Lecture 9

Waves in Gyrotropic Media, Polarization

9.1 Gyrotropic Media

This section presents deriving the permittivity tensor of a gyrotropic medium in the ionosphere. Our ionosphere is always biased by a static magnetic field due to the Earth’s magnetic field [67]. But in this derivation, one assumes that the ionosphere has a static magnetic field polarized in the $z$ direction, namely that $\mathbf{B} = \hat{z}B_0$. Now, the equation of motion from the Lorentz force law for an electron with $q = -e$, in accordance with Newton’s law, becomes

$$m_e \frac{d\mathbf{v}}{dt} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (9.1.1)$$

Next, let us assume that the electric field is polarized in the $xy$ plane. The derivative of $\mathbf{v}$ is the acceleration of the electron, and also, $\mathbf{v} = \frac{d\mathbf{r}}{dt}$. And in the frequency domain, the above equation becomes

$$m_e \omega^2 x = e(\mathbf{E}_x + j\omega B_0 y) \quad (9.1.2)$$
$$m_e \omega^2 y = e(\mathbf{E}_y - j\omega B_0 x) \quad (9.1.3)$$

The above equations cannot be solved easily for $x$ and $y$ in terms of the electric field because they correspond to a two-by-two matrix system with cross coupling between the unknowns $x$ and $y$. But they can be simplified as follows: We can multiply (9.1.3) by $\pm j$ and add it to (9.1.2) to get two decoupled equations [68]:

$$m_e \omega^2 (x + jy) = e[(\mathbf{E}_x + j\mathbf{E}_y) + \omega B_0 (x + jy)] \quad (9.1.4)$$
$$m_e \omega^2 (x - jy) = e[(\mathbf{E}_x - j\mathbf{E}_y) - \omega B_0 (x - jy)] \quad (9.1.5)$$

Defining new variables such that

$$s_\pm = x \pm jy \quad (9.1.6)$$
$$E_\pm = E_x \pm jE_y \quad (9.1.7)$$
then (9.1.4) and (9.1.5) become

\[ m_e \omega^2 s_\pm = e(E_\pm \pm \omega B_0 s_\pm) \] (9.1.8)

Thus, solving the above yields

\[ s_\pm = \frac{e}{m_e \omega^2 \mp e B_0 \omega} E_\pm = C_\pm E_\pm \] (9.1.9)

where

\[ C_\pm = \frac{e}{m_e \omega^2 \pm e B_0 \omega} \] (9.1.10)

By this manipulation, the above equations (9.1.2) and (9.1.3) transform to new equations where there is no cross coupling between \( s_\pm \) and \( E_\pm \). The mathematical parlance for this is the diagnolization of a matrix equation [69]. Thus, the new equation can be solved easily.

Next, one can define \( P_x = -Ne x \), \( P_y = -Ne y \), and that \( P_\pm = P_x \pm jP_y = -Ne s_\pm \). Then it can be shown that

\[ P_\pm = \varepsilon_0 \chi_\pm E_\pm \] (9.1.11)

The expression for \( \chi_\pm \) can be derived, and they are given as

\[ \chi_\pm = -\frac{Ne C_\pm}{\varepsilon_0} = \frac{Ne}{\varepsilon_0} \frac{e}{m_e \omega^2 \pm e B_0 \omega} = -\frac{\omega_p^2}{\omega^2 \mp \Omega \omega} \] (9.1.12)

where \( \Omega \) and \( \omega_p \) are the cyclotron frequency\(^1\) and plasma frequency, respectively.

\[ \Omega = \frac{eB_0}{m_e}, \quad \omega_p = \frac{Ne^2}{m_e \varepsilon_0} \] (9.1.13)

At the cyclotron frequency, a solution exists to the equation of motion (9.1.1) without a forcing term, which in this case is the electric field \( E = 0 \). Thus, at this frequency, the solution blows up if the forcing term, \( E_\pm \) is not zero. This is like what happens to an LC tank circuit at resonance whose current or voltage tends to infinity when the forcing term, like the voltage or current is nonzero.

Now, one can rewrite (9.1.11) in terms of the original variables \( P_x, P_y, E_x, E_y \), or

\[ P_x = \frac{P_+ + P_-}{2} = \frac{\varepsilon_0}{2} (\chi_+ E_+ + \chi_- E_-) = \frac{\varepsilon_0}{2} [\chi_+(E_x + jE_y) + \chi_-(E_x - jE_y)] \]

\[ = \frac{\varepsilon_0}{2} [(\chi_+ + \chi_-)E_x + j(\chi_+ - \chi_-)E_y] \] (9.1.14)

\[ P_y = \frac{P_+ - P_-}{2j} = \frac{\varepsilon_0}{2j} (\chi_+ E_+ - \chi_- E_-) = \frac{\varepsilon_0}{2j} [\chi_+(E_x + jE_y) - \chi_-(E_x - jE_y)] \]

\[ = \frac{\varepsilon_0}{2j} [(\chi_+ - \chi_-)E_x + j(\chi_+ + \chi_-)E_y] \] (9.1.15)

\(^1\)This is also called the gyrofrequency.
The above relationship can be expressed using a tensor where

\[ P = \varepsilon_0 \chi \cdot E \] (9.1.16)

where \( P = [P_x, P_y] \), and \( E = [E_x, E_y] \). From the above, \( \chi \) is of the form

\[ \chi = \frac{1}{2} \begin{pmatrix} (\chi_x + \chi_y) & j(\chi_x - \chi_y) \\ -j(\chi_x - \chi_y) & (\chi_x + \chi_y) \end{pmatrix} = \begin{pmatrix} -\omega_p^2 / \omega^2 & j\omega_p^2 \Omega / \omega \omega^2 \\ j\omega_p^2 \Omega / \omega \omega^2 & -\omega_p^2 / \omega^2 \end{pmatrix} \] (9.1.17)

Notice that in the above, when the \( B \) field is turned off or \( \Omega = 0 \), the above resembles the solution of a collisionless, cold plasma.

For the electric field in the \( z \) direction, it will drive a motion of the electron to be in the \( z \) direction. In this case, \( \mathbf{v} \times \mathbf{B} \) term is zero, and the electron motion is unaffected by the magnetic field as can be seen from the Lorentz force law or (9.1.1). Hence, it behaves like a simple collisionless plasma without a biasing magnetic field. Consequently, the above can be generalized to 3D to give

\[ \chi_p = -\omega_p^2 / \omega^2. \]

Using the fact that \( \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (\mathbf{I} + \chi) \cdot \mathbf{E} = \varepsilon_0 \cdot \mathbf{E} \), the above implies that

\[ \varepsilon = \varepsilon_0 \begin{pmatrix} 1 + \chi_0 & j\chi_1 & 0 \\ -j\chi_1 & 1 + \chi_0 & 0 \\ 0 & 0 & 1 + \chi_p \end{pmatrix} \] (9.1.19)

Please notice that the above tensor is a hermitian tensor. We shall learn later that this is the hallmark of a lossless medium.

Another characteristic of a gyrotropic medium is that a linearly polarized wave will rotate when passing through it. This is the Faraday rotation effect [68], which we shall learn later. This phenomenon poses a severe problem to Earth-to-satellite communication, using linearly polarized wave as it requires the alignment of the Earth-to-satellite antennas. This can be avoided using a rotatingly polarized wave, called a circularly polarized wave that we shall learn in the next section. Also, the ionosphere of the Earth is highly dependent on temperature, and the effect of the Sun. The fluctuation of particles in the ionosphere gives rise to scintillation effects that affect radio wave communication systems [70].

### 9.2 Wave Polarization

Studying wave polarization is very important for communication purposes [31]. A wave whose electric field is pointing in the \( x \) direction while propagating in the \( z \) direction is a linearly polarized (LP) wave. The same can be said of one with electric field polarized in the \( y \) direction. It turns out that a linearly polarized wave suffers from Faraday rotation when
Electromagnetic Field Theory

it propagates through the ionosphere. For instance, an $x$ polarized wave can become a $y$ polarized due to Faraday rotation. So its polarization becomes ambiguous: to overcome this, Earth to satellite communication is done with circularly polarized (CP) waves [71]. So even if the electric field vector is rotated by Faraday’s rotation, it remains to be a CP wave. We will study these polarized waves next.

We can write a general uniform plane wave propagating in the $z$ direction as

$$E = \hat{x}E_x(z,t) + \hat{y}E_y(z,t) \quad (9.2.1)$$

Clearly, $\nabla \cdot E = 0$, and $E_x(z,t)$ and $E_y(z,t)$ are solutions to the one-dimensional wave equation. For a time harmonic field, the two components may not be in phase, and we have in general

$$E_x(z,t) = E_1 \cos(\omega t - \beta z) \quad (9.2.2)$$
$$E_y(z,t) = E_2 \cos(\omega t - \beta z + \alpha) \quad (9.2.3)$$

where $\alpha$ denotes the phase difference between these two wave components. We shall study how the linear superposition of these two components behaves for different $\alpha$’s. First, we set $z = 0$ to observe this field. Then

$$E = \hat{x}E_1 \cos(\omega t) + \hat{y}E_2 \cos(\omega t + \alpha) \quad (9.2.4)$$

For $\alpha = \frac{\pi}{2}$

$$E_x = E_1 \cos(\omega t), \quad E_y = E_2 \cos(\omega t + \pi/2) \quad (9.2.5)$$

Next, we evaluate the above for different $\omega t$’s

$$\begin{align*}
\omega t &= 0, & E_x &= E_1, & E_y &= 0 \\
\omega t &= \pi/4, & E_x &= E_1/\sqrt{2}, & E_y &= -E_2/\sqrt{2} \\
\omega t &= \pi/2, & E_x &= 0, & E_y &= -E_2 \\
\omega t &= 3\pi/4, & E_x &= -E_1/\sqrt{2}, & E_y &= -E_2/\sqrt{2} \\
\omega t &= \pi, & E_x &= -E_1, & E_y &= 0 
\end{align*} \quad (9.2.6)$$

The tip of the vector field $E$ traces out an ellipse as show in Figure 9.1. With the thumb pointing in the $z$ direction, and the wave rotating in the direction of the fingers, such a wave is called left-hand elliptically polarized (LHEP) wave.
When $E_1 = E_2$, the ellipse becomes a circle, and we have a left-hand circularly polarized (LHCP) wave. When $\alpha = -\pi/2$, the wave rotates in the counter-clockwise direction, and the wave is either right-hand elliptically polarized (RHEP), or right-hand circularly polarized (RHCP) wave depending on the ratio of $E_1/E_2$. Figure 9.2 shows the different polarizations of the wave wave for different phase differences and amplitude ratio.

Figure 9.1: If one follows the tip of the electric field vector, it traces out an ellipse as a function of time $t$. 
Figure 9.2: Due to different phase difference between the $E_x$ and $E_y$ components of the field, and their relative amplitudes $E_2/E_1$, different polarizations will ensure. The arrow indicates the direction of rotation of the field vector. In this figure, $\psi = -\alpha$ in our notes, and $A = E_2/E_1$ (Courtesy of J.A. Kong, Electromagnetic Wave Theory [31]).

Figure 9.3 shows a graphic picture of a CP wave propagating through space.
9.2.1 Arbitrary Polarization Case and Axial Ratio

The axial ratio (AR) is an important figure of merit for designing CP antennas (antennas that will radiate CP or circularly polarized waves). The closer is this ratio to 1, the better the antenna design. We will discuss the general polarization and the axial ratio of a wave.

For the general case for arbitrary $\alpha$, we let

$$E_x = E_1 \cos \omega t, \ E_y = E_2 \cos(\omega t + \alpha) = E_2(\cos \omega t \cos \alpha - \sin \omega t \sin \alpha)$$  (9.2.11)

Then from the above, expressing $E_y$ in terms of $E_x$, one gets

$$E_y = \frac{E_2}{E_1} E_x \cos \alpha - E_2 \left[1 - \left(\frac{E_x}{E_1}\right)^2 \right]^{1/2} \sin \alpha$$  (9.2.12)

Rearranging and squaring, we get

$$aE_x^2 - bE_xE_y + cE_y^2 = 1$$  (9.2.13)

where

$$a = \frac{1}{E_1^2 \sin^2 \alpha}, \quad b = \frac{2 \cos \alpha}{E_1 E_2 \sin \alpha}, \quad c = \frac{1}{E_2^2 \sin^2 \alpha}$$  (9.2.14)

After letting $E_x \rightarrow x$, and $E_y \rightarrow y$, equation (9.2.13) is of the form,

$$ax^2 - bxy + cy^2 = 1$$  (9.2.15)
The equation of an ellipse in its self coordinates is

\[
\left( \frac{x'}{A} \right)^2 + \left( \frac{y'}{B} \right)^2 = 1 \tag{9.2.16}
\]

where \( A \) and \( B \) are axes of the ellipse as shown in Figure 9.4. We can transform the above back to the \((x, y)\) coordinates by letting

\[
x' = x \cos \theta - y \sin \theta \tag{9.2.17}
\]
\[
y' = x \sin \theta + y \cos \theta \tag{9.2.18}
\]
to get

\[
x^2 \left( \frac{\cos^2 \theta + \sin^2 \theta}{A^2} \right) - xy \sin 2\theta \left( \frac{1}{A^2} - \frac{1}{B^2} \right) + y^2 \left( \frac{\sin^2 \theta}{A^2} + \frac{\cos^2 \theta}{B^2} \right) = 1 \tag{9.2.19}
\]

Comparing (9.2.13) and (9.2.19), one gets

\[
\theta = \frac{1}{2} \tan^{-1} \left( \frac{2 \cos \alpha E_1 E_2}{E_2^2 - E_1^2} \right) \tag{9.2.20}
\]
\[
AR = \left( \frac{1 + \Delta}{1 - \Delta} \right)^{1/2} > 1 \tag{9.2.21}
\]

where \( AR \) is the axial ratio where

\[
\Delta = \left( 1 - \frac{4E_1^2 E_2^2 \sin^2 \alpha}{(E_1^2 + E_2^2)^2} \right)^{1/2} \tag{9.2.22}
\]
9.3 Polarization and Power Flow

For a linearly polarized wave,

\[ E = \hat{x} E_0 \cos(\omega t - \beta z), \quad H = \hat{y} \frac{E_0}{\eta} \cos(\omega t - \beta z) \]  

(9.3.1)

Hence, the instantaneous power becomes

\[ S(t) = E(t) \times H(t) = \hat{z} \frac{E_0^2}{\eta} \cos^2(\omega t - \beta z) \]  

(9.3.2)

indicating that for a linearly polarized wave, the instantaneous power is function of both time and space. It travels as lumps of energy through space. In the above \( E_0 \) is the amplitude of the linearly polarized wave.

Next, we look at power flow for for elliptically and circularly polarized waves. It is to be noted that in the phasor world or frequency domain, (9.2.1) becomes

\[ \mathbf{E}(z, \omega) = \hat{x} E_1 e^{-j\beta z} + \hat{y} E_2 e^{-j\beta z + j\alpha} \]  

(9.3.3)
For LHEP,
\[ \mathbf{E}(z, \omega) = e^{-j\beta z}(\hat{x}E_1 + j\hat{y}E_2) \] (9.3.4)
whereas for LHCP
\[ \mathbf{E}(z, \omega) = e^{-j\beta z}E_1(\hat{x} + j\hat{y}) \] (9.3.5)
For RHEP, the above becomes
\[ \mathbf{E}(z, \omega) = e^{-j\beta z}(\hat{x}E_1 - j\hat{y}E_2) \] (9.3.6)
whereas for RHCP, it is
\[ \mathbf{E}(z, \omega) = e^{-j\beta z}E_1(\hat{x} - j\hat{y}) \] (9.3.7)
Focussing on the circularly polarized wave,
\[ \mathbf{E} = (\hat{x} \pm j\hat{y})E_0 e^{-j\beta z} \] (9.3.8)
Using that
\[ \mathbf{H} = \frac{\beta \times \mathbf{E}}{\omega \mu} \],
then
\[ \mathbf{H} = (\mp\hat{x} - j\hat{y})j E_0 \eta e^{-j\beta z} \] (9.3.9)
Therefore,
\[ \mathbf{E}(t) = \hat{x}E_0 \cos(\omega t - \beta z) \pm \hat{y}E_0 \sin(\omega t - \beta z) \] (9.3.10)
\[ \mathbf{H}(t) = \mp \hat{x} \frac{E_0}{\eta} \sin(\omega t - \beta z) + \hat{y} \frac{E_0}{\eta} \cos(\omega t - \beta z) \] (9.3.11)
Then the instantaneous power becomes
\[ \mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t) = \frac{\hat{z} E_0^2}{\eta} \cos^2(\omega t - \beta z) + \frac{\hat{z} E_0^2}{\eta} \sin^2(\omega t - \beta z) = \frac{\hat{z} E_0^2}{\eta} \] (9.3.12)
In other words, a CP wave delivers constant power independent of space and time.
It is to be noted that the complex Poynting vector
\[ \mathbf{S} = \mathbf{E} \times \mathbf{H}^* \] (9.3.13)
are real both for linearly, circularly, and elliptically polarized waves. This is because there is no reactive power in a plane wave of any polarization: the stored energy in the plane wave cannot be returned to the source!
Bibliography


1820, 10 juin 1822, 22 décembre 1823, 12 septembre et 21 novembre 1825. Bachelier, 1825.


Multi-Junction Transmission Lines, Duality Principle


[66] Smithsonian, “This 1600-year-old goblet shows that the romans were nanotechnology pioneers,” https://www.smithsonianmag.com/history/this-1600-year-old-goblet-shows-that-the-romans-were-nanotechnology-pioneers-787224/, accessed: 2019-09-06.


