Lecture 9

Waves in Gyrotropic Media, Polarization

9.1 Gyrotropic Media

This section presents deriving the permittivity tensor of a gyrotropic medium in the ionsphere. Our ionosphere is always biased by a static magnetic field due to the Earth's magnetic field [67]. But in this derivation, one assumes that the ionosphere has a static magnetic field polarized in the z direction, namely that $\mathbf{B} = \hat{z}B_0$. Now, the equation of motion from the Lorentz force law for an electron with $q = -e$, in accordance with Newton's law, becomes

$$m_e \frac{d\mathbf{v}}{dt} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{9.1.1}$$

Next, let us assume that the electric field is polarized in the xy plane. The derivative of $\mathbf{v}$ is the acceleration of the electron, and also, $\mathbf{v} = d\mathbf{r}/dt$. And in the frequency domain, the above equation becomes

$$m_e\omega^2 x = e(E_x + j\omega B_0 y) \tag{9.1.2}$$

$$m_e\omega^2 y = e(E_y - j\omega B_0 x) \tag{9.1.3}$$

The above equations cannot be solved easily for x and y in terms of the electric field because they correspond to a two-by-two matrix system with cross coupling between the unknowns x and y. But they can be simplified as follows: We can multiply (9.1.3) by $\pm j$ and add it to (9.1.2) to get two decoupled equations [68]:

$$m_e\omega^2(x + jy) = e[(E_x + jE_y) + \omega B_0(x + jy)] \tag{9.1.4}$$

$$m_e\omega^2(x - jy) = e[(E_x - jE_y) - \omega B_0(x - jy)] \tag{9.1.5}$$

Defining new variables such that

$$s_\pm = x \pm jy \tag{9.1.6}$$

$$E_\pm = E_x \pm jE_y \tag{9.1.7}$$

then (9.1.4) and (9.1.5) become

$$m_e\omega^2 s_\pm = e(E_\pm \pm \omega B_0 s_\pm) \tag{9.1.8}$$

Thus, solving the above yields

$$s_\pm = \frac{e}{m_e\omega^2 \mp eB_0\omega} E_\pm = C_\pm E_\pm \tag{9.1.9}$$

where

$$C_\pm = \frac{e}{m_e\omega^2 \mp eB_0\omega} \tag{9.1.10}$$

By this manipulation, the above equations (9.1.2) and (9.1.3) transform to new equations where there is no cross coupling between $s_\pm$ and $E_\pm$. The mathematical parlance for this is the diagnolization of a matrix equation [69]. Thus, the new equation can be solved easily.

Next, one can define $P_x = -Nex$, $P_y = -Ney$, and that $P_\pm = P_x \pm jP_y = -Nes_\pm$. Then it can be shown that

$$P_\pm = \varepsilon_0 \chi_\pm E_\pm \tag{9.1.11}$$

The expression for $\chi_\pm$ can be derived, and they are given as

$$\chi_\pm = -\frac{NeC_\pm}{\varepsilon_0} = -\frac{Ne}{\varepsilon_0}\frac{e}{m_e\omega^2 \mp eB_o\omega} = -\frac{{\omega_p}^2}{\omega^2 \mp \Omega\omega} \tag{9.1.12}$$

where Ω and ω_p are the cyclotron frequency[1] and plasma frequency, respectively.

$$\Omega = \frac{eB_0}{m_e}, \quad {\omega_p}^2 = \frac{Ne^2}{m_e\varepsilon_0} \tag{9.1.13}$$

At the cyclotron frequency, a solution exists to the equation of motion (9.1.1) without a forcing term, which in this case is the electric field $\mathbf{E} = 0$. Thus, at this frequency, the solution blows up if the forcing term, $E_\pm$ is not zero. This is like what happens to an LC tank circuit at resonance whose current or voltage tends to infinity when the forcing term, like the voltage or current is nonzero.

Now, one can rewrite (9.1.11) in terms of the original variables $P_x,\ P_y,\ E_x,\ E_y$, or

$$\begin{aligned} P_x = \frac{P_+ + P_-}{2} = \frac{\varepsilon_0}{2}(\chi_+ E_+ + \chi_- E_-) &= \frac{\varepsilon_0}{2}[\chi_+(E_x + jE_y) + \chi_-(E_x - jE_y)] \\ &= \frac{\varepsilon_0}{2}[(\chi_+ + \chi_-)E_x + j(\chi_+ - \chi_-)E_y] \end{aligned} \tag{9.1.14}$$

$$\begin{aligned} P_y = \frac{P_+ - P_-}{2j} = \frac{\varepsilon_0}{2j}(\chi_+ E_+ - \chi_- E_-) &= \frac{\varepsilon_0}{2j}[\chi_+(E_x + jE_y) - \chi_-(E_x - jE_y)] \\ &= \frac{\varepsilon_0}{2j}[(\chi_+ - \chi_-)E_x + j(\chi_+ + \chi_-)E_y] \end{aligned} \tag{9.1.15}$$

[1]This is also called the gyrofrequency.

The above relationship can be expressed using a tensor where

$$\mathbf{P} = \varepsilon_0 \overline{\boldsymbol{\chi}} \cdot \mathbf{E} \tag{9.1.16}$$

where $\mathbf{P} = [P_x, P_y]$, and $\mathbf{E} = [E_x, E_y]$. From the above, $\overline{\boldsymbol{\chi}}$ is of the form

$$\overline{\boldsymbol{\chi}} = \frac{1}{2}\begin{pmatrix} (\chi_+ + \chi_-) & j(\chi_+ - \chi_-) \\ -j(\chi_+ - \chi_-) & (\chi_+ + \chi_-) \end{pmatrix} = \begin{pmatrix} -\frac{{\omega_p}^2}{\omega^2 - \Omega^2} & -j\frac{{\omega_p}^2\Omega}{\omega(\omega^2-\Omega^2)} \\ j\frac{{\omega_p}^2\Omega}{\omega(\omega^2-\Omega^2)} & -\frac{{\omega_p}^2}{\omega^2-\Omega^2} \end{pmatrix} \tag{9.1.17}$$

Notice that in the above, when the $\mathbf{B}$ field is turned off or $\Omega = 0$, the above resembles the solution of a collisionless, cold plasma.

For the electric field in the z direction, it will drive a motion of the electron to be in the z direction. In this case, $\mathbf{v} \times \mathbf{B}$ term is zero, and the electron motion is unaffected by the magnetic field as can be seen from the Lorentz force law or (9.1.1). Hence, it behaves like a simple collisionless plasma without a biasing magnetic field. Consequently, the above can be generalized to 3D to give

$$\overline{\boldsymbol{\chi}} = \begin{bmatrix} \chi_0 & j\chi_1 & 0 \\ -j\chi_1 & \chi_0 & 0 \\ 0 & 0 & \chi_p \end{bmatrix} \tag{9.1.18}$$

where $\chi_p = -\omega_p^2/\omega^2$.

Using the fact that $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0(\overline{\mathbf{I}} + \overline{\boldsymbol{\chi}}) \cdot \mathbf{E} = \overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}$, the above implies that

$$\overline{\boldsymbol{\varepsilon}} = \varepsilon_0 \begin{bmatrix} 1+\chi_0 & j\chi_1 & 0 \\ -j\chi_1 & 1+\chi_0 & 0 \\ 0 & 0 & 1+\chi_p \end{bmatrix} \tag{9.1.19}$$

Please notice that the above tensor is a hermitian tensor. We shall learn later that this is the hallmark of a lossless medium.

Another characteristic of a gyrotropic medium is that a linearly polarized wave will rotate when passing through it. This is the Faraday rotation effect [68], which we shall learn later. This phenomenon poses a severe problem to Earth-to-satellite communication, using linearly polarized wave as it requires the alignment of the Earth-to-satellite antennas. This can be avoided using a rotatingly polarized wave, called a circularly polarized wave that we shall learn in the next section. Also, the ionosphere of the Earth is highly dependent on temperature, and the effect of the Sun. The fluctuation of particles in the ionosphere gives rise to scintillation effects that affect radio wave communication systems [70].

9.2 Wave Polarization

Studying wave polarization is very important for communication purposes [31]. A wave whose electric field is pointing in the x direction while propagating in the z direction is a linearly polarized (LP) wave. The same can be said of one with electric field polarized in the y direction. It turns out that a linearly polarized wave suffers from Faraday rotation when it propagates through the ionosphere. For instance, an x polarized wave can become a y polarized due to Faraday rotation. So its polarization becomes ambiguous: to overcome this, Earth to satellite communication is done with circularly polarized (CP) waves [71]. So even if the electric field vector is rotated by Faraday's rotation, it remains to be a CP wave. We will study these polarized waves next.

We can write a general uniform plane wave propagating in the z direction as

$$\mathbf{E} = \hat{x}E_x(z,t) + \hat{y}E_y(z,t) \tag{9.2.1}$$

Clearly, $\nabla \cdot \mathbf{E} = 0$, and $E_x(z,t)$ and $E_y(z,t)$ are solutions to the one-dimensional wave equation. For a time harmonic field, the two components may not be in phase, and we have in general

$$E_x(z,t) = E_1 \cos(\omega t - \beta z) \tag{9.2.2}$$
$$E_y(z,t) = E_2 \cos(\omega t - \beta z + \alpha) \tag{9.2.3}$$

where α denotes the phase difference between these two wave components. We shall study how the linear superposition of these two components behaves for different α's. First, we set $z = 0$ to observe this field. Then

$$\mathbf{E} = \hat{x}E_1 \cos(\omega t) + \hat{y}E_2 \cos(\omega t + \alpha) \tag{9.2.4}$$

For $\alpha = \frac{\pi}{2}$

$$E_x = E_1 \cos(\omega t),\ E_y = E_z \cos(\omega t + \pi/2) \tag{9.2.5}$$

Next, we evaluate the above for different ωt's

$$\omega t = 0, \qquad E_x = E_1, \qquad E_y = 0 \tag{9.2.6}$$
$$\omega t = \pi/4, \qquad E_x = E_1/\sqrt{2}, \qquad E_y = -E_2/\sqrt{2} \tag{9.2.7}$$
$$\omega t = \pi/2, \qquad E_x = 0, \qquad E_y = -E_2 \tag{9.2.8}$$
$$\omega t = 3\pi/4, \qquad E_x = -E_1/\sqrt{2}, \qquad E_y = -E_2/\sqrt{2} \tag{9.2.9}$$
$$\omega t = \pi, \qquad E_x = -E_1, \qquad E_y = 0 \tag{9.2.10}$$

The tip of the vector field $\mathbf{E}$ traces out an ellipse as show in Figure 9.1. With the thumb pointing in the z direction, and the wave rotating in the direction of the fingers, such a wave is called left-hand elliptically polarized (LHEP) wave.

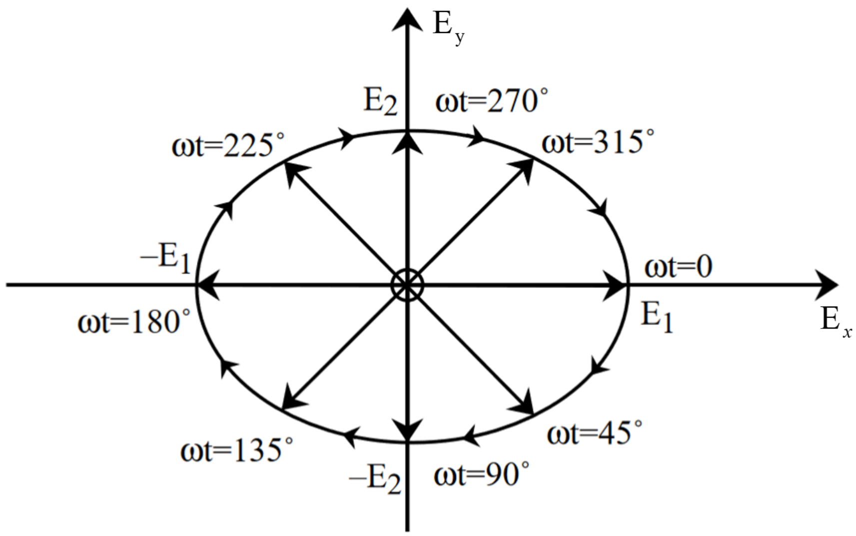

Figure 9.1: If one follows the tip of the electric field vector, it traces out an ellipse as a function of time t.

When $E_1 = E_2$, the ellipse becomes a circle, and we have a left-hand circularly polarized (LHCP) wave. When $\alpha = -\pi/2$, the wave rotates in the counter-clockwise direction, and the wave is either right-hand elliptically polarized (RHEP), or right-hand circularly polarized (RHCP) wave depending on the ratio of E_1/E_2. Figure 9.2 shows the different polarizations of the wave wave for different phase differences and amplitude ratio.

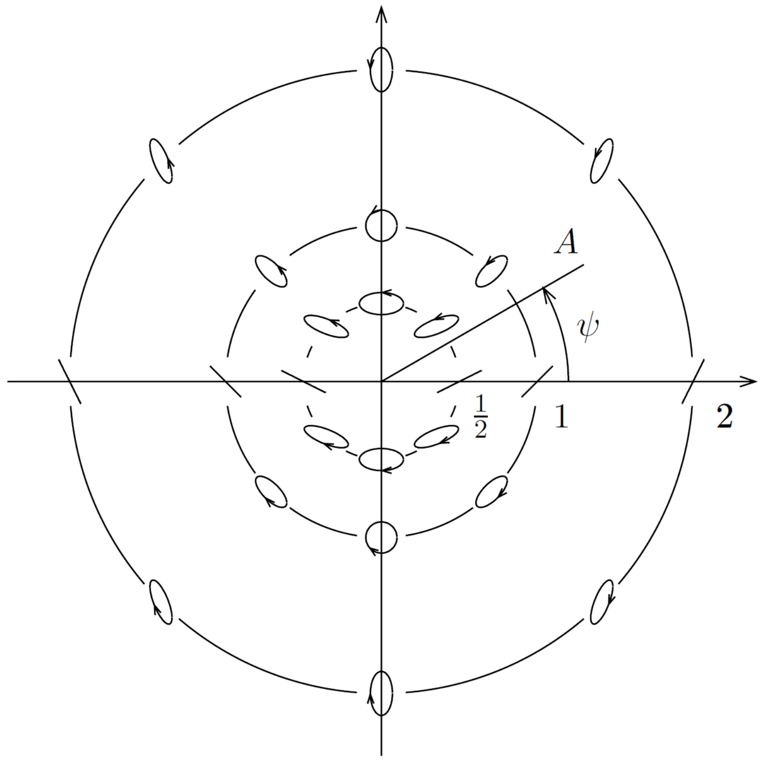

Figure 9.2: Due to different phase difference between the E_x and E_y components of the field, and their relative amplitudes E_2/E_1, different polarizations will ensure. The arrow indicates the direction of rotation of the field vector. In this figure, $\psi = -\alpha$ in our notes, and $A = E_2/E_1$ (Courtesy of J.A. Kong, Electromagnetic Wave Theory [31]).

Figure 9.3 shows a graphic picture of a CP wave propagating through space.

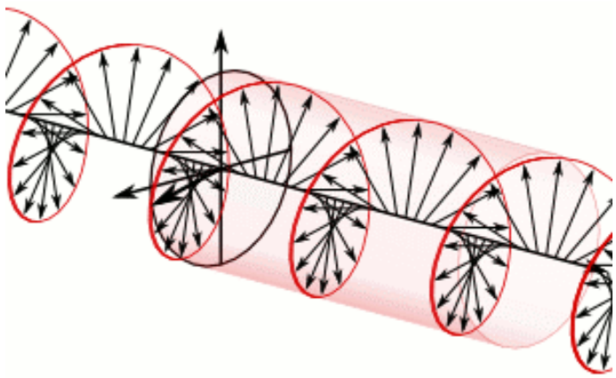

Figure 9.3: The rotation of the field vector of a right-hand circular polarization wave as it propagates in the right direction [72] (Courtesy of Wikipedia).

9.2.1 Arbitrary Polarization Case and Axial Ratio

The axial ratio (AR) is an important figure of merit for designing CP antennas (antennas that will radiate CP or circularly polarized waves). The closer is this ratio to 1, the better the antenna design. We will discuss the general polarization and the axial ratio of a wave.

For the general case for arbitrary α, we let

$$E_x = E_1 \cos\omega t,\ E_y = E_2 \cos(\omega t + \alpha) = E_2(\cos\omega t \cos\alpha - \sin\omega t \sin\alpha) \tag{9.2.11}$$

Then from the above, expressing E_y in terms of E_x, one gets

$$E_y = \frac{E_2}{E_1} E_x \cos\alpha - E_2 \left[1 - \left(\frac{E_x}{E_1}\right)^2\right]^{1/2} \sin\alpha \tag{9.2.12}$$

Rearranging and squaring, we get

$$a{E_x}^2 - bE_xE_y + c{E_y}^2 = 1 \tag{9.2.13}$$

where

$$a = \frac{1}{{E_1}^2 \sin^2\alpha}, \quad b = \frac{2\cos\alpha}{E_1E_2\sin\alpha}, \quad c = \frac{1}{{E_2}^2 \sin^2\alpha} \tag{9.2.14}$$

After letting $E_x \to x$, and $E_y \to y$, equation (9.2.13) is of the form,

$$ax^2 - bxy + cy^2 = 1 \tag{9.2.15}$$

The equation of an ellipse in its self coordinates is

$$\left(\frac{x'}{A}\right)^2 + \left(\frac{y'}{B}\right)^2 = 1 \tag{9.2.16}$$

where A and B are axes of the ellipse as shown in Figure 9.4. We can transform the above back to the (x, y) coordinates by letting

$$x' = x\cos\theta - y\sin\theta \tag{9.2.17}$$

$$y' = x\sin\theta + y\cos\theta \tag{9.2.18}$$

to get

$$x^2\left(\frac{\cos^2\theta}{A^2} + \frac{\sin^2\theta}{B^2}\right) - xy\sin 2\theta\left(\frac{1}{A^2} - \frac{1}{B^2}\right) + y^2\left(\frac{\sin^2\theta}{A^2} + \frac{\cos^2\theta}{B^2}\right) = 1 \tag{9.2.19}$$

Comparing (9.2.13) and (9.2.19), one gets

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\cos\alpha E_1 E_2}{{E_2}^2 - {E_1}^2}\right) \tag{9.2.20}$$

$$\mathrm{AR} = \left(\frac{1+\Delta}{1-\Delta}\right)^{1/2} > 1 \tag{9.2.21}$$

where AR is the axial ratio where

$$\Delta = \left(1 - \frac{4{E_1}^2{E_2}^2\sin^2\alpha}{\left({E_1}^2 + {E_2}^2\right)^2}\right)^{1/2} \tag{9.2.22}$$

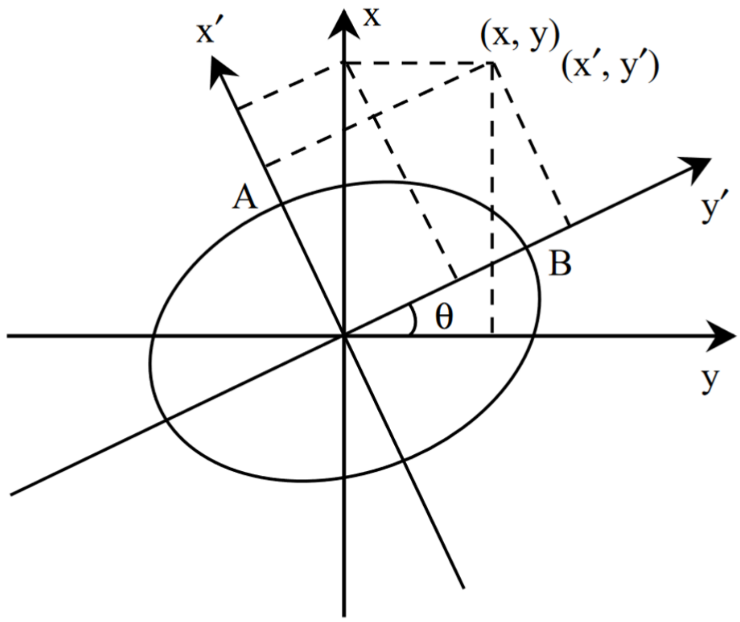

Figure 9.4: This figure shows the parameters used to derive the axial ratio (AR) of an elliptically polarized wave.

9.3 Polarization and Power Flow

For a linearly polarized wave,

$$\mathbf{E} = \hat{x} E_0 \cos(\omega t - \beta z), \quad \mathbf{H} = \hat{y} \frac{E_0}{\eta} \cos(\omega t - \beta z) \tag{9.3.1}$$

Hence, the instantaneous power becomes

$$\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t) = \hat{z} \frac{{E_0}^2}{\eta} \cos^2(\omega t - \beta z) \tag{9.3.2}$$

indicating that for a linearly polarized wave, the instantaneous power is function of both time and space. It travels as lumps of energy through space. In the above E_0 is the amplitude of the linearly polarized wave.

Next, we look at power flow for for elliptically and circularly polarized waves. It is to be noted that in the phasor world or frequency domain, (9.2.1) becomes

$$\mathbf{E}(z, \omega) = \hat{x} E_1 e^{-j\beta z} + \hat{y} E_2 e^{-j\beta z + j\alpha} \tag{9.3.3}$$

For LHEP,

$$\mathbf{E}(z,\omega) = e^{-j\beta z}(\hat{x}E_1 + j\hat{y}E_2) \tag{9.3.4}$$

whereas for LHCP

$$\mathbf{E}(z,\omega) = e^{-j\beta z}E_1(\hat{x} + j\hat{y}) \tag{9.3.5}$$

For RHEP, the above becomes

$$\mathbf{E}(z,\omega) = e^{-j\beta z}(\hat{x}E_1 - j\hat{y}E_2) \tag{9.3.6}$$

whereas for RHCP, it is

$$\mathbf{E}(z,\omega) = e^{-j\beta z}E_1(\hat{x} - j\hat{y}) \tag{9.3.7}$$

Focussing on the circularly polarized wave,

$$\mathbf{E} = (\hat{x} \pm j\hat{y})E_0 e^{-j\beta z} \tag{9.3.8}$$

Using that

$$\mathbf{H} = \frac{\boldsymbol{\beta} \times \mathbf{E}}{\omega\mu},$$

then

$$\mathbf{H} = (\mp\hat{x} - j\hat{y})j\frac{E_0}{\eta}e^{-j\beta z} \tag{9.3.9}$$

Therefore,

$$\mathbf{E}(t) = \hat{x}E_0\cos(\omega t - \beta z) \pm \hat{y}E_0\sin(\omega t - \beta z) \tag{9.3.10}$$

$$\mathbf{H}(t) = \mp\hat{x}\frac{E_0}{\eta}\sin(\omega t - \beta z) + \hat{y}\frac{E_0}{\eta}\cos(\omega t - \beta z) \tag{9.3.11}$$

Then the instantaneous power becomes

$$\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t) = \hat{z}\frac{{E_0}^2}{\eta}\cos^2(\omega t - \beta z) + \hat{z}\frac{{E_0}^2}{\eta}\sin^2(\omega t - \beta z) = \hat{z}\frac{{E_0}^2}{\eta} \tag{9.3.12}$$

In other words, a CP wave delivers constant power independent of space and time.

It is to be noted that the complex Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}^* \tag{9.3.13}$$

are real both for linearly, circularly, and elliptically polarized waves. This is because there is no reactive power in a plane wave of any polarization: the stored energy in the plane wave cannot be returned to the source!

Bibliography

[1] J. A. Kong, *Theory of electromagnetic waves.* New York, Wiley-Interscience, 1975.

[2] A. Einstein *et al.*, "On the electrodynamics of moving bodies," *Annalen der Physik*, vol. 17, no. 891, p. 50, 1905.

[3] P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation," *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 114, no. 767, pp. 243–265, 1927.

[4] R. J. Glauber, "Coherent and incoherent states of the radiation field," *Physical Review*, vol. 131, no. 6, p. 2766, 1963.

[5] C.-N. Yang and R. L. Mills, "Conservation of isotopic spin and isotopic gauge invariance," *Physical review*, vol. 96, no. 1, p. 191, 1954.

[6] G. t'Hooft, *50 years of Yang-Mills theory.* World Scientific, 2005.

[7] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* Princeton University Press, 2017.

[8] F. Teixeira and W. C. Chew, "Differential forms, metrics, and the reflectionless absorption of electromagnetic waves," *Journal of Electromagnetic Waves and Applications*, vol. 13, no. 5, pp. 665–686, 1999.

[9] W. C. Chew, E. Michielssen, J.-M. Jin, and J. Song, *Fast and efficient algorithms in computational electromagnetics.* Artech House, Inc., 2001.

[10] A. Volta, "On the electricity excited by the mere contact of conducting substances of different kinds. in a letter from Mr. Alexander Volta, FRS Professor of Natural Philosophy in the University of Pavia, to the Rt. Hon. Sir Joseph Banks, Bart. KBPR S," *Philosophical transactions of the Royal Society of London*, no. 90, pp. 403–431, 1800.

[11] A.-M. Ampère, *Exposé méthodique des phénomènes électro-dynamiques, et des lois de ces phénomènes.* Bachelier, 1823.

[12] ——, *Mémoire sur la théorie mathématique des phénomènes électro-dynamiques uniquement déduite de l'expérience: dans lequel se trouvent réunis les Mémoires que M. Ampère a communiqués à l'Académie royale des Sciences, dans les séances des 4 et 26 décembre*

1820, 10 juin 1822, 22 décembre 1823, 12 septembre et 21 novembre 1825. Bachelier, 1825.

[13] B. Jones and M. Faraday, *The life and letters of Faraday.* Cambridge University Press, 2010, vol. 2.

[14] G. Kirchhoff, "Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird," *Annalen der Physik*, vol. 148, no. 12, pp. 497–508, 1847.

[15] L. Weinberg, "Kirchhoff's' third and fourth laws'," *IRE Transactions on Circuit Theory*, vol. 5, no. 1, pp. 8–30, 1958.

[16] T. Standage, *The Victorian Internet: The remarkable story of the telegraph and the nineteenth century's online pioneers.* Phoenix, 1998.

[17] J. C. Maxwell, "A dynamical theory of the electromagnetic field," *Philosophical transactions of the Royal Society of London*, no. 155, pp. 459–512, 1865.

[18] H. Hertz, "On the finite velocity of propagation of electromagnetic actions," *Electric Waves*, vol. 110, 1888.

[19] M. Romer and I. B. Cohen, "Roemer and the first determination of the velocity of light (1676)," *Isis*, vol. 31, no. 2, pp. 327–379, 1940.

[20] A. Arons and M. Peppard, "Einstein's proposal of the photon concept–a translation of the Annalen der Physik paper of 1905," *American Journal of Physics*, vol. 33, no. 5, pp. 367–374, 1965.

[21] A. Pais, "Einstein and the quantum theory," *Reviews of Modern Physics*, vol. 51, no. 4, p. 863, 1979.

[22] M. Planck, "On the law of distribution of energy in the normal spectrum," *Annalen der physik*, vol. 4, no. 553, p. 1, 1901.

[23] Z. Peng, S. De Graaf, J. Tsai, and O. Astafiev, "Tuneable on-demand single-photon source in the microwave range," *Nature communications*, vol. 7, p. 12588, 2016.

[24] B. D. Gates, Q. Xu, M. Stewart, D. Ryan, C. G. Willson, and G. M. Whitesides, "New approaches to nanofabrication: molding, printing, and other techniques," *Chemical reviews*, vol. 105, no. 4, pp. 1171–1196, 2005.

[25] J. S. Bell, "The debate on the significance of his contributions to the foundations of quantum mechanics, Bells Theorem and the Foundations of Modern Physics (A. van der Merwe, F. Selleri, and G. Tarozzi, eds.)," 1992.

[26] D. J. Griffiths and D. F. Schroeter, *Introduction to quantum mechanics.* Cambridge University Press, 2018.

[27] C. Pickover, *Archimedes to Hawking: Laws of science and the great minds behind them.* Oxford University Press, 2008.

[28] R. Resnick, J. Walker, and D. Halliday, *Fundamentals of physics.* John Wiley, 1988.

[29] S. Ramo, J. R. Whinnery, and T. Duzer van, *Fields and waves in communication electronics, Third Edition.* John Wiley & Sons, Inc., 1995.

[30] J. L. De Lagrange, "Recherches d'arithmétique," *Nouveaux Mémoires de l'Académie de Berlin*, 1773.

[31] J. A. Kong, *Electromagnetic Wave Theory.* EMW Publishing, 2008.

[32] H. M. Schey, *Div, grad, curl, and all that: an informal text on vector calculus.* WW Norton New York, 2005.

[33] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics, Vols. I, II, & III: The new millennium edition.* Basic books, 2011, vol. 1,2,3.

[34] W. C. Chew, *Waves and fields in inhomogeneous media.* IEEE press, 1995.

[35] V. J. Katz, "The history of Stokes' theorem," *Mathematics Magazine*, vol. 52, no. 3, pp. 146–156, 1979.

[36] W. K. Panofsky and M. Phillips, *Classical electricity and magnetism.* Courier Corporation, 2005.

[37] T. Lancaster and S. J. Blundell, *Quantum field theory for the gifted amateur.* OUP Oxford, 2014.

[38] W. C. Chew, "Fields and waves: Lecture notes for ECE 350 at UIUC," *https://engineering.purdue.edu/wcchew/ece350.html*, 1990.

[39] C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory.* Springer Science & Business Media, 2013.

[40] J. M. Crowley, *Fundamentals of applied electrostatics.* Krieger Publishing Company, 1986.

[41] C. Balanis, *Advanced Engineering Electromagnetics.* Hoboken, NJ, USA: Wiley, 2012.

[42] J. D. Jackson, *Classical electrodynamics.* John Wiley & Sons, 1999.

[43] R. Courant and D. Hilbert, *Methods of Mathematical Physics: Partial Differential Equations.* John Wiley & Sons, 2008.

[44] L. Esaki and R. Tsu, "Superlattice and negative differential conductivity in semiconductors," *IBM Journal of Research and Development*, vol. 14, no. 1, pp. 61–65, 1970.

[45] E. Kudeki and D. C. Munson, *Analog Signals and Systems.* Upper Saddle River, NJ, USA: Pearson Prentice Hall, 2009.

[46] A. V. Oppenheim and R. W. Schafer, *Discrete-time signal processing.* Pearson Education, 2014.

[47] R. F. Harrington, *Time-harmonic electromagnetic fields.* McGraw-Hill, 1961.

[48] E. C. Jordan and K. G. Balmain, *Electromagnetic waves and radiating systems.* Prentice-Hall, 1968.

[49] G. Agarwal, D. Pattanayak, and E. Wolf, "Electromagnetic fields in spatially dispersive media," *Physical Review B*, vol. 10, no. 4, p. 1447, 1974.

[50] S. L. Chuang, *Physics of photonic devices.* John Wiley & Sons, 2012, vol. 80.

[51] B. E. Saleh and M. C. Teich, *Fundamentals of photonics.* John Wiley & Sons, 2019.

[52] M. Born and E. Wolf, *Principles of optics: electromagnetic theory of propagation, interference and diffraction of light.* Elsevier, 2013.

[53] R. W. Boyd, *Nonlinear optics.* Elsevier, 2003.

[54] Y.-R. Shen, *The principles of nonlinear optics.* New York, Wiley-Interscience, 1984.

[55] N. Bloembergen, *Nonlinear optics.* World Scientific, 1996.

[56] P. C. Krause, O. Wasynczuk, and S. D. Sudhoff, *Analysis of electric machinery.* McGraw-Hill New York, 1986.

[57] A. E. Fitzgerald, C. Kingsley, S. D. Umans, and B. James, *Electric machinery.* McGraw-Hill New York, 2003, vol. 5.

[58] M. A. Brown and R. C. Semelka, *MRI.: Basic Principles and Applications.* John Wiley & Sons, 2011.

[59] C. A. Balanis, *Advanced engineering electromagnetics.* John Wiley & Sons, 1999.

[60] Wikipedia, "Lorentz force," https://en.wikipedia.org/wiki/Lorentz_force/, accessed: 2019-09-06.

[61] R. O. Dendy, *Plasma physics: an introductory course.* Cambridge University Press, 1995.

[62] P. Sen and W. C. Chew, "The frequency dependent dielectric and conductivity response of sedimentary rocks," *Journal of microwave power*, vol. 18, no. 1, pp. 95–105, 1983.

[63] D. A. Miller, *Quantum Mechanics for Scientists and Engineers.* Cambridge, UK: Cambridge University Press, 2008.

[64] W. C. Chew, "Quantum mechanics made simple: Lecture notes for ECE 487 at UIUC," *http://wcchew.ece.illinois.edu/chew/course/QMAll20161206.pdf*, 2016.

[65] B. G. Streetman and S. Banerjee, *Solid state electronic devices.* Prentice hall Englewood Cliffs, NJ, 1995.

[66] Smithsonian, "This 1600-year-old goblet shows that the romans were nanotechnology pioneers," https://www.smithsonianmag.com/history/this-1600-year-old-goblet-shows-that-the-romans-were-nanotechnology-pioneers-787224/, accessed: 2019-09-06.

[67] K. G. Budden, *Radio waves in the ionosphere.* Cambridge University Press, 2009.

[68] R. Fitzpatrick, *Plasma physics: an introduction.* CRC Press, 2014.

[69] G. Strang, *Introduction to linear algebra.* Wellesley-Cambridge Press Wellesley, MA, 1993, vol. 3.

[70] K. C. Yeh and C.-H. Liu, "Radio wave scintillations in the ionosphere," *Proceedings of the IEEE*, vol. 70, no. 4, pp. 324–360, 1982.

[71] J. Kraus, *Electromagnetics.* McGraw-Hill, 1984.

[72] Wikipedia, "Circular polarization," https://en.wikipedia.org/wiki/Circular_polarization.

[73] Q. Zhan, "Cylindrical vector beams: from mathematical concepts to applications," *Advances in Optics and Photonics*, vol. 1, no. 1, pp. 1–57, 2009.

[74] H. Haus, *Electromagnetic Noise and Quantum Optical Measurements*, ser. Advanced Texts in Physics. Springer Berlin Heidelberg, 2000.

[75] W. C. Chew, "Lectures on theory of microwave and optical waveguides, for ECE 531 at UIUC," *https://engineering.purdue.edu/wcchew/course/tgwAll20160215.pdf*, 2016.

[76] L. Brillouin, *Wave propagation and group velocity.* Academic Press, 1960.

[77] M. N. Sadiku, *Elements of electromagnetics.* Oxford University Press, 2014.

[78] A. Wadhwa, A. L. Dal, and N. Malhotra, "Transmission media," https://www.slideshare.net/abhishekwadhwa786/transmission-media-9416228.

[79] P. H. Smith, "Transmission line calculator," *Electronics*, vol. 12, no. 1, pp. 29–31, 1939.

[80] F. B. Hildebrand, *Advanced calculus for applications.* Prentice-Hall, 1962.

[81] J. Schutt-Aine, "Experiment02-coaxial transmission line measurement using slotted line," http://emlab.uiuc.edu/ece451/ECE451Lab02.pdf.

[82] D. M. Pozar, E. J. K. Knapp, and J. B. Mead, "ECE 584 microwave engineering laboratory notebook," http://www.ecs.umass.edu/ece/ece584/ECE584_lab_manual.pdf, 2004.

[83] R. E. Collin, *Field theory of guided waves.* McGraw-Hill, 1960.