Lecture 37

Computational Electromagnetics, Numerical Methods

37.1 Computational Electromagnetics and Numerical Methods

Numerical methods exploit the blinding speed of modern digital computers to perform calculations, and hence to solve large system of equations. These equations are partial differential equations or integral equations. When these methods are applied to solving Maxwell's equations and related equations, the field is known as **computational electromagnetics**.

Maxwell's equations are a form of partial differential equations (PDE). Boundary conditions have to be stipulated for these PDE's, and the solution can be sought by solving a boundary value problem (BVP). In solving PDE's the field in every point in space is solved for.

On the other hand, the Green's function method can be used to convert a PDE into an integral equation (IE). Then the fields are expressed in terms of the sources, and the sources are the unknowns to be solved. Sources in IEs are supported by a finite part of space, for instance on the surface of the scatterer, whereas fields from PDEs permeate all of space. Therefore, sources in IEs can be represented by a smaller set of unknowns, and therefore, are easier to solve for compared to fields in PDEs.

37.1.1 Examples of Differential Equations

An example of differential equations written in terms of sources are the scalar wave equation:

$$(\nabla^2 + k^2)\,\phi(\mathbf{r}) = Q(\mathbf{r}), \tag{37.1.1}$$

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An example of vector differential equation for vector electromagnetic field is

$$\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) = i\omega \mathbf{J}(\mathbf{r}) \tag{37.1.2}$$

These equations are linear equations. They have one commonality, i.e., they can be abstractly written as

$$\mathscr{L} f = g \tag{37.1.3}$$

where $\mathscr{L}$ is the differential operator which is linear, and f is the unknown, and g is the driving source. Differential equations, or partial differential equations, as mentioned before, have to be solved with boundary conditions. Otherwise, there is no unique solution to these equations.

In the case of the scalar wave equation (37.1.1), $\mathscr{L} = (\nabla^2 + k^2)$ is a differential operator. In the case of the electromagnetic vector wave equation (37.1.2), $\mathscr{L} = (\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times) - \omega^2 \overline{\boldsymbol{\epsilon}} \cdot$. Furthermore, f will be $\phi(\mathbf{r})$ for the scalar wave equation (37.1.1), while it will be $\mathbf{E}(\mathbf{r})$ in the case of vector wave equation for an electromagnetic system (37.1.2). The g on the right-hand side can represent Q in (37.1.1) or $i\omega \mathbf{J}(\mathbf{r})$ in (37.1.2).

37.1.2   Examples of Integral Equations

This course is replete with PDE's, but we have not come across too many integral equations. Therefore, we shall illustrate the derivation of some integral equations. Since the acoustic wave problem is homomorphic to the electromagnetic wave problem, we will illustrate the derivation of integral equation of scattering using acoustic wave equation.[1]

The surface integral equation method is rather popular in a number of applications, because it employs a homogeneous-medium Green's function which is simple in form, and the unknowns reside on a surface rather than in a volume. In this section, the surface integral equations[2] for scalar and will be studied first. Then, the volume integral equation will be discussed next.

Surface Integral Equations

In an integral equation, the unknown to be sought is embedded in an integral. An integral equation can be viewed as an operator equation as well, just as are differential equations. We shall see how such integral equations with only surface integrals are derived, using the scalar wave equation.

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[1] The cases of electromagnetic wave equations can be found in *Chew, Waves and Fields in Inhomogeneous Media* [34].

[2] These are sometimes called boundary integral equations [208, 209].
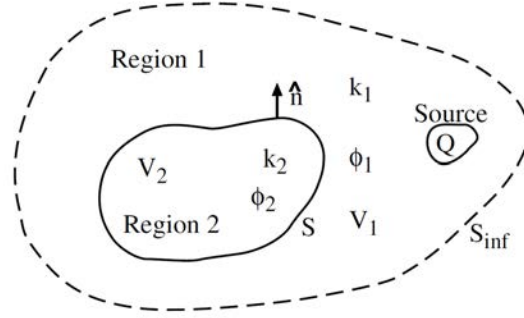
Figure 37.1: A two-region problem can be solved with a surface integral equation.

Consider a scalar wave equation for a two-region problem as shown in Figure 37.1. In region 1, the governing equation for the total field is

$$(\nabla^2 + k_1^2)\,\phi_1(\mathbf{r}) = Q(\mathbf{r}), \tag{37.1.4}$$

For simplicity, we will assume that the scatterer is impenetrable, meaning that the field in region 2 is zero. Therefore, we need only define Green's functions for regions 1 to satisfy the following equations:

$$(\nabla^2 + k_1^2)\,g_1(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'), \tag{37.1.5}$$

The derivation here is similar to the that of Huygens' principle. On multiplying Equation (37.1.1) by $g_1(\mathbf{r}, \mathbf{r}')$ and Equation (37.1.5) by $\phi_1(\mathbf{r})$, subtracting the two resultant equations, and integrating over region 1, we have, for $\mathbf{r}' \in V_1$,

$$\int_{V_1} dV \left[g_1(\mathbf{r}, \mathbf{r}')\nabla^2\phi_1(\mathbf{r}) - \phi_1(\mathbf{r})\nabla^2 g_1(\mathbf{r}, \mathbf{r}') \right]$$

$$= \int_{V_1} dV\, g_1(\mathbf{r}, \mathbf{r}')Q(\mathbf{r}) + \phi_1(\mathbf{r}'), \quad \mathbf{r}' \in V_1. \tag{37.1.6}$$

Since $\nabla \cdot (g\nabla\phi - \phi\nabla g) = g\nabla^2\phi - \phi\nabla^2 g$, by applying Gauss' theorem, the volume integral on the left-hand side of (37.1.6) becomes a surface integral over the surface bounding V_1. Consequently,[3]

$$- \int_{S+S_{inf}} dS\, \hat{n} \cdot \left[g_1(\mathbf{r}, \mathbf{r}')\nabla\phi_1(\mathbf{r}) - \phi_1(\mathbf{r})\nabla g_1(\mathbf{r}, \mathbf{r}') \right]$$

$$= -\phi_{inc}(\mathbf{r}') + \phi_1(\mathbf{r}'), \quad \mathbf{r}' \in V_1. \tag{37.1.7}$$

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[3]The equality of the volume integral on the left-hand side of (37.1.6) and the surface integral on the left-hand side of (37.1.7) is also known as Green's theorem.

In the above, we have let

$$\phi_{inc}(\mathbf{r}') = -\int_{V_1} dV\, g_1(\mathbf{r}, \mathbf{r}') Q(\mathbf{r}), \tag{37.1.8}$$

since it is the incident field generated by the source $Q(\mathbf{r})$.

Note that up to this point, $g_1(\mathbf{r}, \mathbf{r}')$ is not explicitly specified, as long as it is a solution of (37.1.5). A simple choice for $g_1(\mathbf{r}, \mathbf{r}')$ that satisfies the radiation condition is

$$g_1(\mathbf{r}, \mathbf{r}') = \frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{37.1.9}$$

It is the unbounded, homogeneous medium scalar Green's function. In this case, $\phi_{inc}(\mathbf{r})$ is the incident field generated by the source $Q(\mathbf{r})$ in the absence of the scatterer. Moreover, the integral over S_{inf} vanishes when $S_{inf} \to \infty$ by virtue of the radiation condition. Then, after swapping $\mathbf{r}$ and $\mathbf{r}'$, we have

$$\phi_1(\mathbf{r}) = \phi_{inc}(\mathbf{r}) - \int_S dS'\, \hat{n}' \cdot [g_1(\mathbf{r}, \mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla'g_1(\mathbf{r}, \mathbf{r}')], \quad \mathbf{r} \in V_1. \tag{37.1.10}$$

But if $\mathbf{r}' \notin V_1$ in (37.1.6), the second term, $\phi_1(\mathbf{r})$, on the right-hand side of (37.1.6) would be zero, for $\mathbf{r}'$ would be in V_2 where the integration is not performed. Therefore, we can write (37.1.10) as

$$\left.\begin{array}{ll} \text{if } \mathbf{r} \in V_1, & \phi_1(\mathbf{r}) \\ \text{if } \mathbf{r} \in V_2, & 0 \end{array}\right\} = \phi_{inc}(\mathbf{r}) - \int_S dS'\, \hat{n}' \cdot [g_1(\mathbf{r}, \mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla'g_1(\mathbf{r}, \mathbf{r}')]. \tag{37.1.11}$$

The above equation is evocative of Huygens' principle. It says that when the observation point $\mathbf{r}$ is in V_1, then the total field $\phi_1(\mathbf{r})$ consists of the incident field, $\phi_{inc}(\mathbf{r})$, and the contribution of field due to surface sources on S, which is the second term on the right-hand side of (37.1.11). But if the observation point is in V_2, then the surface sources on S generate a field that exactly cancels the incident field $\phi_{inc}(\mathbf{r})$, making the total field in region 2 zero. This fact is the core of the **extinction theorem** as shown in Figure 37.2 (see Born and Wolf 1980).

In (37.1.11), $\hat{n} \cdot \nabla\phi_1(\mathbf{r})$ and $\phi_1(\mathbf{r})$ act as surface sources. Moreover, they are impressed on S, creating a field in region 2 that cancels exactly the incident field in region 2 (see Figure 37.2).
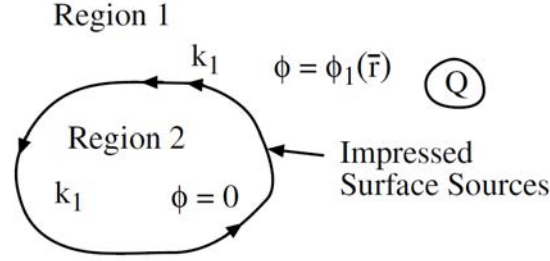
Figure 37.2: The illustration of the extinction theorem.

Applying the extinction theorem, integral equations can now be derived. So, using the lower parts of Equations (37.1.11), we have

$$\phi_{inc}(\mathbf{r}) = \int\limits_{S} dS' \, \hat{n}' \cdot [g_1(\mathbf{r}, \mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla'g_1(\mathbf{r}, \mathbf{r}')], \quad \mathbf{r} \in V_2, \tag{37.1.12}$$

The integral equations above still has two independent unknowns, ϕ_1 and $\hat{n} \cdot \nabla\phi_1$. Next, boundary conditions can be used to eliminate one of these two unknowns.

An acoustic scatterer which is impenetrable either has a hard surface boundary condition where normal velocity is zero, or it has soft surface where the pressure is zero (also called a pressure release surface). Since the force or the velocity of the particle is proportional to the $\nabla\phi$, a hard surface will have $\hat{n} \cdot \nabla\phi_1 = 0$, or a homogeneous Neumann boundary condition, while a soft surface will have $\phi_1 = 0$, a homogeneous Dirichlet boundary condition.

$$\phi_{inc}(\mathbf{r}) = \int\limits_{S} dS' \, \hat{n}' \cdot [g_1(\mathbf{r}, \mathbf{r}')\nabla'\phi_1(\mathbf{r}')], \quad \mathbf{r} \in V_2, \quad \text{soft boundary condition}$$

$$\tag{37.1.13}$$

$$\phi_{inc}(\mathbf{r}) = -\int\limits_{S} dS' \, \phi_1(\mathbf{r}')\nabla'g_1(\mathbf{r}, \mathbf{r}')], \quad \mathbf{r} \in V_2, \quad \text{hard boundary condition}$$

$$\tag{37.1.14}$$

More complicated surface integral equations (SIEs) for penetrable scatterers, as well as vector surface integral equations for the electromagnetics cases are derived in *Chew, Waves and Fields in Inhomogeneous Media* [34, 210]. Also, there is another class of integral equations called volume integral equations (VIEs) [211]. They are also derived in [34].

Nevertheless, all the linear integral equations can be unified under one notation:

$$\mathscr{L}f = g \tag{37.1.15}$$

where $\mathscr{L}$ is a linear operator. This is similar to the differential equation case. The difference is that the unknown f represents the source of the problem, while g is the incident field impinging on the scatterer or object. Furthermore, f does not need to satisfy any boundary condition, since the field radiated via the Green's function satisfies the radiation condition.

37.2 Subspace Projection Methods

Several operator equations have been derived in the previous sections. They are all of the form

$$\mathscr{L}f = g \tag{37.2.1}$$

37.2.1 Function as a Vector

In the above, f is a functional vector which is the analogy of the vector $\mathbf{f}$ in matrix theory or linear algebra. In linear algebra, the vector $\mathbf{f}$ is of length N in an N dimensional space. It can be indexed by a set of countable index, say i, and we can described such a vector with N numbers such as $f_i, i = 1, \ldots, N$ explicitly. This is shown in Figure 37.3(a).

A function $f(x)$, however, can be thought of as being indexed by x in the 1D case. However, the index in this case is a continuum, and countably infinite. Hence, it corresponds to a vector of infinite dimension and it lives in an infinite dimensional space.[4]

To make such functions economical in storage, for instance, we replace the function $f(x)$ by its sampled values at N locations, such that $f(x_i), \quad i = 1, \ldots, N$. Then the values of the function in between the stored points $f(x_i)$ can be obtained by interpolation. Therefore, a function vector $f(x)$, even though it is infinite dimensional, can be approximated by a finite length vector, $\mathbf{f}$. This concept is illustrated in Figure 37.3(b) and (c). This concept can be generalized to a function of 3D space $f(\mathbf{r})$. If $\mathbf{r}$ is sampled over a 3D volume, it can provide an index to a vector $f_i = f(\mathbf{r}_i)$, and hence, $f(\mathbf{r})$ can be thought of as a vector as well.

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[4]When these functions are square integrable implying finite "energy", these infinite dimensional spaces are called Hilbert spaces.
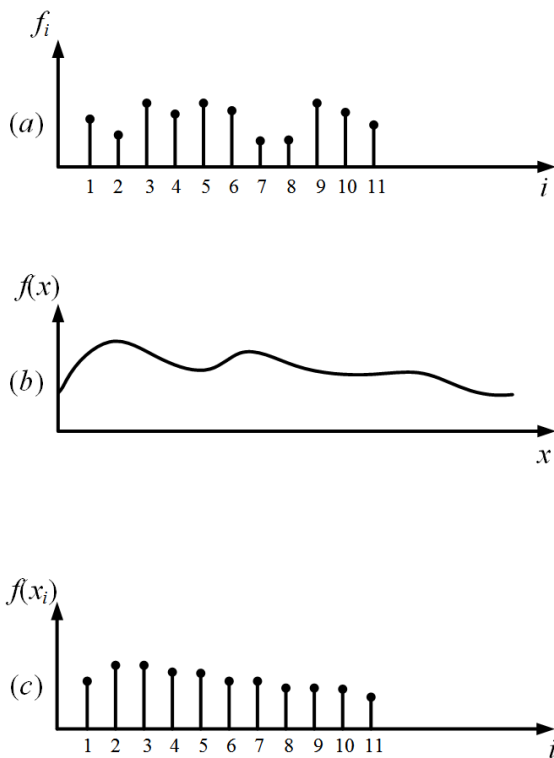
Figure 37.3: A function can be thought of as a vector.

37.2.2 Operator as a Map

An operator like $\mathscr{L}$ above can be thought of as a map or a transformation. It maps a function f defined in a Hilbert space V to g defined in another Hilbert space W. Mathematically, this is written as

$$\mathscr{L}: V \to W \tag{37.2.2}$$

Indicating that $\mathscr{L}$ is a map of vectors in the space V to the space W. Here, V is also called the **domain space** (or domain) of $\mathscr{L}$ while W is the **range space** (or range) of $\mathscr{L}$.

37.2.3 Approximating Operator Equations with Matrix Equations

One main task of numerical method is first to approximate an operator equation $\mathscr{L}f = g$ by a matrix equation $\overline{\mathbf{L}} \cdot \mathbf{f} = \mathbf{g}$. To convert the above, we first let

$$f \cong \sum_{n=1}^{N} a_n f_n = g \tag{37.2.3}$$

In the above, $f_n, n, \ldots, N$ are known functions called basis functions. Now, a_n's are the new unknowns to be sought. Also the above is an approximation, and the accuracy of the approximation depends very much on the original function f. A set of very popular basis functions are functions that form a piece-wise linear interpolation of the function from its nodes. These basis functions are shown in Figure 37.4.
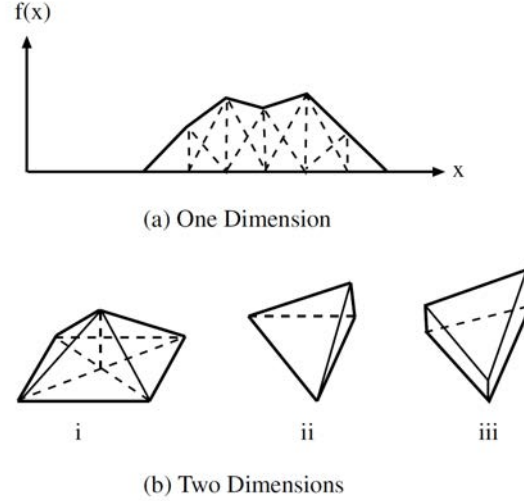

(a) One Dimension

(b) Two Dimensions

Figure 37.4: Examples of basis function in (a) one dimension, (b) two dimension. Each of these functions are define over a finite domain. Hence, they are also called sub-domain basis functions.

Upon substituting (37.2.3) into (37.2.1), we obtain

$$\sum_{n=1}^{N} a_n \mathscr{L} f_n = g \tag{37.2.4}$$

Then, multiplying (37.2.4) by w_m and integrate over the space that $w_m(\mathbf{r})$ is defined, then we have

$$\sum_{n=1}^{N} a_n \langle w_m, \mathscr{L} f_n \rangle = \langle w_m, g \rangle, m = 1, \ldots, N \tag{37.2.5}$$

In the above, the inner product is defined as

$$\langle f_1, f_2 \rangle = \int d\mathbf{r} f_1(\mathbf{r}) f_2(\mathbf{r}) \tag{37.2.6}$$

where the integration is over the support of the functions, or the space over which the functions are defined.[5] For PDEs these functions are defined over a 3D space, while in SIEs, these

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[5]This is known as the reaction inner product [34, 47, 120]. As oppose to most math and physics literature, the energy inner product is used [120].

functions are defined over a surface. In a 1D problems, these functions are defined over a 1D space.

The functions $w_m, m = 1, \ldots, N$ is known as the weighting functions or testing functions. The testing functions should be chosen so that they can approximate well a function that lives in the range space W of the operator $\mathscr{L}$. Such set of testing functions lives in the **dual space** of the range space. For example, if f_r lives in the range space of the operator $\mathscr{L}$, the set of function f_d, such that the inner product $\langle f_d, f_r \rangle$ exists, forms the dual space of W.

The above is a matrix equation of the form

$$\overline{\mathbf{L}} \cdot \mathbf{a} = \mathbf{g} \tag{37.2.7}$$

where

$$
\begin{aligned}
\left[\overline{\mathbf{L}} \right]_{mn} &= \langle w_m, \mathscr{L} f_n \rangle \\
[\mathbf{a}]_n &= a_n, [\mathbf{g}]_m = \langle w_m, g \rangle
\end{aligned}
\tag{37.2.8}
$$

What has effectively happened here is that given an operator $\mathscr{L}$ that maps a function that lives in an infinite dimensional Hilbert space V, to another function that lives in another infinite dimensional Hilbert space W, via the operator equation $\mathscr{L} f = g$, we have approximated the Hilbert spaces with finite dimensional spaces (subspaces), and finally, obtain a finite dimensional matrix equation that is the representation of the original infinite dimensional operator equation.

In the above, $\overline{\mathbf{L}}$ is the matrix representation of the operator $\mathscr{L}$ in the subspaces, and $\mathbf{a}$ and $\mathbf{g}$ are the vector representations of f and g, respectively, in their respective subspaces.

When such a method is applied to integral equations, it is usually called the method of moments (MOM). (Surface integral equations are also called boundary integral equations (BIEs) in other fields [209]. When finite discrete basis are used to represent the surface unknowns, it is also called the boundary element method (BEM) [212]. But when this method is applied to solve PDEs, it is called the finite element method (FEM) [213–216], which is a rather popular method due to its simplicity.

37.2.4   Mesh Generation

In order to approximate a function defined on an arbitrary shaped surface or volume by sum of basis functions, it is best to mesh (tessellate or discretize) the surface and volume by meshes. In 2D, all shapes can be tessellated by unions of triangles, while a 3D volume can be meshed (tessellated) by unions of tetrahedrons. Such meshes are used not only in CEM, but in other fields such as solid mechanics. Hence, there are many commercial software available to generate sophisticated meshes.

When a surface is curved, or of arbitrary shape, it can be meshed by union of triangles as shown in Figure 37.5. When a volume is of arbitrary shape of a volume is around an arbitrary shape object, it can be meshed by tetrahedrons as shown in Figure 37.6. Then basis functions are defined to interpolate the field between nodal values or values defined on the edges of a triangle or a tetrahedron.
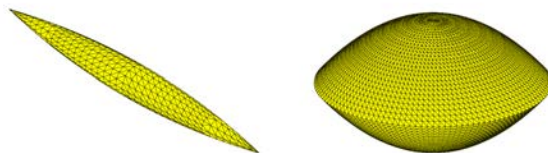
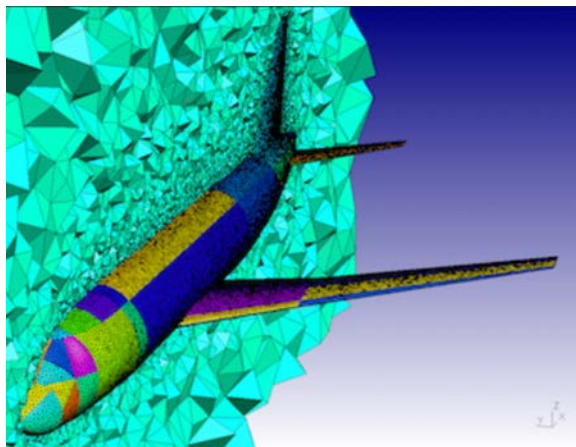Figure 37.5: An arbitrary surface can be meshed by a union of triangles.

Figure 37.6: A volume region can be meshed by a union of tetrahedra. But the surface of the aircraft is meshed with a union of trianlges (courtesy of gmsh.info).

37.3   Solving Matrix Equation by Optimization

When a matrix system get exceedingly large, it is preferable that a direct inversion of the matrix equation not performed. Direct inversions (e.g., using Gaussian elimination [217] or Kramer's rule [218]) have computational complexity[6] of $O(N^3)$, and requiring storage of $O(N^2)$. Hence, when N is large, other methods have to be sought.

To this end, it is better to convert the solving of a matrix equation into an optimization problem. These methods can be designed so that a much larger system can be solved with a digital computer. Optimization problem results in finding the stationary point of a functional.[7] First, we will figure out how to find such a functional.

Consider a matrix equation given by

$$\overline{\mathbf{L}} \cdot \mathbf{f} = \mathbf{g} \tag{37.3.1}$$

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[6]The scaling of computer time with respect to the number of unknowns (degrees of freedom) is known in the computer parlance as computational complexity.

[7]Functional is usually defined as a function of a function [34, 43]. Here, we include a function of a vector to be a functional as well.

For simplicity, we consider $\overline{\mathbf{L}}$ as a symmetric matrix.[8] Then one can define a functional

$$I = \mathbf{f}^t \cdot \overline{\mathbf{L}} \cdot \mathbf{f} - 2\mathbf{f}^t \cdot \mathbf{g} \tag{37.3.2}$$

Such a functional is called a quadratic functional because it is analogous to $I = Lx^2 - 2xg$, which is quadratic, in its simplest 1D rendition.

Taking the first variation with respect to $\mathbf{f}$, namely, we let $\mathbf{f} = \mathbf{f}_o + \delta\mathbf{f}$, and find the leading order approximation of the functional. Therefore, one gets

$$\delta I = \delta\mathbf{f}^t \cdot \overline{\mathbf{L}} \cdot \mathbf{f}_o + \mathbf{f}_o^t \cdot \overline{\mathbf{L}} \cdot \delta\mathbf{f} - 2\delta\mathbf{f}^t \cdot \mathbf{g} \tag{37.3.3}$$

If $\overline{\mathbf{L}}$ is symmetric, the first two terms are the same, and the above becomes

$$\delta I = 2\delta\mathbf{f}^t \cdot \overline{\mathbf{L}} \cdot \mathbf{f}_o - 2\delta\mathbf{f}^t \cdot \mathbf{g} \tag{37.3.4}$$

For $\mathbf{f}_o$ to be the optimal point or the stationary point, then its first variation has to be zero, or that $\delta I = 0$. Thus we conclude that

$$\overline{\mathbf{L}} \cdot \mathbf{f}_o = \mathbf{g} \tag{37.3.5}$$

Hence, the optimal point to the functional I in (37.3.2) is the solution to (37.3.1).

Such method, when applied to an infinite dimensional Hilbert space problem, is called variational method, but the main ideas are similar. The wonderful idea about such a method is that instead of doing direct inversion, one can search for the optimal point or stationary point of the quadratic functional using gradient search or gradient descent methods or some optimization method.

37.3.1   Gradient of a Functional

It turns out that the gradient of a quadratic functional can be found quite easily. Also it is cheaper to computer the gradient of a functional than to find the inverse of a matrix operator. To do this, it is better to write out functional using index (or indicial, or Einstein) notation [219]. In this notation, the functional first variation δI in (37.3.4) becomes

$$\delta I = 2\delta f_j L_{ij} f_i - 2\delta f_j g_j \tag{37.3.6}$$

In the above, we neglect to distinguish between $\mathbf{f}_o$ and $\mathbf{f}$ and $\mathbf{f}$ represents the optimal point is implied. Also, in this notation, the summation symbol is dropped, and summations over repeated indices are implied. In this notation, it is easier to see what a functional derivative is. We can differentiate the above with respect to f_j easily to arrive at

$$\frac{\partial I}{\partial f_j} = 2L_{ij} f_i - 2g_j \tag{37.3.7}$$

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[8]Functional for the asymmetric case can be found in *Chew, Waves and Fields in Inhomogeneous Media,* Chapter 5 [34].

Notice that the remaining equation has one index j remaining in index notation, meaning that it is a vector equation. We can reconstitute the above using our more familiar matrix notation that

$$\nabla_{\mathbf{f}} I = 2\overline{\mathbf{L}} \cdot \mathbf{f} - 2\mathbf{g} \tag{37.3.8}$$

The left-hand side is a notation for the gradient of a functional in a multi-dimensional space defined by $\mathbf{f}$, and the right-hand side is the expression for calculating this gradient. One needs only to perform a matrix-vector product to find this gradient. Hence, the computational complexity of finding this gradient is $O(N^2)$ at worst, and $O(N)$ for many sparse matrices. In a gradient search method, such a gradient is calculated repeated until the optimal point is found. Such methods are called iterative methods.

If the optimal point can be found in N_{iter} iterations, then the CPU time scales as $N_{iter}N^{\alpha}$ where $1 < \alpha < 2$. There is a clever gradient search algorithm, called the **conjugate gradient method** that can find the optimal point in N_{iter} in exact arithmetics. In many gradient search methods, $N_{\text{iter}} \ll N$ resulting in great savings in computer time.

What is more important is that this method does not require the storage of the matrix $\overline{\mathbf{L}}$, but a computer code that produces the vector $\mathbf{g}_o = \overline{\mathbf{L}} \cdot \mathbf{f}$ as an output, with $\mathbf{f}$ as an input. Both $\mathbf{f}$ and $\mathbf{g}_o$ require only $O(N)$ memory storage. Such methods are called matrix-free methods. Even when $\overline{\mathbf{L}}$ is a dense matrix, but is the matrix representation of some Green's function, fast methods now exist to perform the dense matrix-vector product in $O(N \log N)$ operations.[9]

The value I is also called the cost function, and its minimum is sought in the seeking of the solution by gradient search methods. Detail discussion of these methods is given in [220]. Figure 37.7 shows the contour plot of a cost function in 2D. When the condition number[10] of the matrix $\overline{\mathbf{L}}$ is large (implying that the matrix is ill-conditioned), the contour plot will resemble a deep valley. And hence, the gradient search method will tend to zig-zag along the way as it finds the solution. Therefore, convergence is slow for matrices with large condition numbers

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[9]Chew et al, *Fast and Efficient Algorithms in CEM* [9].
[10]This is the ratio of the largest eigenvalue of the matrix to its smallest eigenvalue.
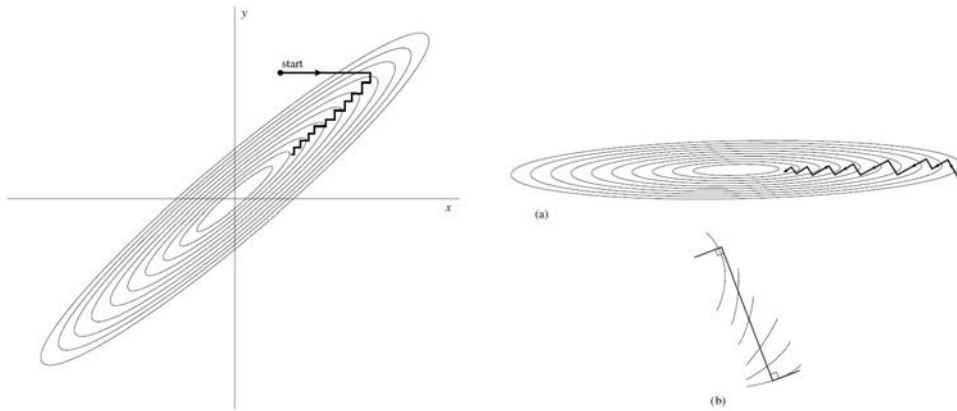
Figure 37.7: Plot of a 2D cost function for an ill-conditioned system (courtesy of Numerical Recipe [220]).

Figure 37.8 shows a cartoon picture in 2D of the histories of different search paths from a machine-learning example where a cost functional similar to I has to be minimized. Finding the optimal point or the minimum point of a general functional is still a hot topic of research: it is important in artificial intelligence.
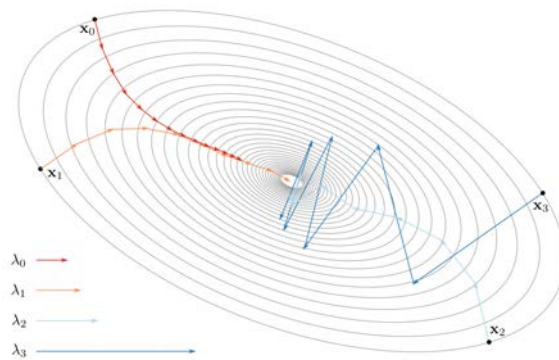

Figure 37.8: Gradient search or gradient descent method is finding an optimal point (courtesy of Y. Ioannou: https://blog.yani.io/sgd/).

Bibliography

[1] J. A. Kong, *Theory of electromagnetic waves*. New York, Wiley-Interscience, 1975.

[2] A. Einstein *et al.*, "On the electrodynamics of moving bodies," *Annalen der Physik*, vol. 17, no. 891, p. 50, 1905.

[3] P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation," *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 114, no. 767, pp. 243–265, 1927.

[4] R. J. Glauber, "Coherent and incoherent states of the radiation field," *Physical Review*, vol. 131, no. 6, p. 2766, 1963.

[5] C.-N. Yang and R. L. Mills, "Conservation of isotopic spin and isotopic gauge invariance," *Physical review*, vol. 96, no. 1, p. 191, 1954.

[6] G. t'Hooft, *50 years of Yang-Mills theory*. World Scientific, 2005.

[7] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. Princeton University Press, 2017.

[8] F. Teixeira and W. C. Chew, "Differential forms, metrics, and the reflectionless absorption of electromagnetic waves," *Journal of Electromagnetic Waves and Applications*, vol. 13, no. 5, pp. 665–686, 1999.

[9] W. C. Chew, E. Michielssen, J.-M. Jin, and J. Song, *Fast and efficient algorithms in computational electromagnetics*. Artech House, Inc., 2001.

[10] A. Volta, "On the electricity excited by the mere contact of conducting substances of different kinds. in a letter from Mr. Alexander Volta, FRS Professor of Natural Philosophy in the University of Pavia, to the Rt. Hon. Sir Joseph Banks, Bart. KBPR S," *Philosophical transactions of the Royal Society of London*, no. 90, pp. 403–431, 1800.

[11] A.-M. Ampère, *Exposé méthodique des phénomènes électro-dynamiques, et des lois de ces phénomènes*. Bachelier, 1823.

[12] ——, *Mémoire sur la théorie mathématique des phénomènes électro-dynamiques uniquement déduite de l'expérience: dans lequel se trouvent réunis les Mémoires que M. Ampère a communiqués à l'Académie royale des Sciences, dans les séances des 4 et*

26 décembre 1820, 10 juin 1822, 22 décembre 1823, 12 septembre et 21 novembre 1825. Bachelier, 1825.

[13] B. Jones and M. Faraday, *The life and letters of Faraday*.   Cambridge University Press, 2010, vol. 2.

[14] G. Kirchhoff, "Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird," *Annalen der Physik*, vol. 148, no. 12, pp. 497–508, 1847.

[15] L. Weinberg, "Kirchhoff's' third and fourth laws'," *IRE Transactions on Circuit Theory*, vol. 5, no. 1, pp. 8–30, 1958.

[16] T. Standage, *The Victorian Internet: The remarkable story of the telegraph and the nineteenth century's online pioneers*.   Phoenix, 1998.

[17] J. C. Maxwell, "A dynamical theory of the electromagnetic field," *Philosophical transactions of the Royal Society of London*, no. 155, pp. 459–512, 1865.

[18] H. Hertz, "On the finite velocity of propagation of electromagnetic actions," *Electric Waves*, vol. 110, 1888.

[19] M. Romer and I. B. Cohen, "Roemer and the first determination of the velocity of light (1676)," *Isis*, vol. 31, no. 2, pp. 327–379, 1940.

[20] A. Arons and M. Peppard, "Einstein's proposal of the photon concept–a translation of the Annalen der Physik paper of 1905," *American Journal of Physics*, vol. 33, no. 5, pp. 367–374, 1965.

[21] A. Pais, "Einstein and the quantum theory," *Reviews of Modern Physics*, vol. 51, no. 4, p. 863, 1979.

[22] M. Planck, "On the law of distribution of energy in the normal spectrum," *Annalen der physik*, vol. 4, no. 553, p. 1, 1901.

[23] Z. Peng, S. De Graaf, J. Tsai, and O. Astafiev, "Tuneable on-demand single-photon source in the microwave range," *Nature communications*, vol. 7, p. 12588, 2016.

[24] B. D. Gates, Q. Xu, M. Stewart, D. Ryan, C. G. Willson, and G. M. Whitesides, "New approaches to nanofabrication: molding, printing, and other techniques," *Chemical reviews*, vol. 105, no. 4, pp. 1171–1196, 2005.

[25] J. S. Bell, "The debate on the significance of his contributions to the foundations of quantum mechanics, Bells Theorem and the Foundations of Modern Physics (A. van der Merwe, F. Selleri, and G. Tarozzi, eds.)," 1992.

[26] D. J. Griffiths and D. F. Schroeter, *Introduction to quantum mechanics*.   Cambridge University Press, 2018.

[27] C. Pickover, *Archimedes to Hawking: Laws of science and the great minds behind them*. Oxford University Press, 2008.

[28] R. Resnick, J. Walker, and D. Halliday, *Fundamentals of physics.* John Wiley, 1988.

[29] S. Ramo, J. R. Whinnery, and T. Duzer van, *Fields and waves in communication electronics, Third Edition.* John Wiley & Sons, Inc., 1995, also 1965, 1984.

[30] J. L. De Lagrange, "Recherches d'arithmétique," *Nouveaux Mémoires de l'Académie de Berlin*, 1773.

[31] J. A. Kong, *Electromagnetic Wave Theory.* EMW Publishing, 2008, also 1985.

[32] H. M. Schey, *Div, grad, curl, and all that: an informal text on vector calculus.* WW Norton New York, 2005.

[33] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics, Vols. I, II, & III: The new millennium edition.* Basic books, 2011, also 1963, 2006, vol. 1,2,3.

[34] W. C. Chew, *Waves and fields in inhomogeneous media.* IEEE Press, 1995, also 1990.

[35] V. J. Katz, "The history of Stokes' theorem," *Mathematics Magazine*, vol. 52, no. 3, pp. 146–156, 1979.

[36] W. K. Panofsky and M. Phillips, *Classical electricity and magnetism.* Courier Corporation, 2005.

[37] T. Lancaster and S. J. Blundell, *Quantum field theory for the gifted amateur.* OUP Oxford, 2014.

[38] W. C. Chew, "Fields and waves: Lecture notes for ECE 350 at UIUC," *https://engineering.purdue.edu/wcchew/ece350.html*, 1990.

[39] C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory.* Springer Science & Business Media, 2013.

[40] J. M. Crowley, *Fundamentals of applied electrostatics.* Krieger Publishing Company, 1986.

[41] C. Balanis, *Advanced Engineering Electromagnetics.* Hoboken, NJ, USA: Wiley, 2012.

[42] J. D. Jackson, *Classical electrodynamics.* John Wiley & Sons, 1999.

[43] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Volumes 1 and 2.* Interscience Publ., 1962.

[44] L. Esaki and R. Tsu, "Superlattice and negative differential conductivity in semiconductors," *IBM Journal of Research and Development*, vol. 14, no. 1, pp. 61–65, 1970.

[45] E. Kudeki and D. C. Munson, *Analog Signals and Systems.* Upper Saddle River, NJ, USA: Pearson Prentice Hall, 2009.

[46] A. V. Oppenheim and R. W. Schafer, *Discrete-time signal processing.* Pearson Education, 2014.

[47] R. F. Harrington, *Time-harmonic electromagnetic fields.* McGraw-Hill, 1961.

[48] E. C. Jordan and K. G. Balmain, *Electromagnetic waves and radiating systems.* Prentice-Hall, 1968.

[49] G. Agarwal, D. Pattanayak, and E. Wolf, "Electromagnetic fields in spatially dispersive media," *Physical Review B*, vol. 10, no. 4, p. 1447, 1974.

[50] S. L. Chuang, *Physics of photonic devices.* John Wiley & Sons, 2012, vol. 80.

[51] B. E. Saleh and M. C. Teich, *Fundamentals of photonics.* John Wiley & Sons, 2019.

[52] M. Born and E. Wolf, *Principles of optics: electromagnetic theory of propagation, interference and diffraction of light.* Elsevier, 2013, also 1959 to 1986.

[53] R. W. Boyd, *Nonlinear optics.* Elsevier, 2003.

[54] Y.-R. Shen, *The principles of nonlinear optics.* New York, Wiley-Interscience, 1984.

[55] N. Bloembergen, *Nonlinear optics.* World Scientific, 1996.

[56] P. C. Krause, O. Wasynczuk, and S. D. Sudhoff, *Analysis of electric machinery.* McGraw-Hill New York, 1986.

[57] A. E. Fitzgerald, C. Kingsley, S. D. Umans, and B. James, *Electric machinery.* McGraw-Hill New York, 2003, vol. 5.

[58] M. A. Brown and R. C. Semelka, *MRI.: Basic Principles and Applications.* John Wiley & Sons, 2011.

[59] C. A. Balanis, *Advanced engineering electromagnetics.* John Wiley & Sons, 1999, also 1989.

[60] Wikipedia, "Lorentz force," https://en.wikipedia.org/wiki/Lorentz_force/, accessed: 2019-09-06.

[61] R. O. Dendy, *Plasma physics: an introductory course.* Cambridge University Press, 1995.

[62] P. Sen and W. C. Chew, "The frequency dependent dielectric and conductivity response of sedimentary rocks," *Journal of microwave power*, vol. 18, no. 1, pp. 95–105, 1983.

[63] D. A. Miller, *Quantum Mechanics for Scientists and Engineers.* Cambridge, UK: Cambridge University Press, 2008.

[64] W. C. Chew, "Quantum mechanics made simple: Lecture notes for ECE 487 at UIUC," *http://wcchew.ece.illinois.edu/chew/course/QMAll20161206.pdf*, 2016.

[65] B. G. Streetman and S. Banerjee, *Solid state electronic devices.* Prentice hall Englewood Cliffs, NJ, 1995.

[66] Smithsonian, "This 1600-year-old goblet shows that the romans were nanotechnology pioneers," https://www.smithsonianmag.com/history/this-1600-year-old-goblet-shows-that-the-romans-were-nanotechnology-pioneers-787224/, accessed: 2019-09-06.

[67] K. G. Budden, *Radio waves in the ionosphere.* Cambridge University Press, 2009.

[68] R. Fitzpatrick, *Plasma physics: an introduction.* CRC Press, 2014.

[69] G. Strang, *Introduction to linear algebra.* Wellesley-Cambridge Press Wellesley, MA, 1993, vol. 3.

[70] K. C. Yeh and C.-H. Liu, "Radio wave scintillations in the ionosphere," *Proceedings of the IEEE*, vol. 70, no. 4, pp. 324–360, 1982.

[71] J. Kraus, *Electromagnetics.* McGraw-Hill, 1984, also 1953, 1973, 1981.

[72] Wikipedia, "Circular polarization," https://en.wikipedia.org/wiki/Circular_polarization.

[73] Q. Zhan, "Cylindrical vector beams: from mathematical concepts to applications," *Advances in Optics and Photonics*, vol. 1, no. 1, pp. 1–57, 2009.

[74] H. Haus, *Electromagnetic Noise and Quantum Optical Measurements*, ser. Advanced Texts in Physics. Springer Berlin Heidelberg, 2000.

[75] W. C. Chew, "Lectures on theory of microwave and optical waveguides, for ECE 531 at UIUC," *https://engineering.purdue.edu/wcchew/course/tgwAll20160215.pdf*, 2016.

[76] L. Brillouin, *Wave propagation and group velocity.* Academic Press, 1960.

[77] R. Plonsey and R. E. Collin, *Principles and applications of electromagnetic fields.* McGraw-Hill, 1961.

[78] M. N. Sadiku, *Elements of electromagnetics.* Oxford University Press, 2014.

[79] A. Wadhwa, A. L. Dal, and N. Malhotra, "Transmission media," https://www.slideshare.net/abhishekwadhwa786/transmission-media-9416228.

[80] P. H. Smith, "Transmission line calculator," *Electronics*, vol. 12, no. 1, pp. 29–31, 1939.

[81] F. B. Hildebrand, *Advanced calculus for applications.* Prentice-Hall, 1962.

[82] J. Schutt-Aine, "Experiment02-coaxial transmission line measurement using slotted line," http://emlab.uiuc.edu/ece451/ECE451Lab02.pdf.

[83] D. M. Pozar, E. J. K. Knapp, and J. B. Mead, "ECE 584 microwave engineering laboratory notebook," http://www.ecs.umass.edu/ece/ece584/ECE584_lab_manual.pdf, 2004.

[84] R. E. Collin, *Field theory of guided waves.* McGraw-Hill, 1960.

[85] Q. S. Liu, S. Sun, and W. C. Chew, "A potential-based integral equation method for low-frequency electromagnetic problems," *IEEE Transactions on Antennas and Propagation*, vol. 66, no. 3, pp. 1413–1426, 2018.

[86] Wikipedia, "Snell's law," https://en.wikipedia.org/wiki/Snell's_law.

[87] G. Tyras, *Radiation and propagation of electromagnetic waves.* Academic Press, 1969.

[88] L. Brekhovskikh, *Waves in layered media.* Academic Press, 1980.

[89] Scholarpedia, "Goos-hanchen effect," http://www.scholarpedia.org/article/Goos-Hanchen_effect.

[90] K. Kao and G. A. Hockham, "Dielectric-fibre surface waveguides for optical frequencies," in *Proceedings of the Institution of Electrical Engineers*, vol. 113, no. 7. IET, 1966, pp. 1151–1158.

[91] E. Glytsis, "Slab waveguide fundamentals," http://users.ntua.gr/eglytsis/IO/Slab_Waveguides_p.pdf, 2018.

[92] Wikipedia, "Optical fiber," https://en.wikipedia.org/wiki/Optical_fiber.

[93] Atlantic Cable, "1869 indo-european cable," https://atlantic-cable.com/Cables/1869IndoEur/index.htm.

[94] Wikipedia, "Submarine communications cable," https://en.wikipedia.org/wiki/Submarine_communications_cable.

[95] D. Brewster, "On the laws which regulate the polarisation of light by reflexion from transparent bodies," *Philosophical Transactions of the Royal Society of London*, vol. 105, pp. 125–159, 1815.

[96] Wikipedia, "Brewster's angle," https://en.wikipedia.org/wiki/Brewster's_angle.

[97] H. Raether, "Surface plasmons on smooth surfaces," in *Surface plasmons on smooth and rough surfaces and on gratings.* Springer, 1988, pp. 4–39.

[98] E. Kretschmann and H. Raether, "Radiative decay of non radiative surface plasmons excited by light," *Zeitschrift für Naturforschung A*, vol. 23, no. 12, pp. 2135–2136, 1968.

[99] Wikipedia, "Surface plasmon," https://en.wikipedia.org/wiki/Surface_plasmon.

[100] Wikimedia, "Gaussian wave packet," https://commons.wikimedia.org/wiki/File:Gaussian_wave_packet.svg.

[101] Wikipedia, "Charles K. Kao," https://en.wikipedia.org/wiki/Charles_K._Kao.

[102] H. B. Callen and T. A. Welton, "Irreversibility and generalized noise," *Physical Review*, vol. 83, no. 1, p. 34, 1951.

[103] R. Kubo, "The fluctuation-dissipation theorem," *Reports on progress in physics*, vol. 29, no. 1, p. 255, 1966.

[104] C. Lee, S. Lee, and S. Chuang, "Plot of modal field distribution in rectangular and circular waveguides," *IEEE transactions on microwave theory and techniques*, vol. 33, no. 3, pp. 271–274, 1985.

[105] W. C. Chew, *Waves and Fields in Inhomogeneous Media*. IEEE Press, 1996.

[106] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. Courier Corporation, 1965, vol. 55.

[107] ——, "Handbook of mathematical functions: with formulas, graphs, and mathematical tables," http://people.math.sfu.ca/~cbm/aands/index.htm.

[108] W. C. Chew, W. Sha, and Q. I. Dai, "Green's dyadic, spectral function, local density of states, and fluctuation dissipation theorem," *arXiv preprint arXiv:1505.01586*, 2015.

[109] Wikipedia, "Very Large Array," https://en.wikipedia.org/wiki/Very_Large_Array.

[110] C. A. Balanis and E. Holzman, "Circular waveguides," *Encyclopedia of RF and Microwave Engineering*, 2005.

[111] M. Al-Hakkak and Y. Lo, "Circular waveguides with anisotropic walls," *Electronics Letters*, vol. 6, no. 24, pp. 786–789, 1970.

[112] Wikipedia, "Horn Antenna," https://en.wikipedia.org/wiki/Horn_antenna.

[113] P. Silvester and P. Benedek, "Microstrip discontinuity capacitances for right-angle bends, t junctions, and crossings," *IEEE Transactions on Microwave Theory and Techniques*, vol. 21, no. 5, pp. 341–346, 1973.

[114] R. Garg and I. Bahl, "Microstrip discontinuities," *International Journal of Electronics Theoretical and Experimental*, vol. 45, no. 1, pp. 81–87, 1978.

[115] P. Smith and E. Turner, "A bistable fabry-perot resonator," *Applied Physics Letters*, vol. 30, no. 6, pp. 280–281, 1977.

[116] A. Yariv, *Optical electronics*. Saunders College Publ., 1991.

[117] Wikipedia, "Klystron," https://en.wikipedia.org/wiki/Klystron.

[118] ——, "Magnetron," https://en.wikipedia.org/wiki/Cavity_magnetron.

[119] ——, "Absorption Wavemeter," https://en.wikipedia.org/wiki/Absorption_wavemeter.

[120] W. C. Chew, M. S. Tong, and B. Hu, "Integral equation methods for electromagnetic and elastic waves," *Synthesis Lectures on Computational Electromagnetics*, vol. 3, no. 1, pp. 1–241, 2008.

[121] A. D. Yaghjian, "Reflections on Maxwell's treatise," *Progress In Electromagnetics Research*, vol. 149, pp. 217–249, 2014.

[122] L. Nagel and D. Pederson, "Simulation program with integrated circuit emphasis," in *Midwest Symposium on Circuit Theory*, 1973.

[123] S. A. Schelkunoff and H. T. Friis, *Antennas: theory and practice*.   Wiley New York, 1952, vol. 639.

[124] H. G. Schantz, "A brief history of uwb antennas," *IEEE Aerospace and Electronic Systems Magazine*, vol. 19, no. 4, pp. 22–26, 2004.

[125] E. Kudeki, "Fields and Waves," http://remote2.ece.illinois.edu/~erhan/FieldsWaves/ECE350lectures.html.

[126] Wikipedia, "Antenna Aperture," https://en.wikipedia.org/wiki/Antenna_aperture.

[127] C. A. Balanis, *Antenna theory: analysis and design*.   John Wiley & Sons, 2016.

[128] R. W. P. King, G. S. Smith, M. Owens, and T. Wu, "Antennas in matter: Fundamentals, theory, and applications," *NASA STI/Recon Technical Report A*, vol. 81, 1981.

[129] H. Yagi and S. Uda, "Projector of the sharpest beam of electric waves," *Proceedings of the Imperial Academy*, vol. 2, no. 2, pp. 49–52, 1926.

[130] Wikipedia, "Yagi-Uda Antenna," https://en.wikipedia.org/wiki/Yagi-Uda_antenna.

[131] Antenna-theory.com, "Slot Antenna," http://www.antenna-theory.com/antennas/aperture/slot.php.

[132] A. D. Olver and P. J. Clarricoats, *Microwave horns and feeds*.   IET, 1994, vol. 39.

[133] B. Thomas, "Design of corrugated conical horns," *IEEE Transactions on Antennas and Propagation*, vol. 26, no. 2, pp. 367–372, 1978.

[134] P. J. B. Clarricoats and A. D. Olver, *Corrugated horns for microwave antennas*.   IET, 1984, no. 18.

[135] P. Gibson, "The vivaldi aerial," in *1979 9th European Microwave Conference*.   IEEE, 1979, pp. 101–105.

[136] Wikipedia, "Vivaldi Antenna," https://en.wikipedia.org/wiki/Vivaldi_antenna.

[137] ——, "Cassegrain Antenna," https://en.wikipedia.org/wiki/Cassegrain_antenna.

[138] ——, "Cassegrain Reflector," https://en.wikipedia.org/wiki/Cassegrain_reflector.

[139] W. A. Imbriale, S. S. Gao, and L. Boccia, *Space antenna handbook*.   John Wiley & Sons, 2012.

[140] J. A. Encinar, "Design of two-layer printed reflectarrays using patches of variable size," *IEEE Transactions on Antennas and Propagation*, vol. 49, no. 10, pp. 1403–1410, 2001.

[141] D.-C. Chang and M.-C. Huang, "Microstrip reflectarray antenna with offset feed," *Electronics Letters*, vol. 28, no. 16, pp. 1489–1491, 1992.

[142] G. Minatti, M. Faenzi, E. Martini, F. Caminita, P. De Vita, D. González-Ovejero, M. Sabbadini, and S. Maci, "Modulated metasurface antennas for space: Synthesis, analysis and realizations," *IEEE Transactions on Antennas and Propagation*, vol. 63, no. 4, pp. 1288–1300, 2014.

[143] X. Gao, X. Han, W.-P. Cao, H. O. Li, H. F. Ma, and T. J. Cui, "Ultrawideband and high-efficiency linear polarization converter based on double v-shaped metasurface," *IEEE Transactions on Antennas and Propagation*, vol. 63, no. 8, pp. 3522–3530, 2015.

[144] D. De Schweinitz and T. L. Frey Jr, "Artificial dielectric lens antenna," Nov. 13 2001, US Patent 6,317,092.

[145] K.-L. Wong, "Planar antennas for wireless communications," *Microwave Journal*, vol. 46, no. 10, pp. 144–145, 2003.

[146] H. Nakano, M. Yamazaki, and J. Yamauchi, "Electromagnetically coupled curl antenna," *Electronics Letters*, vol. 33, no. 12, pp. 1003–1004, 1997.

[147] K. Lee, K. Luk, K.-F. Tong, S. Shum, T. Huynh, and R. Lee, "Experimental and simulation studies of the coaxially fed U-slot rectangular patch antenna," *IEE Proceedings-Microwaves, Antennas and Propagation*, vol. 144, no. 5, pp. 354–358, 1997.

[148] K. Luk, C. Mak, Y. Chow, and K. Lee, "Broadband microstrip patch antenna," *Electronics letters*, vol. 34, no. 15, pp. 1442–1443, 1998.

[149] M. Bolic, D. Simplot-Ryl, and I. Stojmenovic, *RFID systems: research trends and challenges.* John Wiley & Sons, 2010.

[150] D. M. Dobkin, S. M. Weigand, and N. Iyer, "Segmented magnetic antennas for near-field UHF RFID," *Microwave Journal*, vol. 50, no. 6, p. 96, 2007.

[151] Z. N. Chen, X. Qing, and H. L. Chung, "A universal UHF RFID reader antenna," *IEEE transactions on microwave theory and techniques*, vol. 57, no. 5, pp. 1275–1282, 2009.

[152] C.-T. Chen, *Linear system theory and design.* Oxford University Press, Inc., 1998.

[153] S. H. Schot, "Eighty years of Sommerfeld's radiation condition," *Historia mathematica*, vol. 19, no. 4, pp. 385–401, 1992.

[154] A. Ishimaru, *Electromagnetic wave propagation, radiation, and scattering from fundamentals to applications.* Wiley Online Library, 2017, also 1991.

[155] A. E. H. Love, "I. the integration of the equations of propagation of electric waves," *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, vol. 197, no. 287-299, pp. 1–45, 1901.

[156] Wikipedia, "Christiaan Huygens," https://en.wikipedia.org/wiki/Christiaan_Huygens.

[157] ——, "George Green (mathematician)," https://en.wikipedia.org/wiki/George_Green_(mathematician).

[158] C.-T. Tai, *Dyadic Greens Functions in Electromagnetic Theory.*   PA: International Textbook, Scranton, 1971.

[159] ——, *Dyadic Green functions in electromagnetic theory.*   Institute of Electrical & Electronics Engineers (IEEE), 1994.

[160] W. Franz, "Zur formulierung des huygensschen prinzips," *Zeitschrift für Naturforschung A*, vol. 3, no. 8-11, pp. 500–506, 1948.

[161] J. A. Stratton, *Electromagnetic Theory.*   McGraw-Hill Book Company, Inc., 1941.

[162] J. D. Jackson, *Classical Electrodynamics.*   John Wiley & Sons, 1962.

[163] W. Meissner and R. Ochsenfeld, "Ein neuer effekt bei eintritt der supraleitfähigkeit," *Naturwissenschaften*, vol. 21, no. 44, pp. 787–788, 1933.

[164] Wikipedia, "Superconductivity," https://en.wikipedia.org/wiki/Superconductivity.

[165] D. Sievenpiper, L. Zhang, R. F. Broas, N. G. Alexopolous, and E. Yablonovitch, "High-impedance electromagnetic surfaces with a forbidden frequency band," *IEEE Transactions on Microwave Theory and techniques*, vol. 47, no. 11, pp. 2059–2074, 1999.

[166] Wikipedia, "Snell's law," https://en.wikipedia.org/wiki/Snell's_law.

[167] H. Lamb, "On sommerfeld's diffraction problem; and on reflection by a parabolic mirror," *Proceedings of the London Mathematical Society*, vol. 2, no. 1, pp. 190–203, 1907.

[168] W. J. Smith, *Modern optical engineering.*   McGraw-Hill New York, 1966, vol. 3.

[169] D. C. O'Shea, T. J. Suleski, A. D. Kathman, and D. W. Prather, *Diffractive optics: design, fabrication, and test.*   Spie Press Bellingham, WA, 2004, vol. 62.

[170] J. B. Keller and H. B. Keller, "Determination of reflected and transmitted fields by geometrical optics," *JOSA*, vol. 40, no. 1, pp. 48–52, 1950.

[171] G. A. Deschamps, "Ray techniques in electromagnetics," *Proceedings of the IEEE*, vol. 60, no. 9, pp. 1022–1035, 1972.

[172] R. G. Kouyoumjian and P. H. Pathak, "A uniform geometrical theory of diffraction for an edge in a perfectly conducting surface," *Proceedings of the IEEE*, vol. 62, no. 11, pp. 1448–1461, 1974.

[173] R. Kouyoumjian, "The geometrical theory of diffraction and its application," in *Numerical and Asymptotic Techniques in Electromagnetics.*   Springer, 1975, pp. 165–215.

[174] S.-W. Lee and G. Deschamps, "A uniform asymptotic theory of electromagnetic diffraction by a curved wedge," *IEEE Transactions on Antennas and Propagation*, vol. 24, no. 1, pp. 25–34, 1976.

[175] Wikipedia, "Fermat's principle," https://en.wikipedia.org/wiki/Fermat's_principle.

[176] N. Yu, P. Genevet, M. A. Kats, F. Aieta, J.-P. Tetienne, F. Capasso, and Z. Gaburro, "Light propagation with phase discontinuities: generalized laws of reflection and refraction," *Science*, vol. 334, no. 6054, pp. 333–337, 2011.

[177] A. Sommerfeld, *Partial differential equations in physics*. Academic Press, 1949, vol. 1.

[178] R. Haberman, *Elementary applied partial differential equations*. Prentice Hall Englewood Cliffs, NJ, 1983, vol. 987.

[179] G. A. Deschamps, "Gaussian beam as a bundle of complex rays," *Electronics letters*, vol. 7, no. 23, pp. 684–685, 1971.

[180] J. Enderlein and F. Pampaloni, "Unified operator approach for deriving hermite–gaussian and laguerre–gaussian laser modes," *JOSA A*, vol. 21, no. 8, pp. 1553–1558, 2004.

[181] D. L. Andrews, *Structured light and its applications: An introduction to phase-structured beams and nanoscale optical forces*. Academic Press, 2011.

[182] J. W. Strutt, "Xv. on the light from the sky, its polarization and colour," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 41, no. 271, pp. 107–120, 1871.

[183] L. Rayleigh, "X. on the electromagnetic theory of light," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 12, no. 73, pp. 81–101, 1881.

[184] R. C. Wittmann, "Spherical wave operators and the translation formulas," *IEEE Transactions on Antennas and Propagation*, vol. 36, no. 8, pp. 1078–1087, 1988.

[185] S. Sun, Y. G. Liu, W. C. Chew, and Z. Ma, "Calderón multiplicative preconditioned efie with perturbation method," *IEEE Transactions on Antennas and Propagation*, vol. 61, no. 1, pp. 247–255, 2012.

[186] G. Mie, "Beiträge zur optik trüber medien, speziell kolloidaler metallösungen," *Annalen der physik*, vol. 330, no. 3, pp. 377–445, 1908.

[187] Wikipedia, "Mie scattering," https://en.wikipedia.org/wiki/Mie_scattering.

[188] R. E. Collin, *Foundations for microwave engineering*. John Wiley & Sons, 2007, also 1966.

[189] L. B. Felsen and N. Marcuvitz, *Radiation and scattering of waves*. John Wiley & Sons, 1994, also 1973, vol. 31.

[190] P. P. Ewald, "Die berechnung optischer und elektrostatischer gitterpotentiale," *Annalen der physik*, vol. 369, no. 3, pp. 253–287, 1921.

[191] E. Whitaker and G. Watson, *A Course of Modern Analysis*. Cambridge Mathematical Library, 1927.

[192] A. Sommerfeld, *Über die Ausbreitung der Wellen in der drahtlosen Telegraphie.* Verlag der Königlich Bayerischen Akademie der Wissenschaften, 1909.

[193] J. Kong, "Electromagnetic fields due to dipole antennas over stratified anisotropic media," *Geophysics*, vol. 37, no. 6, pp. 985–996, 1972.

[194] K. Yee, "Numerical solution of initial boundary value problems involving maxwell's equations in isotropic media," *IEEE Transactions on Antennas and Propagation*, vol. 14, no. 3, pp. 302–307, 1966.

[195] A. Taflove, "Review of the formulation and applications of the finite-difference time-domain method for numerical modeling of electromagnetic wave interactions with arbitrary structures," *Wave Motion*, vol. 10, no. 6, pp. 547–582, 1988.

[196] A. Taflove and S. C. Hagness, *Computational electrodynamics: the finite-difference time-domain method.* Artech house, 2005, also 1995.

[197] W. Yu, R. Mittra, T. Su, Y. Liu, and X. Yang, *Parallel finite-difference time-domain method.* Artech House Norwood, 2006.

[198] D. Potter, "Computational physics," 1973.

[199] W. F. Ames, *Numerical methods for partial differential equations.* Academic press, 2014, also 1977.

[200] K. W. Morton, *Revival: Numerical Solution Of Convection-Diffusion Problems (1996).* CRC Press, 2019.

[201] K. Aki and P. G. Richards, *Quantitative seismology*, 2002.

[202] W. C. Chew, "Electromagnetic theory on a lattice," *Journal of Applied Physics*, vol. 75, no. 10, pp. 4843–4850, 1994.

[203] J. v. Neumann, *Mathematische Grundlagen der Quantenmechanik, Berlın.* Springer, New York, Dover Publications, 1943.

[204] R. Courant, K. Friedrichs, and H. Lewy, "Über die partiellen differenzengleichungen der mathematischen physik," *Mathematische annalen*, vol. 100, no. 1, pp. 32–74, 1928.

[205] J.-P. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," *Journal of computational physics*, vol. 114, no. 2, pp. 185–200, 1994.

[206] W. C. Chew and W. H. Weedon, "A 3d perfectly matched medium from modified maxwell's equations with stretched coordinates," *Microwave and optical technology letters*, vol. 7, no. 13, pp. 599–604, 1994.

[207] W. C. Chew, J. Jin, and E. Michielssen, "Complex coordinate system as a generalized absorbing boundary condition," in *IEEE Antennas and Propagation Society International Symposium 1997. Digest*, vol. 3. IEEE, 1997, pp. 2060–2063.

[208] W. C. H. McLean, *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, 2000.

[209] G. C. Hsiao and W. L. Wendland, *Boundary integral equations.* Springer, 2008.

[210] K. F. Warnick, *Numerical analysis for electromagnetic integral equations.* Artech House, 2008.

[211] M. M. Botha, "Solving the volume integral equations of electromagnetic scattering," *Journal of Computational Physics*, vol. 218, no. 1, pp. 141–158, 2006.

[212] P. K. Banerjee and R. Butterfield, *Boundary element methods in engineering science.* McGraw-Hill London, 1981, vol. 17.

[213] O. C. Zienkiewicz, R. L. Taylor, P. Nithiarasu, and J. Zhu, *The finite element method.* McGraw-Hill London, 1977, vol. 3.

[214] J.-F. Lee, R. Lee, and A. Cangellaris, "Time-domain finite-element methods," *IEEE Transactions on Antennas and Propagation*, vol. 45, no. 3, pp. 430–442, 1997.

[215] J. L. Volakis, A. Chatterjee, and L. C. Kempel, *Finite element method electromagnetics: antennas, microwave circuits, and scattering applications.* John Wiley & Sons, 1998, vol. 6.

[216] J.-M. Jin, *The finite element method in electromagnetics.* John Wiley & Sons, 2015.

[217] G. Strang, *Linear algebra and its applications.* Academic Press, 1976.

[218] Cramer and Gabriel, *Introduction a l'analyse des lignes courbes algebriques par Gabriel Cramer...* chez les freres Cramer & Cl. Philibert, 1750.

[219] J. A. Schouten, *Tensor analysis for physicists.* Courier Corporation, 1989.

[220] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical recipes 3rd edition: The art of scientific computing.* Cambridge University Press, 2007.