# Lecture 4

# Magnetostatics, Boundary Conditions, and Jump Conditions

## 4.1 Magnetostatics

The magnetostatic equations where $\partial/\partial t = 0$ are [29,31,40]

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{4.1.1}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{4.1.2}$$

One way to satisfy the second equation is to let

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4.1.3}$$

because

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{4.1.4}$$

The above is zero for the same reason that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. In this manner, Gauss's law is automatically satisfied.

From (4.1.1), we have

$$\nabla \times \left( \frac{\mathbf{B}}{\mu} \right) = \mathbf{J} \tag{4.1.5}$$

Then using (4.1.3)

$$\nabla \times \left( \frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J} \tag{4.1.6}$$

In a homogeneous medium, $\mu$ is a constant and hence

$$\nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{J} \tag{4.1.7}$$

We use the vector identity that (see previous lecture)

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A} \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \end{aligned} \tag{4.1.8}$$

As a result, we arrive at [41]

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} \tag{4.1.9}$$

By imposing the Coulomb's gauge that $\nabla \cdot \mathbf{A} = 0$, which will be elaborated in the next section, we arrive at

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \tag{4.1.10}$$

The above is also known as the vector Poisson's equation. In cartesian coordinates, the above can be viewed as three scalar Poisson's equations. Each of the Poisson's equation can be solved using the Green's function method previously described. Consequently, in free space

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r}')}{R} dV' \tag{4.1.11}$$

where

$$R = |\mathbf{r} - \mathbf{r}'| \tag{4.1.12}$$

and $dV' = dx'dy'dz'$. It is also variously written as $d\mathbf{r}'$ or $d^3\mathbf{r}'$.

### 4.1.1 More on Coulomb's Gauge

Gauge is a very important concept in physics [42], and we will further elaborate it here. First, notice that $\mathbf{A}$ in (4.1.3) is not unique because one can always define

$$\mathbf{A}' = \mathbf{A} - \nabla \Psi \tag{4.1.13}$$

Then

$$\nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} - \nabla \Psi) = \nabla \times \mathbf{A} = \mathbf{B} \tag{4.1.14}$$

where we have made use of that $\nabla \times \nabla \Psi = 0$. Hence, the $\nabla \times$ of both $\mathbf{A}$ and $\mathbf{A}'$ produce the same $\mathbf{B}$.

To find $\mathbf{A}$ uniquely, we have to define or set the divergence of $\mathbf{A}$ or provide a gauge condition. One way is to set the divergence of $\mathbf{A}$ to be zero, namely

$$\nabla \cdot \mathbf{A} = 0 \tag{4.1.15}$$

Then

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} - \nabla^2 \Psi \neq \nabla \cdot \mathbf{A} \tag{4.1.16}$$

The last non-equal sign follows if $\nabla^2 \Psi \neq 0$. However, if we further stipulate that $\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} = 0$, then $-\nabla^2 \Psi = 0$. This does not necessary imply that $\Psi = 0$, but if we impose that condition that $\Psi \to 0$ when $\mathbf{r} \to \infty$, then $\Psi = 0$ everywhere.[1] By so doing, $\mathbf{A}$ and $\mathbf{A}'$ are equal to each other, and we obtain (4.1.10) and (4.1.11).

## 4.2 Boundary Conditions–1D Poisson's Equation

Boundary conditions are embedded in the partial differential equations that the potential or the field satisfy. Two important concepts to keep in mind are:

- Differentiation of a function with discontinuous slope will give rise to step discontinuity.
- Differentiation of a function with step discontinuity will give rise to a Dirac delta function. This is also called the jump condition, a term often used by the mathematics community [43].

Take for example a one dimensional Poisson's equation that

$$\frac{d}{dx} \varepsilon(x) \frac{d}{dx} \Phi(x) = -\varrho(x) \tag{4.2.1}$$

where $\varepsilon(x)$ represents material property that has the form given in Figure 4.1. One can actually say a lot about $\Phi(x)$ given $\varrho(x)$ on the right-hand side. If $\varrho(x)$ has a delta function singularity, it implies that $\varepsilon(x) \frac{d}{dx} \Phi(x)$ has a step discontinuity. If $\varrho(x)$ is finite everywhere, then $\varepsilon(x) \frac{d}{dx} \Phi(x)$ must be continuous everywhere.

Furthermore, if $\varepsilon(x) \frac{d}{dx} \Phi(x)$ is finite everywhere, it implies that $\Phi(x)$ must be continuous everywhere.

[1]It is a property of the Laplace boundary value problem that if $\Psi = 0$ on a closed surface $S$, then $\Psi = 0$ everywhere inside $S$. Earnshaw's theorem is useful for proving this assertion.

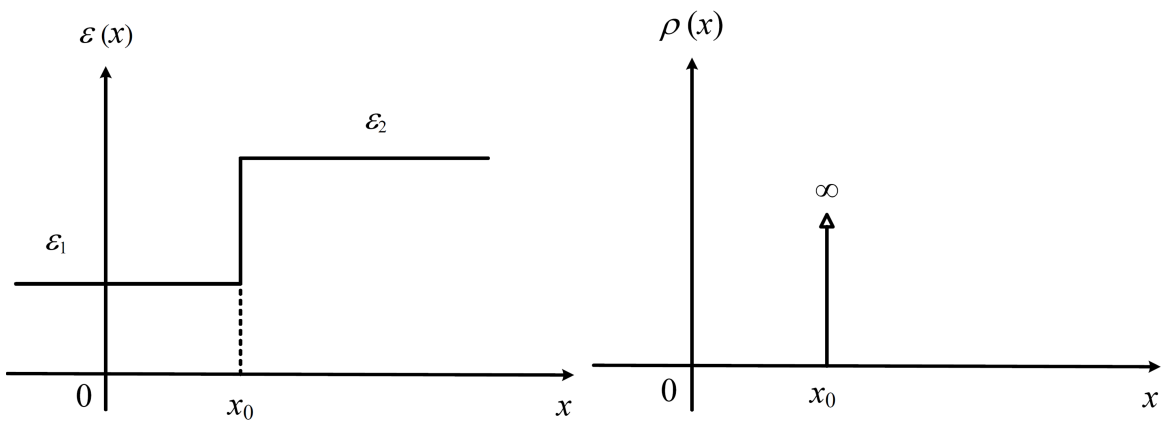


Figure 4.1: A figure showing a charge sheet at the interface between two dielectric media. Because it is a surface charge sheet, the volume charge density $\varrho(x)$ is infinite at the sheet location $x_0$.

To see this in greater detail, we illustrate it with the following example. In the above, $\varrho(x)$ represents a charge distribution given by $\varrho(x) = \varrho_s \delta(x - x_0)$. In this case, the charge distribution is everywhere zero except at the location of the surface charge sheet, where the charge density is infinite: it is represented mathematically by a delta function[2] in space.

To find the boundary condition of the potential $\Phi(x)$ at $x_0$ , we integrate (4.2.1) over an infinitesimal width around $x_0$, the location of the charge sheet, namely

$$\int_{x_0-\Delta}^{x_0+\Delta} dx \frac{d}{dx}\varepsilon(x)\frac{d}{dx}\Phi(x) = -\int_{x_0-\Delta}^{x_0+\Delta} dx \varrho(x) \tag{4.2.2}$$

or on the left-hand side, we get

$$\varepsilon(x)\frac{d}{dx}\Phi(x)\bigg|_{x_0-\Delta}^{x_0+\Delta} \cong -\varrho_s \tag{4.2.3}$$

whereas on the right-hand side, we pick up the contribution from the delta function. Evaluating the left-hand side at their limits, one arrives at

$$\lim_{\Delta\to 0} \varepsilon(x_0^+)\frac{d}{dx}\Phi(x_0^+) - \varepsilon(x_0^-)\frac{d}{dx}\Phi(x_0^-) \cong -\varrho_s, \tag{4.2.4}$$

In other words, the jump discontinuity is in $\varepsilon(x)\frac{d}{dx}\Phi(x)$ and the amplitude of the jump discontinuity is proportional to the amplitude of the delta function.

Since $\mathbf{E} = \nabla\Phi$, or

$$E_x(x) = -\frac{d}{dx}\Phi(x), \tag{4.2.5}$$

[2]This function has been attributed to Dirac who used in pervasively, but Cauchy was aware of such a function.

The above implies that

$$\varepsilon(x_0^+)E_x(x_0^+) - \varepsilon(x_0^-)E_x(x_0^-) = \varrho_s \tag{4.2.6}$$

or

$$D_x(x_0^+) - D_x(x_0^-) = \varrho_s \tag{4.2.7}$$

where

$$D_x(x) = \varepsilon(x)E_x(x) \tag{4.2.8}$$

The lesson learned from above is that boundary condition is obtained by integrating the pertinent differential equation over an infinitesimal small segment. In this mathematical way of looking at the boundary condition, one can also eyeball the differential equation and ascertain the terms that will have the jump discontinuity that will yield the delta function on the right-hand side.

## 4.3 Boundary Conditions–Maxwell's Equations

As seen previously, boundary conditions for a field is embedded in the differential equation that the field satisfies. Hence, boundary conditions can be derived from the differential operator forms of Maxwell's equations. In most textbooks, boundary conditions are obtained by integrating Maxwell's equations over a small pill box [29,31,41]. To derive these boundary conditions, we will take an unconventional view: namely to see what sources can induce jump conditions on the pertinent fields. Boundary conditions are needed at media interfaces, as well as across current or charge sheets.

### 4.3.1 Faraday's Law

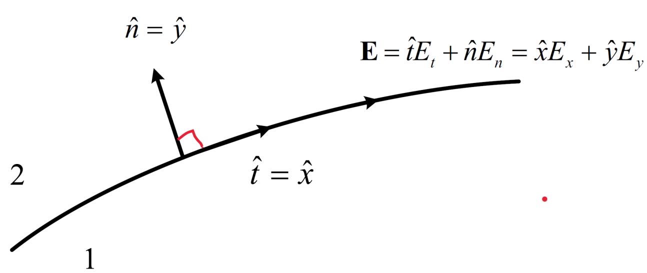


Figure 4.2: This figure is for the derivation of Faraday's law. A local coordinate system can be used to see the boundary condition more lucidly. Here, the normal $\hat{n} = \hat{y}$ and the tangential component $\hat{t} = \hat{x}$.

For this, we start with Faraday's law, which implies that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.3.1}$$

One quick answer we could have is that if the right-hand side of the above equation is everywhere finite, then there could not be any jump discontinuity on the field **E** on the left hand side. To see this quickly, one can project the tangential field component and normal field component to a local coordinate system. In other words, one can think of $\hat{t}$ and $\hat{n}$ as the local $\hat{x}$ and $\hat{y}$ coordinates. Then writing the curl operator in this local coordinates, one gets

$$\nabla \times \mathbf{E} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right) \times (\hat{x}E_x + \hat{y}E_y) \tag{4.3.2}$$

$$= \hat{z}\frac{\partial}{\partial x}E_y - \hat{z}\frac{\partial}{\partial y}E_x \tag{4.3.3}$$

In simplifying the above, we have used the distributive property of cross product, and evaluating the cross product in cartesian coordinates. The cross product produces four terms, but only two of the four terms are non-zero as shown above.

Since the right-hand side of (4.3.1) is finite, the above implies that $\frac{\partial}{\partial x}E_y$ and $\frac{\partial}{\partial y}E_x$ have to be finite. In order words, $E_x$ is continuous in the $y$ direction and $E_y$ is continuous in the $x$ direction. Since in the local coordinate system, $E_x = E_t$, then $E_t$ is continuous across the boundary. The above implies that

$$E_{1t} = E_{2t} \tag{4.3.4}$$

or

$$\hat{n} \times \mathbf{E}_1 = \hat{n} \times \mathbf{E}_2 \tag{4.3.5}$$

where $\hat{n}$ is the unit normal at the interface, and $\hat{n} \times \mathbf{E}$ always bring out the tangential component of a vector **E** (convince yourself).

### 4.3.2 Gauss's Law

From Gauss's law, we have

$$\nabla \cdot \mathbf{D} = \varrho \tag{4.3.6}$$

where $\varrho$ is the volume charge density.

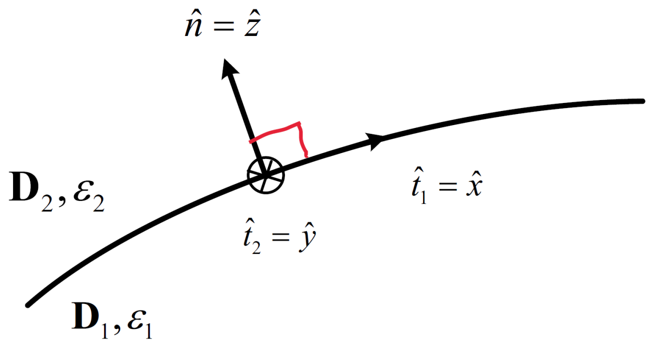


Figure 4.3: A figure showing the derivation of boundary condition for Gauss's law. Again, a local coordinate system can be introduced for convenience.

Expressing the above in local coordinates, then

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x} D_x + \frac{\partial}{\partial y} D_y + \frac{\partial}{\partial z} D_z = \varrho \tag{4.3.7}$$

If there is a surface layer charge at the interface, then the volume charge density must be infinitely large, and can be expressed in terms of a delta function, or $\varrho = \varrho_s \delta(z)$ in local coordinates. By looking at the above expression, the only term that can produce a $\delta(z)$ is from $\frac{\partial}{\partial z} D_z$. In other words, $D_z$ has a jump discontinuity at $z = 0$; the other terms do not. Then

$$\frac{\partial}{\partial z} D_z = \varrho_s \delta(z) \tag{4.3.8}$$

Integrating the above from $0 - \Delta$ to $0 + \Delta$, we get

$$D_z(z)\Big|_{0-\Delta}^{0+\Delta} = \varrho_s \tag{4.3.9}$$

or

$$D_z(0^+) - D_z(0^-) = \varrho_s \tag{4.3.10}$$

where $0^+ = \lim_{\Delta \to 0} 0 + \Delta$, $0^- = \lim_{\Delta \to 0} 0 - \Delta$. Since $D_z(0^+) = D_{2n}$, $D_z(0^-) = D_{1n}$, the above becomes

$$D_{2n} - D_{1n} = \varrho_s \tag{4.3.11}$$

or that

$$\hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \varrho_s \tag{4.3.12}$$

In other words, a charge sheet $\varrho_s$ can give rise to a jump discontinuity in the normal component of the electric flux **D**. Figure 4.4 shows an intuitive sketch as to why a charge sheet gives rise to a discontinuous normal component of the electric flux **D**.

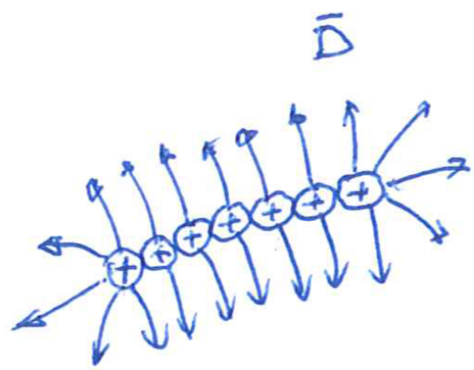


Figure 4.4: A figure intuitively showing why a sheet of charge gives rise to a jump discontinuiy in the normal component of the electric flux **D**.

### 4.3.3 Ampere's Law

Ampere's law, or the generalized one, stipulates that

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{4.3.13}$$

Again if the right-hand side is everywhere finite, then **H** is a continuous field everywhere. However, if the right-hand side has a delta function singularity, then this is not so. For instance, we can project the above equation onto a local coordinates just as we did for Faraday's law.

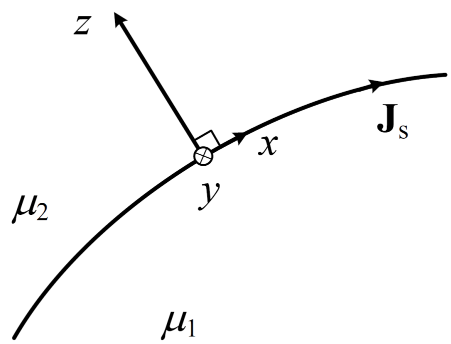


Figure 4.5: A figure showing the derivation of boundary condition for Ampere's law. A local coordinate system is used for simplicity.

To be general, we also include the presence of a current sheet at the interface. A current sheet, or a surface current density becomes a delta function singularity when expressed as a volume current density; Thus, rewriting (4.3.13) in a local coordinate system, assuming that $\mathbf{J} = \hat{x} J_{sx} \delta(z)$, then

$$\nabla \times \mathbf{H} = \hat{x} \left( \frac{\partial}{\partial y} H_z - \frac{\partial}{\partial z} H_y \right) = \hat{x} J_{sx} \delta(z) \tag{4.3.14}$$

The displacement current term on the right-hand side is ignored since it is regular or finite, and will not induce a jump discontinuity on the field; hence, we have the form of the right-hand side of the above equation. From the above, the only term that can produce a $\delta(z)$ singularity on the left-hand side is the $-\frac{\partial}{\partial z} H_y$ term. Therefore, we conclude that

$$-\frac{\partial}{\partial z} H_y = J_{sx} \delta(z) \tag{4.3.15}$$

In other words, $H_y$ has to have a jump discontinuity at the interface where the current sheet resides. Or that

$$H_y(z = 0^+) - H_y(z = 0^-) = -J_{sx} \tag{4.3.16}$$

The above implies that

$$H_{2y} - H_{1y} = -J_{sx} \tag{4.3.17}$$

But $H_y$ is just the tangential component of the $\mathbf{H}$ field. Now if we repeat the same exercise with $\mathbf{J} = \hat{y} J_{sy} \delta(z)$, at the interface, we have

$$H_{2x} - H_{1x} = J_{sy} \tag{4.3.18}$$

Now, (4.3.17) and (4.3.18) can be rewritten using a cross product as

$$\hat{z} \times (\hat{y} H_{2y} - \hat{y} H_{1y}) = \hat{x} J_{sx} \tag{4.3.19}$$

$$\hat{z} \times (\hat{x} H_{2x} - \hat{x} H_{1x}) = \hat{y} J_{sy} \tag{4.3.20}$$

The above two equations can be combined as one to give

$$\hat{z} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s \tag{4.3.21}$$

Taking $\hat{z} = \hat{n}$ in general, we have

$$\hat{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s \tag{4.3.22}$$

In other words, a current sheet $\mathbf{J}_s$ can give rise to a jump discontinuity in the tangential components of the magnetic field, $\hat{n} \times \mathbf{H}$. This is illustrated intuitively in Figure 4.6


Figure 4.6: A figure intuitively showing that with the understanding of how a single line current source generates a magnetic field (right), a cluster of them forming a current sheet will generate a jump discontinuity in the tangential component of the magnetic field $\mathbf{H}$ (left).

### 4.3.4 Gauss's Law for Magnetic Flux

Similarly, from Gauss's law for magnetic flux, or that

$$\nabla \cdot \mathbf{B} = 0 \tag{4.3.23}$$

one deduces that

$$\hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \tag{4.3.24}$$

or that the normal magnetic fluxes are continuous at an interface. In other words, since magnetic charges do not exist, the normal component of the magnetic flux has to be continuous.

# Bibliography


[1] J. A. Kong, "Theory of electromagnetic waves," *New York, Wiley-Interscience, 1975. 348 p.*, 1975.

[2] A. Einstein *et al.*, "On the electrodynamics of moving bodies," *Annalen der Physik*, vol. 17, no. 891, p. 50, 1905.

[3] P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation," *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 114, no. 767, pp. 243–265, 1927.

[4] R. J. Glauber, "Coherent and incoherent states of the radiation field," *Physical Review*, vol. 131, no. 6, p. 2766, 1963.

[5] C.-N. Yang and R. L. Mills, "Conservation of isotopic spin and isotopic gauge invariance," *Physical review*, vol. 96, no. 1, p. 191, 1954.

[6] G. t'Hooft, *50 years of Yang-Mills theory.* World Scientific, 2005.

[7] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* Princeton University Press, 2017.

[8] F. Teixeira and W. C. Chew, "Differential forms, metrics, and the reflectionless absorption of electromagnetic waves," *Journal of Electromagnetic Waves and Applications*, vol. 13, no. 5, pp. 665–686, 1999.

[9] W. C. Chew, E. Michielssen, J.-M. Jin, and J. Song, *Fast and efficient algorithms in computational electromagnetics.* Artech House, Inc., 2001.

[10] A. Volta, "On the electricity excited by the mere contact of conducting substances of different kinds. in a letter from Mr. Alexander Volta, FRS Professor of Natural Philosophy in the University of Pavia, to the Rt. Hon. Sir Joseph Banks, Bart. KBPR S," *Philosophical transactions of the Royal Society of London*, no. 90, pp. 403–431, 1800.

[11] A.-M. Ampère, *Exposé méthodique des phénomènes électro-dynamiques, et des lois de ces phénomènes.* Bachelier, 1823.

[12] ——, *Mémoire sur la théorie mathématique des phénomènes électro-dynamiques uniquement déduite de l'expérience: dans lequel se trouvent réunis les Mémoires que M. Ampère a communiqués à l'Académie royale des Sciences, dans les séances des 4 et 26 décembre 1820, 10 juin 1822, 22 décembre 1823, 12 septembre et 21 novembre 1825.* Bachelier, 1825.

[13] B. Jones and M. Faraday, *The life and letters of Faraday.* Cambridge University Press, 2010, vol. 2.

[14] G. Kirchhoff, "Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird," *Annalen der Physik*, vol. 148, no. 12, pp. 497–508, 1847.

[15] L. Weinberg, "Kirchhoff's' third and fourth laws'," *IRE Transactions on Circuit Theory*, vol. 5, no. 1, pp. 8–30, 1958.

[16] T. Standage, *The Victorian Internet: The remarkable story of the telegraph and the nineteenth century's online pioneers.* Phoenix, 1998.

[17] J. C. Maxwell, "A dynamical theory of the electromagnetic field," *Philosophical transactions of the Royal Society of London*, no. 155, pp. 459–512, 1865.

[18] H. Hertz, "On the finite velocity of propagation of electromagnetic actions," *Electric Waves*, vol. 110, 1888.

[19] M. Romer and I. B. Cohen, "Roemer and the first determination of the velocity of light (1676)," *Isis*, vol. 31, no. 2, pp. 327–379, 1940.

[20] A. Arons and M. Peppard, "Einstein's proposal of the photon concept–a translation of the Annalen der Physik paper of 1905," *American Journal of Physics*, vol. 33, no. 5, pp. 367–374, 1965.

[21] A. Pais, "Einstein and the quantum theory," *Reviews of Modern Physics*, vol. 51, no. 4, p. 863, 1979.

[22] M. Planck, "On the law of distribution of energy in the normal spectrum," *Annalen der physik*, vol. 4, no. 553, p. 1, 1901.

[23] Z. Peng, S. De Graaf, J. Tsai, and O. Astafiev, "Tuneable on-demand single-photon source in the microwave range," *Nature communications*, vol. 7, p. 12588, 2016.

[24] B. D. Gates, Q. Xu, M. Stewart, D. Ryan, C. G. Willson, and G. M. Whitesides, "New approaches to nanofabrication: molding, printing, and other techniques," *Chemical reviews*, vol. 105, no. 4, pp. 1171–1196, 2005.

[25] J. S. Bell, "The debate on the significance of his contributions to the foundations of quantum mechanics, Bells Theorem and the Foundations of Modern Physics (A. van der Merwe, F. Selleri, and G. Tarozzi, eds.)," 1992.

[26] D. J. Griffiths and D. F. Schroeter, *Introduction to quantum mechanics.* Cambridge University Press, 2018.

[27] C. Pickover, *Archimedes to Hawking: Laws of science and the great minds behind them.* Oxford University Press, 2008.

[28] R. Resnick, J. Walker, and D. Halliday, *Fundamentals of physics.* John Wiley, 1988.

[29] S. Ramo, J. R. Whinnery, and T. Duzer van, *Fields and waves in communication electronics, Third Edition.* John Wiley & Sons, Inc., 1995.

[30] J. L. De Lagrange, "Recherches d'arithmétique," *Nouveaux Mémoires de l'Académie de Berlin*, 1773.

[31] J. A. Kong, *Electromagnetic Wave Theory.* EMW Publishing, 2008.

[32] H. M. Schey and H. M. Schey, *Div, grad, curl, and all that: an informal text on vector calculus.* WW Norton New York, 2005.

[33] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics, Vol. I: The new millennium edition: mainly mechanics, radiation, and heat.* Basic books, 2011, vol. 1.

[34] W. C. Chew, *Waves and fields in inhomogeneous media.* IEEE press, 1995.

[35] V. J. Katz, "The history of Stokes' theorem," *Mathematics Magazine*, vol. 52, no. 3, pp. 146–156, 1979.

[36] W. K. Panofsky and M. Phillips, *Classical electricity and magnetism.* Courier Corporation, 2005.

[37] T. Lancaster and S. J. Blundell, *Quantum field theory for the gifted amateur.* OUP Oxford, 2014.

[38] W. C. Chew, "Ece 350x lecture notes," *http://wcchew.ece.illinois.edu/chew/ece350.html*, 1990.

[39] C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory.* Springer Science & Business Media, 2013.

[40] J. M. Crowley, *Fundamentals of applied electrostatics.* Krieger Publishing Company, 1986.

[41] C. Balanis, *Advanced Engineering Electromagnetics.* Hoboken, NJ, USA: Wiley, 2012.

[42] J. D. Jackson, *Classical electrodynamics.* AAPT, 1999.

[43] R. Courant and D. Hilbert, *Methods of Mathematical Physics: Partial Differential Equations.* John Wiley & Sons, 2008.

[44] L. Esaki and R. Tsu, "Superlattice and negative differential conductivity in semiconductors," *IBM Journal of Research and Development*, vol. 14, no. 1, pp. 61–65, 1970.

[45] E. Kudeki and D. C. Munson, *Analog Signals and Systems.* Upper Saddle River, NJ, USA: Pearson Prentice Hall, 2009.

[46] A. V. Oppenheim and R. W. Schafer, *Discrete-time signal processing.* Pearson Education, 2014.

[47] R. F. Harrington, *Time-harmonic electromagnetic fields.* McGraw-Hill, 1961.

[48] E. C. Jordan and K. G. Balmain, *Electromagnetic waves and radiating systems.* Prentice-Hall, 1968.

[49] G. Agarwal, D. Pattanayak, and E. Wolf, "Electromagnetic fields in spatially dispersive media," *Physical Review B*, vol. 10, no. 4, p. 1447, 1974.

[50] S. L. Chuang, *Physics of photonic devices.* John Wiley & Sons, 2012, vol. 80.

[51] B. E. Saleh and M. C. Teich, *Fundamentals of photonics.* John Wiley & Sons, 2019.

[52] M. Born and E. Wolf, *Principles of optics: electromagnetic theory of propagation, interference and diffraction of light.* Elsevier, 2013.

[53] R. W. Boyd, *Nonlinear optics.* Elsevier, 2003.

[54] Y.-R. Shen, "The principles of nonlinear optics," *New York, Wiley-Interscience, 1984, 575 p.*, 1984.

[55] N. Bloembergen, *Nonlinear optics.* World Scientific, 1996.

[56] P. C. Krause, O. Wasynczuk, and S. D. Sudhoff, *Analysis of electric machinery.* McGraw-Hill New York, 1986, vol. 564.

[57] A. E. Fitzgerald, C. Kingsley, S. D. Umans, and B. James, *Electric machinery.* McGraw-Hill New York, 2003, vol. 5.

[58] M. A. Brown and R. C. Semelka, *MRI.: Basic Principles and Applications.* John Wiley & Sons, 2011.

[59] C. A. Balanis, *Advanced engineering electromagnetics.* John Wiley & Sons, 1999.

[60] Wikipedia, "Lorentz force," 2019.

[61] R. O. Dendy, *Plasma physics: an introductory course.* Cambridge University Press, 1995.

[62] P. Sen and W. C. Chew, "The frequency dependent dielectric and conductivity response of sedimentary rocks," *Journal of microwave power*, vol. 18, no. 1, pp. 95–105, 1983.

[63] D. A. Miller, *Quantum Mechanics for Scientists and Engineers.* Cambridge, UK: Cambridge University Press, 2008.

[64] W. C. Chew, "Quantum mechanics made simple: Lecture notes," *http://wcchew.ece.illinois.edu/chew/course/QMAll20161206.pdf*, 2016.

[65] B. G. Streetman, S. Banerjee *et al.*, *Solid state electronic devices.* Prentice hall Englewood Cliffs, NJ, 1995, vol. 4.

[66] Smithsonian, https://www.smithsonianmag.com/history/this-1600-year-old-goblet-shows-that-the-romans-were-nanotechnology-pioneers-787224/, accessed: 2019-09-06.