Lecture 6

Time-Harmonic Fields, Complex Power and Poynting's Theorem

6.1 Time-Harmonic Fields—Linear Systems

The analysis of Maxwell's equations can be greatly simplified by assuming the fields to be time harmonic, or sinusoidal (cosinusoidal). Electrical engineers use a method called phasor technique [31, 45], to simplify equations involving time-harmonic signals. This is also a poor-man's Fourier transform [46]. That is one begets the benefits of Fourier transform technique without knowledge of Fourier transform. Since only one time-harmonic frequency is involved, this is also called frequency domain analysis.[1]
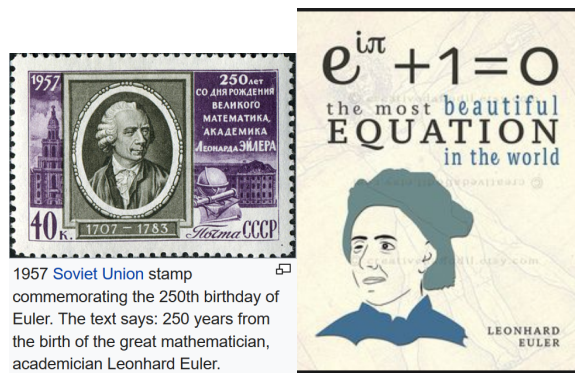

1957 Soviet Union stamp commemorating the 250th birthday of Euler. The text says: 250 years from the birth of the great mathematician, academician Leonhard Euler.

Figure 6.1: Courtesy of Wikipedia and Pinterest.

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[1]It is simple only for linear systems: for nonlinear systems, such analysis can be quite unwieldy. But rest assured, as we will not discuss nonlinear systems in this course.

To learn phasor techniques, one makes use of the formula due to Euler (1707–1783) (Wikipedia)

$$e^{j\alpha} = \cos\alpha + j\sin\alpha \tag{6.1.1}$$

where $j = \sqrt{-1}$ is an imaginary number. But lo and behold, in other disciplines, $\sqrt{-1}$ is denoted by "i", but "i" is too close to the symbol for current. So the preferred symbol for electrical engineering for an imaginary number is j: a quirkness of convention, just as positive charges do not carry current in a wire.

From Euler's formula one gets

$$\cos\alpha = \Re e(e^{j\alpha}) \tag{6.1.2}$$

Hence, all time harmonic quantity can be written as

$$V(x, y, z, t) = V'(x, y, z)\cos(\omega t + \alpha) \tag{6.1.3}$$
$$= V'(\mathbf{r})\Re e(e^{j(\omega t + \alpha)}) \tag{6.1.4}$$
$$= \Re e\left(V'(\mathbf{r})e^{j\alpha}e^{j\omega t}\right) \tag{6.1.5}$$
$$= \Re e\left(\underset{\sim}{V}(\mathbf{r})e^{j\omega t}\right) \tag{6.1.6}$$

Now $\underset{\sim}{V}(\mathbf{r}) = V'(\mathbf{r})e^{j\alpha}$ is a complex number called the phasor representation or phasor of $V(\mathbf{r}, t)$ a time-harmonic quantity.[2] Here, the phase $\alpha = \alpha(\mathbf{r})$ can also be a function of position $\mathbf{r}$, or x, y, z. Consequently, any component of a field can be expressed as

$$E_x(x, y, z, t) = E_x(\mathbf{r}, t) = \Re e[\underset{\sim}{E_x}(\mathbf{r})e^{j\omega t}] \tag{6.1.7}$$

The above can be repeated for y and z components. Compactly, one can write

$$\mathbf{E}(\mathbf{r}, t) = \Re e[\underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}] \tag{6.1.8}$$

$$\mathbf{H}(\mathbf{r}, t) = \Re e[\underset{\sim}{\mathbf{H}}(\mathbf{r})e^{j\omega t}] \tag{6.1.9}$$

where $\underset{\sim}{\mathbf{E}}$ and $\underset{\sim}{\mathbf{H}}$ are complex vector fields. Such phasor representations of time-harmonic fields simplify Maxwell's equations. For instance, if one writes

$$\mathbf{B}(\mathbf{r}, t) = \Re e\left(\underset{\sim}{\mathbf{B}}(\mathbf{r})e^{j\omega t}\right) \tag{6.1.10}$$

then

$$\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t) = \frac{\partial}{\partial t}\Re e[\underset{\sim}{\mathbf{B}}(\mathbf{r})e^{j\omega t}]$$
$$= \Re e\left(\frac{\partial}{\partial t}\underset{\sim}{\mathbf{B}}(\mathbf{r})j\omega e^{j\omega t}\right)$$
$$= \Re e\left(\underset{\sim}{\mathbf{B}}(\mathbf{r})j\omega e^{j\omega t}\right) \tag{6.1.11}$$

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[2]We will use under tilde to denote a complex number or a phasor here, but this notation will be dropped later. Whether a variable is complex or real is clear from the context.

Therefore, a time derivative can be effected very simply for a time-harmonic field. One just needs to multiply $j\omega$ to the phasor representation of a field or a signal. Therefore, given Faraday's law that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{M} \tag{6.1.12}$$

assuming that all quantities are time harmonic, then

$$\mathbf{E}(\mathbf{r}, t) = \Re e[\underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}] \tag{6.1.13}$$

$$\mathbf{M}(\mathbf{r}, t) = \Re e[\underset{\sim}{\mathbf{M}}(\mathbf{r})e^{j\omega t}] \tag{6.1.14}$$

using (6.1.11), and (6.1.14), into (6.1.12), one gets

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = \Re e[\nabla \times \underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}] \tag{6.1.15}$$

and that

$$\Re e[\nabla \times \underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}] = -\Re e[\underset{\sim}{\mathbf{B}}(\mathbf{r})j\omega e^{j\omega t}] - \Re e[\underset{\sim}{\mathbf{M}}(\mathbf{r})e^{j\omega t}] \tag{6.1.16}$$

Since if

$$\Re e[Ae^{j\omega t}] = \Re e[B(\mathbf{r})e^{j\omega t}], \quad \forall t \tag{6.1.17}$$

then $A = B$, it must be true from (6.1.16) that

$$\nabla \times \underset{\sim}{\mathbf{E}}(\mathbf{r}) = -j\omega \underset{\sim}{\mathbf{B}}(\mathbf{r}) - \underset{\sim}{\mathbf{M}}(\mathbf{r}) \tag{6.1.18}$$

Hence, finding the phasor representation of an equation is clear: whenever we have $\frac{\partial}{\partial t}$, we replace it by $j\omega$. Applying this methodically to the other Maxwell's equations, we have

$$\nabla \times \underset{\sim}{\mathbf{H}}(\mathbf{r}) = j\omega \underset{\sim}{\mathbf{D}}(\mathbf{r}) + \underset{\sim}{\mathbf{J}}(\mathbf{r}) \tag{6.1.19}$$

$$\nabla \cdot \underset{\sim}{\mathbf{D}}(\mathbf{r}) = \underset{\sim}{\varrho_e}(\mathbf{r}) \tag{6.1.20}$$

$$\nabla \cdot \underset{\sim}{\mathbf{B}}(\mathbf{r}) = \underset{\sim}{\varrho_m}(\mathbf{r}) \tag{6.1.21}$$

In the above, the phasors are functions of frequency. For instance, $\underset{\sim}{\mathbf{H}}(\mathbf{r})$ should rightly be written as $\underset{\sim}{\mathbf{H}}(\mathbf{r}, \omega)$, but the ω dependence is implied.

6.2 Fourier Transform Technique

In the phasor representation, Maxwell's equations has no time derivatives; hence the equations are simplified. We can also arrive at the above simplified equations using Fourier transform

technique. To this end, we use Faraday's law as an example. By letting

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) e^{j\omega t} d\omega \qquad (6.2.1)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{B}(\mathbf{r}, \omega) e^{j\omega t} d\omega \qquad (6.2.2)$$

$$\mathbf{M}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{M}(\mathbf{r}, \omega) e^{j\omega t} d\omega \qquad (6.2.3)$$

Substituting the above into Faraday's law given by (6.1.12), we get

$$\nabla \times \int_{-\infty}^{\infty} d\omega e^{j\omega t} \mathbf{E}(\mathbf{r}, \omega) = -\frac{\partial}{\partial t} \int_{-\infty}^{\infty} d\omega e^{j\omega t} \mathbf{B}(\mathbf{r}, \omega) - \int_{-\infty}^{\infty} d\omega e^{j\omega t} \mathbf{M}(\mathbf{r}, \omega) \qquad (6.2.4)$$

Using the fact that

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} d\omega e^{j\omega t} \mathbf{B}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} d\omega \frac{\partial}{\partial t} e^{j\omega t} \mathbf{B}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} d\omega e^{j\omega t} j\omega \mathbf{B}(\mathbf{r}, \omega) \qquad (6.2.5)$$

and that

$$\nabla \times \int_{-\infty}^{\infty} d\omega e^{j\omega t} \mathbf{E}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} d\omega e^{j\omega t} \nabla \times \mathbf{E}(\mathbf{r}, \omega) \qquad (6.2.6)$$

Furthermore, using the fact that

$$\int_{-\infty}^{\infty} d\omega e^{j\omega t} A(\omega) = \int_{-\infty}^{\infty} d\omega e^{j\omega t} B(\omega), \quad \forall t \qquad (6.2.7)$$

implies that $A(\omega) = B(\omega)$, and using (6.2.5) and (6.2.6) in (6.2.4), and the property (6.2.7), one gets

$$\nabla \times \mathbf{E}(\mathbf{r}, \omega) = -j\omega \mathbf{B}(\mathbf{r}, \omega) - \mathbf{M}(\mathbf{r}, \omega) \qquad (6.2.8)$$

These equations look exactly like the phasor equations we have derived previously, save that the field $\mathbf{E}(\mathbf{r}, \omega)$, $\mathbf{B}(\mathbf{r}, \omega)$, and $\mathbf{M}(\mathbf{r}, \omega)$ are now the Fourier transforms of the field $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$, and $\mathbf{M}(\mathbf{r}, t)$. Moreover, the Fourier transform variables can be complex just like phasors. Repeating the exercise above for the other Maxwell's equations, we obtain equations that look similar to those for their phasor representations. Hence, Maxwell's equations can be simplified either by using phasor technique or Fourier transform technique. However, the dimensions of the phasors are different from the dimensions of the Fourier-transformed fields: $\underline{\mathbf{E}}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r}, \omega)$ do not have the same dimension on closer examination.

6.3 Complex Power

Consider now that in the phasor representations, $\underset{\sim}{\mathbf{E}}(\mathbf{r})$ and $\underset{\sim}{\mathbf{H}}(\mathbf{r})$ are complex vectors, and their cross product, $\underset{\sim}{\mathbf{E}}(\mathbf{r}) \times \underset{\sim}{\mathbf{H}}^*(\mathbf{r})$, which still has the unit of power density, has a different physical meaning. First, consider the instantaneous Poynting's vector

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \tag{6.3.1}$$

where all the quantities are real valued. Now, we can use phasor technique to analyze the above. Assuming time-harmonic fields, the above can be rewritten as

$$\mathbf{S}(\mathbf{r}, t) = \Re e[\underset{\sim}{\mathbf{E}}(\mathbf{r})e^{j\omega t}] \times \Re e[\underset{\sim}{\mathbf{H}}(\mathbf{r})e^{j\omega t}]$$

$$= \frac{1}{2}[\underset{\sim}{\mathbf{E}}e^{j\omega t} + (\underset{\sim}{\mathbf{E}}e^{j\omega t})^*] \times \frac{1}{2}[\underset{\sim}{\mathbf{H}}e^{j\omega t} + (\underset{\sim}{\mathbf{H}}e^{j\omega t})^*] \tag{6.3.2}$$

where we have made use of the formula that

$$\Re e(Z) = \frac{1}{2}(Z + Z^*) \tag{6.3.3}$$

Then more elaborately, on expanding (6.3.2), we get

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{4}\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}e^{2j\omega t} + \frac{1}{4}\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}^* + \frac{1}{4}\underset{\sim}{\mathbf{E}}^* \times \underset{\sim}{\mathbf{H}} + \frac{1}{4}\underset{\sim}{\mathbf{E}}^* \times \underset{\sim}{\mathbf{H}}^* e^{-2j\omega t} \tag{6.3.4}$$

Then rearranging terms and using (6.3.3) yield

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{2}\Re e[\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}^*] + \frac{1}{2}\Re e[\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}e^{2j\omega t}] \tag{6.3.5}$$

where the first term is independent of time, while the second term is sinusoidal in time. If we define a time-average quantity such that

$$\mathbf{S}_{av} = \langle \mathbf{S}(\mathbf{r}, t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{S}(\mathbf{r}, t)dt \tag{6.3.6}$$

then it is quite clear that the second term of (6.3.5) time averages to zero, and

$$\mathbf{S}_{av} = \langle \mathbf{S}(\mathbf{r}, t) \rangle = \frac{1}{2}\Re e[\underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}}^*] \tag{6.3.7}$$

Hence, in the phasor representation, the quantity

$$\underset{\sim}{\mathbf{S}} = \underset{\sim}{\mathbf{E}} \times \underset{\sim}{\mathbf{H}} \tag{6.3.8}$$

is termed the complex Poynting's vector. The power flow associated with it is termed complex power.

$$V = L\frac{dI}{dt}$$

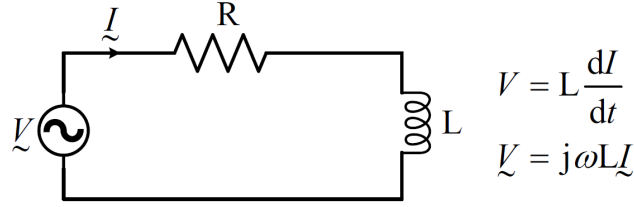$$\underset{\sim}{V} = j\omega L \underset{\sim}{I}$$

Figure 6.2:

To understand what complex power is , it is fruitful if we revisit complex power [47, 48] in our circuit theory course. The circuit in Figure 6.2 can be easily solved by using phasor technique. The impedance of the circuit is $Z = R + j\omega L$. Hence,

$$\underset{\sim}{V} = (R + j\omega L)\underset{\sim}{I} \tag{6.3.9}$$

where $\underset{\sim}{V}$ and $\underset{\sim}{I}$ are the phasors of the voltage and current for time-harmonic signals. Just as in the electromagnetic case, the complex power is taken to be

$$\underset{\sim}{P} = \underset{\sim}{V}\underset{\sim}{I}^* \tag{6.3.10}$$

But the instantaneous power is given by

$$P_{inst}(t) = V(t)I(t) \tag{6.3.11}$$

where $V(t) = \Re e\{\underset{\sim}{V}e^{j\omega t}\}$ and $I(t) = \Re e\{\underset{\sim}{I}e^{j\omega t}\}$. As shall be shown below,

$$P_{av} = \langle P_{inst}(t)\rangle = \frac{1}{2}\Re e[\underset{\sim}{P}] \tag{6.3.12}$$

It is clear that if $V(t)$ is sinusoidal, it can be written as

$$V(t) = V_0 \cos(\omega t) = \Re e[\underset{\sim}{V}e^{j\omega t}] \tag{6.3.13}$$

where, without loss of generality, we assume that $\underset{\sim}{V} = V_0$. Then from (6.3.9), it is clear that $V(t)$ and $I(t)$ are not in phase. Namely that

$$I(t) = I_0 \cos(\omega t + \alpha) = \Re e[\underset{\sim}{I}e^{j\omega t}] \tag{6.3.14}$$

where $\underset{\sim}{I} = I_0 e^{j\alpha}$. Then

$$\begin{aligned} P_{inst}(t) &= V_0 I_0 \cos(\omega t)\cos(\omega t + \alpha) \\ &= V_0 I_0 \cos(\omega t)[\cos(\omega t)\cos(\alpha) - \sin(\omega t)\sin\alpha] \\ &= V_0 I_0 \cos^2(\omega t)\cos\alpha - V_0 I_0 \cos(\omega t)\sin(\omega t)\sin\alpha \end{aligned} \tag{6.3.15}$$

It can be seen that the first term does not time-average to zero, but the second term does. Now taking the time average of (6.3.15), we get

$$P_{av} = \langle P_{inst} \rangle = \frac{1}{2} V_0 I_0 \cos \alpha = \frac{1}{2} \Re e[\underset{\sim}{V} \underset{\sim}{I}^*] \tag{6.3.16}$$

$$= \frac{1}{2} \Re e[\underset{\sim}{P}] \tag{6.3.17}$$

On the other hand, the reactive power

$$P_{reactive} = \frac{1}{2} \Im m[\underset{\sim}{P}] = \frac{1}{2} \Im m[\underset{\sim}{V} \underset{\sim}{I}^*] = \frac{1}{2} \Im m[V_0 I_0 e^{-j\alpha}] = -\frac{1}{2} V_0 I_0 \sin \alpha \tag{6.3.18}$$

One sees that amplitude of the time-varying term in (6.3.15) is precisely proportional to $\Im m[\underset{\sim}{P}]$.[3]

The reason for the existence of imaginary part of $\underset{\sim}{P}$ is because $V(t)$ and $I(t)$ are out of phase or $\underset{\sim}{V} = V_0$, but $\underset{\sim}{I} = I_0 e^{j\alpha}$. The reason why they are out of phase is because the circuit has a reactive part to it. Hence the imaginary part of complex power is also called the reactive power [34,47,48]. In a reactive circuit, the plot of the instantaneous power is shown in Figure 6.3. The reactive power corresponds to part of the instantaneous power that time averages to zero. This part is there when $\alpha \neq 0$ or when a reactive component like an inductor or capacitor exists in the circuit. When a power company delivers power to our home, the power is complex because the current and voltage are not in phase. Even though the reactive power time averages to zero, the power company still needs to deliver it to our home to run our washing machine, dish washer, fans, and air conditioner etc, and hence, charges us for it.
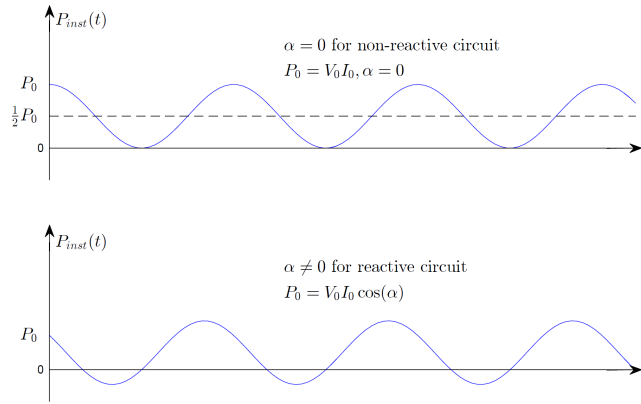

Figure 6.3:

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[3]Because that complex power is proportional to $\underset{\sim}{V}\underset{\sim}{I}^*$, it is the relative phase between $\underset{\sim}{V}$ and $\underset{\sim}{I}$ that matters. Therefore, α above is the relative phase between the phasor current and phasor voltage.

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