\omega\mu) = \boldsymbol{\beta} \times \mathbf{E}/(\omega\mu)$. By so doing[4]

$$\mathbf{H}_i = \frac{\boldsymbol{\beta}_i \times \mathbf{E}_i}{\omega\mu_1} = \frac{\boldsymbol{\beta}_i \times \hat{y}}{\omega\mu_1} E_0 e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}} = \frac{\hat{z}\beta_{ix} - \hat{x}\beta_{iz}}{\omega\mu_1} E_0 e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}} \tag{14.1.6}$$

$$\mathbf{H}_r = \frac{\boldsymbol{\beta}_r \times \mathbf{E}_r}{\omega\mu_1} = \frac{\boldsymbol{\beta}_r \times \hat{y}}{\omega\mu_1} R^{TE} E_0 e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}} = \frac{\hat{z}\beta_{rx} + \hat{x}\beta_{rz}}{\omega\mu_1} R^{TE} E_0 e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}} \tag{14.1.7}$$

$$\mathbf{H}_t = \frac{\boldsymbol{\beta}_t \times \mathbf{E}_t}{\omega\mu_2} = \frac{\boldsymbol{\beta}_t \times \hat{y}}{\omega\mu_2} T^{TE} E_0 e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}} = \frac{\hat{z}\beta_{tx} - \hat{x}\beta_{tz}}{\omega\mu_2} T^{TE} E_0 e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}} \tag{14.1.8}$$

Imposing $\hat{n} \times \mathbf{H}$ continuous or H_x continuous at $z = 0$, we have

$$-\frac{\beta_{iz}}{\omega\mu_1} E_0 e^{-j\beta_{ix}x} + \frac{\beta_{rz}}{\omega\mu_1} R^{TE} E_0 e^{-j\beta_{rx}x} = -\frac{\beta_{tz}}{\omega\mu_2} T^{TE} E_0 e^{-j\beta_{tx}x} \tag{14.1.9}$$

As mentioned before, the phase-matching condition requires that $\beta_{ix} = \beta_{rx} = \beta_{tx}$. The dispersion relation for plane waves requires that in their respective media,

$$\beta_{ix}^2 + \beta_{iz}^2 = \beta_{rx}^2 + \beta_{rz}^2 = \omega^2\mu_1\varepsilon_1 = \beta_1^2, \qquad \text{medium 1} \tag{14.1.10}$$

$$\beta_{tx}^2 + \beta_{tz}^2 = \omega^2\mu_2\varepsilon_2 = \beta_2^2, \qquad \text{medium 2} \tag{14.1.11}$$

[2]Here, we will treat this problem as a boundary value problem where the unknowns are sought from equations obtained from boundary conditions.

[3]The phase-matching condition can also be proved by taking the Fourier transform of the equation with respect to x. Among the physics community, this is also known as momentum matching, as the wavenumber of a wave is related to the momentum of the particle.

[4]We note here that field theory is a lot more complicated than transmission line theory. That is the triumph of transmission line theory as well.

Since

$$\beta_{ix} = \beta_{rx} = \beta_{tx} = \beta_x \tag{14.1.12}$$

the above implies that

$$\beta_{iz} = \beta_{rz} = \beta_{1z} \tag{14.1.13}$$

Moreover, $\beta_{tz} = \beta_{2z} \neq \beta_{1z}$ usually since $\beta_1 \neq \beta_2$. Then (14.1.9) simplifies to

$$\frac{\beta_{1z}}{\mu_1}(1 - R^{TE}) = \frac{\beta_{2z}}{\mu_2} T^{TE} \tag{14.1.14}$$

where $\beta_{1z} = \sqrt{\beta_1^2 - \beta_x^2}$, and $\beta_{2z} = \sqrt{\beta_2^2 - \beta_x^2}$.

Solving (14.1.5) and (14.1.14) for R^{TE} and T^{TE} yields

$$R^{TE} = \left(\frac{\beta_{1z}}{\mu_1} - \frac{\beta_{2z}}{\mu_2}\right) \Bigg/ \left(\frac{\beta_{1z}}{\mu_1} + \frac{\beta_{2z}}{\mu_2}\right) \tag{14.1.15}$$

$$T^{TE} = 2\left(\frac{\beta_{1z}}{\mu_1}\right) \Bigg/ \left(\frac{\beta_{1z}}{\mu_1} + \frac{\beta_{2z}}{\mu_2}\right) \tag{14.1.16}$$

14.1.2 TM Polarization (Parallel or H Polarization)[5]

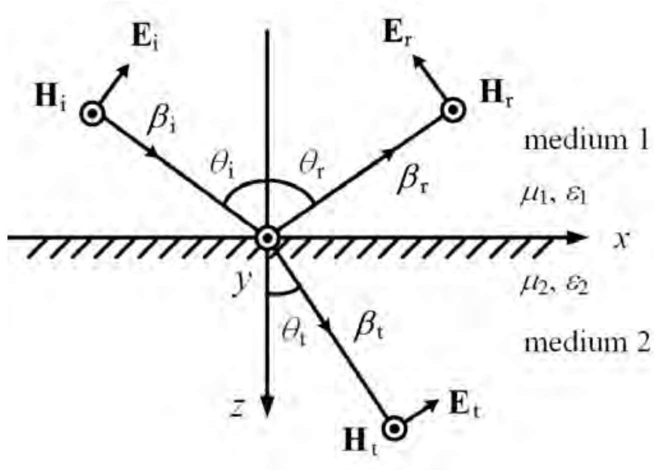

Figure 14.2: A similar schematic showing the reflection of the TM polarization wave impinging on a dielectric interface. The solution to this problem can be easily obtained by invoking duality principle.

[5]Also known as TM_z polarization.

The solution to the TM polarization case can be obtained by invoking duality principle where we do the substitution $\mathbf{E} \to \mathbf{H}$, $\mathbf{H} \to -\mathbf{E}$, and $\mu \rightleftharpoons \varepsilon$ as shown in Figure 14.2. The reflection coefficient for the TM magnetic field is then

$$R^{TM} = \left(\frac{\beta_{1z}}{\varepsilon_1} - \frac{\beta_{2z}}{\varepsilon_2}\right) \Big/ \left(\frac{\beta_{1z}}{\varepsilon_1} + \frac{\beta_{2z}}{\varepsilon_2}\right) \tag{14.1.17}$$

$$T^{TM} = 2\left(\frac{\beta_{1z}}{\varepsilon_1}\right) \Big/ \left(\frac{\beta_{1z}}{\varepsilon_1} + \frac{\beta_{2z}}{\varepsilon_2}\right) \tag{14.1.18}$$

Please remember that R^{TM} and T^{TM} are reflection and transmission coefficients for the magnetic fields, whereas R^{TE} and T^{TE} are those for the electric fields. Some textbooks may define these reflection coefficients based on electric field only, and they will look different, and duality principle cannot be applied.

14.2 Interesting Physical Phenomena

Three interesting physical phenomena emerge from the solutions of the single-interface problem. They are **total internal reflection**, **Brewster angle effect**, and **surface plasmonic resonance**. We will look at them next.

14.2.1 Total Internal Reflection

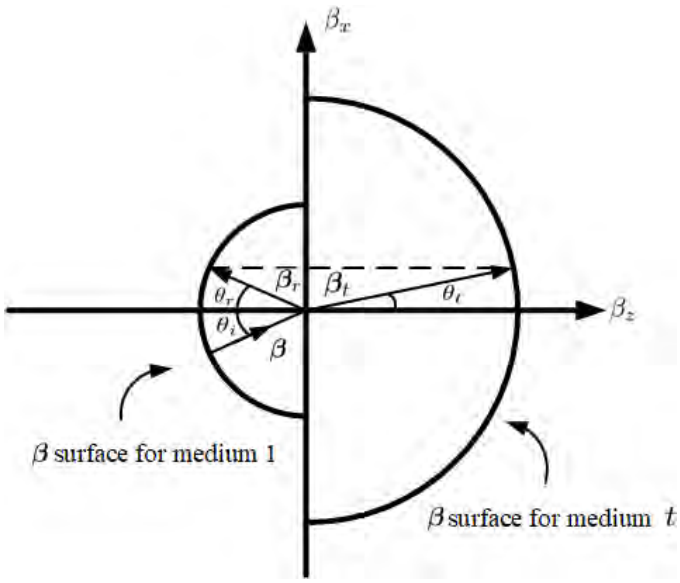

Figure 14.3: Wave-number surfaces in two regions showing the phase matching condition. The wave number in medium t is larger than the wave number in medium 0. The wave vectors for the incident wave, reflected wave, and transmitted wave have to be aligned in such a way that their components parallel to the interface are equal in order to satisfy the phase-matching condition. One can see that Snell's law is satisfied when the phase-matching condition is satisfied.

Total internal reflection comes about because of phase matching (also called momentum matching). This phase-matching condition can be illustrated using β-surfaces (same as k-surfaces in some literature), as shown in Figure 14.3. It turns out that because of phase matching, for certain interfaces, β_{2z} becomes pure imaginary.

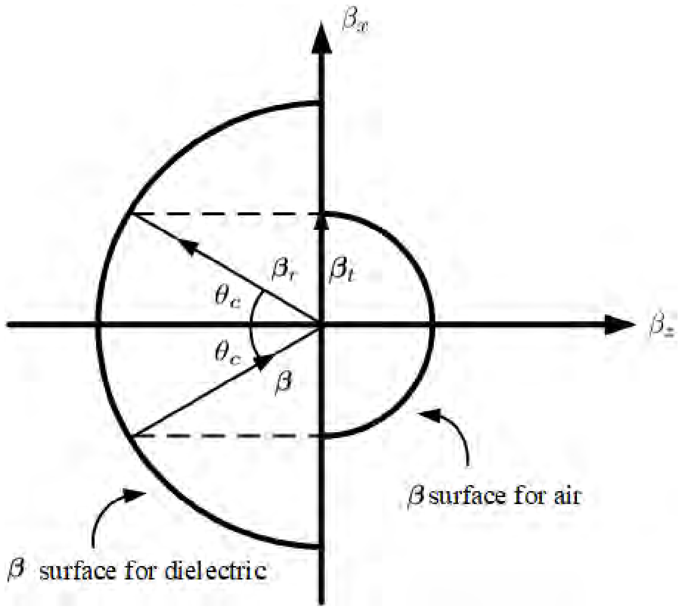

Figure 14.4: Wave-number surfaces in two regions showing the phase matching condition. The wave number in medium t is smaller than the wave number in medium 0. The figure shows in incident wave vector coming in at the critical angle. Then the transmitted wave vector is parallel to the interface as shown. When the incident angle is larger than the critical angle, β_z becomes an imaginary number the wave vector in Region t is complex and cannot be drawn.

As shown in Figures 14.3 and 14.4, because of the dispersion relation that $\beta_{rx}^2 + \beta_{rz}^2 = \beta_{ix}^2 + \beta_{iz}^2 = \beta_1^2$, $\beta_{tx}^2 + \beta_{tz}^2 = \beta_2^2$, they are equations of two circles in 2D whose radii are β_1 and β_2, respectively. (The tips of the $\boldsymbol{\beta}$ vectors for Regions 1 and 2 have to be on a spherical surface in the β_x, β_y, and β_z space in the general 3D case, but in this figure, we only show a cross section of the sphere assuming that $\beta_y = 0$.)

Phase matching implies that the x-component of the $\boldsymbol{\beta}$ vectors are equal to each other as shown. One sees that $\theta_i = \theta_r$ in Figure 14.4, and also as θ_i increases, θ_t increases. For an optically less dense medium where $\beta_2 < \beta_1$, according to the Snell's law of refraction, the transmitted $\boldsymbol{\beta}$ will refract away from the normal, as seen in the figure (where k is synonymous with our β). Therefore, eventually the vector $\boldsymbol{\beta}_t$ becomes parallel to the x axis when $\beta_{ix} = \beta_{rx} = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$ and $\theta_t = \pi/2$. The incident angle at which this happens is termed the critical angle θ_c (see Figure 14.4).

Since $\beta_{ix} = \beta_1 \sin\theta_i = \beta_{rx} = \beta_1 \sin\theta_r = \beta_2$, or

$$\sin\theta_r = \sin\theta_i = \sin\theta_c = \frac{\beta_2}{\beta_1} = \frac{\sqrt{\mu_2\varepsilon_2}}{\sqrt{\mu_1\varepsilon_1}} = \frac{n_2}{n_1} \tag{14.2.1}$$

where n_1 is the reflective index defined as $c_0/v_i = \sqrt{\mu_i\varepsilon_i}/\sqrt{\mu_0\varepsilon_0}$ where v_i is the phase velocity of the wave in Region i. Hence,

$$\theta_c = \sin^{-1}(n_2/n_1) \tag{14.2.2}$$

When $\theta_i > \theta_c$. $\beta_x > \beta_2$ and $\beta_{2z} = \sqrt{\beta_2{}^2 - \beta_x{}^2}$ becomes pure imaginary. When β_{2z} becomes pure imaginary, the wave cannot propagate in Region 2, or $\beta_{2z} = -j\alpha_{2z}$, and the wave becomes evanescent. The reflection coefficient (14.1.15) becomes of the form

$$R^{TE} = (A - jB)/(A + jB) \tag{14.2.3}$$

Since the numerator is the complex conjugate of the denominator. It is clear that $|R^{TE}| = 1$ and that $R^{TE} = e^{j\theta_{TE}}$. Therefore, a total internally reflected wave suffers a phase shift. A phase shift in the frequency domain corresponds to a time delay in the time domain. Such a time delay is achieved by the wave traveling laterally in Region 2 before being refracted back to Region 1. Such a lateral shift is called the Goos-Hanschen shift as shown in Figure 14.5 [58]. A wave that travels laterally along the surface of two media is also known as lateral waves [95, 96].

(Please be reminded that total internal reflection comes about entirely due to the phase-matching condition when Region 2 is a faster medium than Region 1. Hence, it will occur with all manner of waves, such as elastic waves, sound waves, seismic waves, quantum waves etc.)

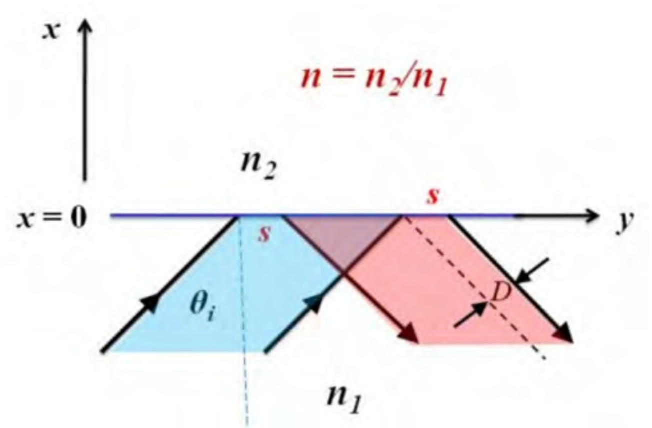

Figure 14.5: Goos-Hanschen Shift. A phase delay is equivalent to a time delay (courtesy of Paul R. Berman (2012), Scholarpedia, 7(3):11584 [97]).

The guidance of a wave in a dielectric slab is due to total internal reflection at the dielectric-to-air interface. The wave bounces between the two interfaces of the slab, and creates evanescent waves outside, as shown in Figure 14.6. The guidance of waves in an optical fiber works by similar mechanism of total internal reflection, as shown in Figure 14.7. Due to the tremendous impact the optical fiber has on modern-day communications, Charles Kao, the father of the optical fiber, was awarded the Nobel Prize in 2009. His work was first published in [98].

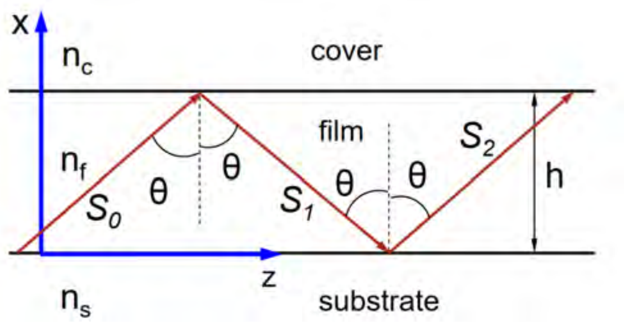

Figure 14.6: The total internal reflections at the two interfaces of a thin-film waveguide can be used to guide an optical wave (courtesy of E.N. Glytsis, NTUA, Greece [99]).

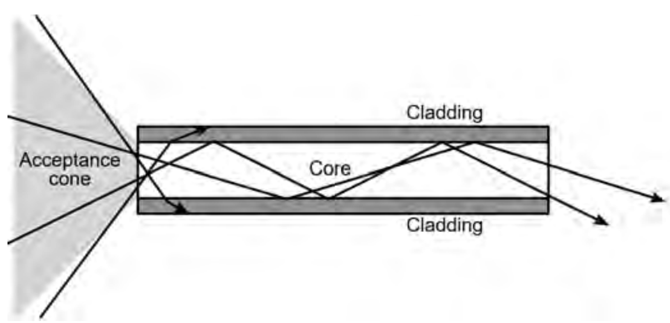

Figure 14.7: An optical fiber consists of a core and a cladding region. Total internal reflections occur at the core-cladding interface. They can guide an optical wave in the fiber (courtesy of Wikepedia [100]).

Waveguides have affected international communications for over a hundred year now. Since telegraphy was in place before the full advent of Maxwell's equations, submarine cables for global communications were laid as early as 1850's. Figure 14.8 shows a submarine cable from 1869 using coaxial cable, and one used in the modern world using optical fiber.

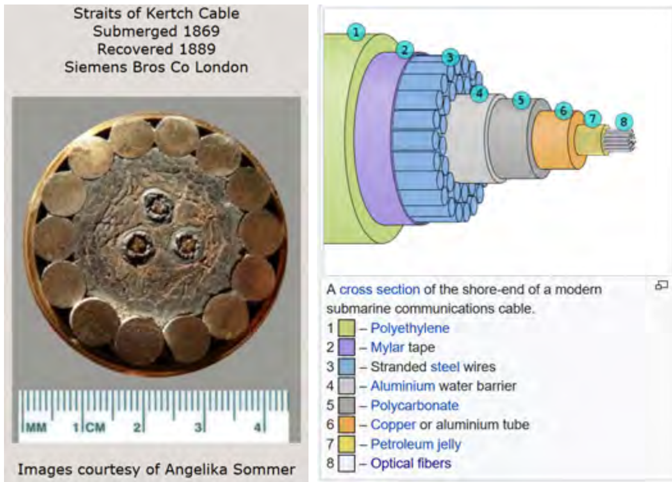

Figure 14.8: The picture of an old 1869 submarine cable made of coaxial cables (left), and modern submarine cable made of optical fibers (right) (courtesy of Atlantic-Cable [101], and Wikipedia [102]).