Lecture 29

Uniqueness Theorem

29.1 Uniqueness Theorem

The uniqueness of a solution to a linear system of equations is an important concept in mathematics. Under certain conditions, ordinary differential equation partial differential equation and matrix equations will have unique solutions. But uniqueness is not always guaranteed as we shall see. This issue is discussed in many math books and linear algebra books [69, 81]. The prove of uniqueness for Laplace and Poisson equations are given in [29, 48] which is slightly different for electrodynamic problems. In this section, we will prove uniqueness theorem for electrodynamic problems [31, 34, 47, 59, 75]. First, let us assume that there exist two solutions in the presence of one set of common impressed sources $\mathbf{J}_i$ and $\mathbf{M}_i$. Namely, these two solutions are $\mathbf{E}^a$, $\mathbf{H}^a$, $\mathbf{E}^b$, $\mathbf{H}^b$. Both of them satisfy Maxwell's equations and the same boundary conditions. Are $\mathbf{E}^a = \mathbf{E}^b$, $\mathbf{H}^a = \mathbf{H}^b$?

To study the uniqueness theorem, we consider general linear anisotropic inhomogeneous media, where the tensors $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\varepsilon}}$ can be complex so that lossy media can be included, it follows that

$$\nabla \times \mathbf{E}^a = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}^a - \mathbf{M}_i \tag{29.1.1}$$

$$\nabla \times \mathbf{E}^b = -j\omega\overline{\boldsymbol{\mu}} \cdot \mathbf{H}^b - \mathbf{M}_i \tag{29.1.2}$$

$$\nabla \times \mathbf{H}^a = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}^a + \mathbf{J}_i \tag{29.1.3}$$

$$\nabla \times \mathbf{H}^b = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \mathbf{E}^b + \mathbf{J}_i \tag{29.1.4}$$

By taking the difference of these two solutions, we have

$$\nabla \times (\mathbf{E}^a - \mathbf{E}^b) = -j\omega\overline{\boldsymbol{\mu}} \cdot (\mathbf{H}^a - \mathbf{H}^b) \tag{29.1.5}$$

$$\nabla \times (\mathbf{H}^a - \mathbf{H}^b) = j\omega\overline{\boldsymbol{\varepsilon}} \cdot (\mathbf{E}^a - \mathbf{E}^b) \tag{29.1.6}$$

Or alternatively, defining $\delta\mathbf{E} = \mathbf{E}^a - \mathbf{E}^b$ and $\delta\mathbf{H} = \mathbf{H}^a - \mathbf{H}^b$, we have

$$\nabla \times \delta\mathbf{E} = -j\omega\overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} \tag{29.1.7}$$

$$\nabla \times \delta\mathbf{H} = j\omega\overline{\boldsymbol{\varepsilon}} \cdot \delta\mathbf{E} \tag{29.1.8}$$

The difference solutions satisfy the original source-free Maxwell's equations.

By taking the left dot product of $\delta\mathbf{H}^*$ with (29.1.7), and then the left dot product of $\delta\mathbf{E}^*$ with the complex conjugation of (29.1.8), we obtain

$$\begin{aligned}\delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} &= -j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H}\\ \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* &= -j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*\end{aligned} \tag{29.1.9}$$

Now, taking the difference of the above, we get

$$\begin{aligned}\delta\mathbf{H}^* \cdot \nabla \times \delta\mathbf{E} - \delta\mathbf{E} \cdot \nabla \times \delta\mathbf{H}^* &= \nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*)\\ &= -j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*\end{aligned} \tag{29.1.10}$$

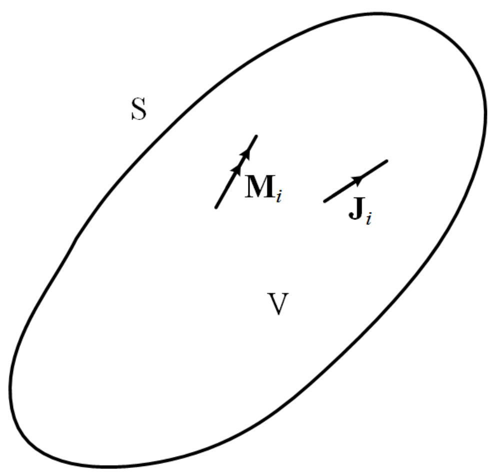

Figure 29.1: Geometry for proving the uniqueness theorem. We like to know the boundary conditions needed on S in order to guarantee the uniqueness of the solution in V.

Next, integrating the above equation over a volume V bounded by a surface S as shown in Figure 29.1. Two scenarios are possible: one that the volume V contains the impressed sources, and two, that the sources are outside the volume V. After making use of Gauss' divergence theorem, we arrive at

$$\begin{aligned}\iint_V \nabla \cdot (\delta\mathbf{E} \times \delta\mathbf{H}^*) dV &= \oiint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot d\mathbf{S}\\ &= \iiint_V [-j\omega\delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H} + j\omega\delta\mathbf{E} \cdot \overline{\boldsymbol{\varepsilon}}^* \cdot \delta\mathbf{E}^*] dV\end{aligned} \tag{29.1.11}$$

And next, we would like to know the kind of boundary conditions that would make the left-hand side equal to zero. It is seen that the surface integral on the left-hand side will be zero if:[1]

1. If $\hat{n}\times\mathbf{E}$ is specified over S so that $\hat{n}\times\mathbf{E}_a=\hat{n}\times\mathbf{E}_b$, then $\hat{n}\times\delta\mathbf{E}=0$ or the PEC boundary condition for $\delta\mathbf{E}$, and then[2]
$\oiint_S(\delta\mathbf{E}\times\delta\mathbf{H}^*)\cdot\hat{n}dS=\oiint_S(\hat{n}\times\delta\mathbf{E})\cdot\delta\mathbf{H}^*dS=0.$

2. If $\hat{n}\times\mathbf{H}$ is specified over S so that $\hat{n}\times\mathbf{H}_a=\hat{n}\times\mathbf{H}_b$, then $\hat{n}\times\delta\mathbf{H}=0$ or the PMC boundary condition for $\delta\mathbf{H}$, and then
$\oiint_S(\delta\mathbf{E}\times\delta\mathbf{H}^*)\cdot\hat{n}dS=-\oiint_S(\hat{n}\times\delta\mathbf{H}^*)\cdot\delta\mathbf{E}dS=0.$

3. If $\hat{n}\times\mathbf{E}$ is specified over S_1, and $\hat{n}\times\mathbf{H}$ is specified over S_2 (where $S_1\cup S_2=S$), then $\hat{n}\times\delta\mathbf{E}=0$ (PEC boundary condition) on S_1, and $\hat{n}\times\delta\mathbf{H}=0$ (PMC boundary condition) on S_2, then the left-hand side becomes

$$\begin{aligned}\oiint_S(\delta\mathbf{E}\times\delta\mathbf{H}^*)\cdot\hat{n}dS=\iint_{S_1}+\iint_{S_2}=&\iint_{S_1}(\hat{n}\times\delta\mathbf{E})\cdot\delta\mathbf{H}^*dS\\&-\iint_{S_2}(\hat{n}\times\delta\mathbf{H}^*)\cdot\delta\mathbf{E}dS=0.\end{aligned}$$

Thus, under the above three scenarios, the left-hand side of (29.1.11) is zero, and then the right-hand side of (29.1.11) becomes

$$\iiint_V[-j\omega\delta\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\delta\mathbf{H}+j\omega\delta\mathbf{E}\cdot\overline{\boldsymbol{\varepsilon}}^*\cdot\delta\mathbf{E}^*]dV=0 \tag{29.1.12}$$

For lossless media, $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\varepsilon}}$ are hermitian tensors (or matrices[3]), then it can be seen, using the properties of hermitian matrices or tensors, that $\delta\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\delta\mathbf{H}$ and $\delta\mathbf{E}\cdot\overline{\boldsymbol{\varepsilon}}^*\cdot\delta\mathbf{E}^*$ are purely real. Taking the imaginary part of the above equation yields

$$\iiint_V[-\delta\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\delta\mathbf{H}+\delta\mathbf{E}\cdot\overline{\boldsymbol{\varepsilon}}^*\cdot\delta\mathbf{E}^*]dV=0 \tag{29.1.13}$$

The above two terms correspond to stored magnetic field energy and stored electric field energy in the difference solutions $\delta\mathbf{H}$ and $\delta\mathbf{E}$, respectively. The above being zero does not imply that $\delta\mathbf{H}$ and $\delta\mathbf{E}$ are zero.

For resonance solutions, the stored electric energy can balance the stored magnetic energy. The above resonance solutions are those of the difference solutions satisfying PEC or PMC boundary condition or mixture thereof. Therefore, $\delta\mathbf{H}$ and $\delta\mathbf{E}$ need not be zero, even though (29.1.13) is zero. This happens when we encounter solutions that are the resonant modes of the volume V bounded by surface S.

Uniqueness can only be guaranteed if the medium is lossy as shall be shown later. It is also guaranteed if lossy impedance boundary conditions are imposed.[4] First we begin with the isotropic case.

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[1] In the following, please be reminded that PEC stands for "perfect electric conductor", while PMC stands for "perfect magnetic conductor". PMC is the dual of PEC. Also, a fourth case of impedance boundary condition is possible, which is beyond the scope of this course. Interested readers may consult Chew, Theory of Microwave and Optical Waveguides [75].

[2] Using the vector identity that $\mathbf{a}\cdot(\mathbf{b}\times c)=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$.

[3] Tensors are a special kind of matrices.

[4] See Chew, Theory of Microwave and Optical Waveguides.

29.1.1 Isotropic Case

It is easier to see this for lossy isotropic media. Then (29.1.12) simplifies to

$$\iiint_V [-j\omega\mu|\delta\mathbf{H}|^2 + j\omega\varepsilon^*|\delta\mathbf{E}|^2]dV = 0 \tag{29.1.14}$$

For isotropic lossy media, $\mu = \mu' - j\mu''$ and $\varepsilon = \varepsilon' - j\varepsilon''$. Taking the real part of the above, we have from (29.1.14) that

$$\iiint_V [-\omega\mu''|\delta\mathbf{H}|^2 - \omega\varepsilon''|\delta\mathbf{E}|^2]dV = 0 \tag{29.1.15}$$

Since the integrand in the above is always negative definite, the integral can be zero only if

$$\delta\mathbf{E} = 0, \quad \delta\mathbf{H} = 0 \tag{29.1.16}$$

everywhere in V, implying that $\mathbf{E}_a = \mathbf{E}_b$, and that $\mathbf{H}_a = \mathbf{H}_b$. Hence, it is seen that uniqueness is guaranteed only if the medium is lossy. The physical reason is that when the medium is lossy, a pure time-harmonic solution cannot exist due to loss. The modes, which are the source-free solutions of Maxwell's equations, are decaying sinusoids.

Notice that the same conclusion can be drawn if we make μ'' and ε'' negative. This corresponds to active media, and uniqueness can be guaranteed for a time-harmonic solution. In this case, no time-harmonic solution exists, and the resonant solution is a growing sinusoid.

29.1.2 General Anisotropic Case

The proof for general anisotropic media is more complicated. For the lossless anisotropic media, we see that (29.1.12) is purely imaginary. However, when the medium is lossy, this same equation will have a real part. Hence, we need to find the real part of (29.1.12) for the general lossy case.

About taking the Real and Imaginary Parts of a Complex Expression

To this end, we digress on taking the real and imaginary parts of a complex expression. Here, we need to find the complex conjugate[5] of (29.1.12), which is scalar, and add it to itself to get its real part. The complex conjugate of the scalar

$$c = \delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}} \cdot \delta\mathbf{H}$$

is[6]

$$c^* = \delta\mathbf{H} \cdot \overline{\boldsymbol{\mu}}^* \cdot \delta\mathbf{H}^* = \delta\mathbf{H}^* \cdot \overline{\boldsymbol{\mu}}^\dagger \cdot \delta\mathbf{H}$$

[5] Also called hermitian conjugate.

[6] To arrive at these expressions, one makes use of the matrix algebra rule that if $\overline{\mathbf{D}} = \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \cdot \overline{\mathbf{C}}$, then $\overline{\mathbf{D}}^t = \overline{\mathbf{C}}^t \cdot \overline{\mathbf{B}}^t \cdot \overline{\mathbf{A}}^t$. This is true even for non-square matrices. But for our case here, $\overline{\mathbf{A}}$ is a 1×3 row vector, and $\overline{\mathbf{C}}$ is a 3×1 column vector, and $\overline{\mathbf{B}}$ is a 3×3 matrix. In vector algebra, the transpose of a vector is implied. Also, in our case here, $\overline{\mathbf{D}}$ is a scalar, and hence, its transpose is itself.

Similarly, the complex conjugate of the scalar

$$d = \delta\mathbf{E}\cdot\overline{\boldsymbol{\varepsilon}}^*\cdot\delta\mathbf{E}^* = \delta\mathbf{E}^*\cdot\overline{\boldsymbol{\varepsilon}}^\dagger\cdot\delta\mathbf{E}$$

is

$$d^* = \delta\mathbf{E}^*\cdot\overline{\boldsymbol{\varepsilon}}^\dagger\cdot\delta\mathbf{E}$$

Therefore,

$$\Im m\left(\delta\mathbf{H}^*\cdot\overline{\boldsymbol{\mu}}\cdot\delta\mathbf{H}\right) = \frac{1}{2j}\delta\mathbf{H}^*\cdot(\overline{\boldsymbol{\mu}} - \overline{\mu}^\dagger)\cdot\delta\mathbf{H}$$

$$\Im m\left(\delta\mathbf{E}\cdot\overline{\varepsilon}\cdot\delta\mathbf{E}^*\right) = \frac{1}{2j}\delta\mathbf{E}^*\cdot(\overline{\varepsilon} - \overline{\varepsilon}^\dagger)\cdot\delta\mathbf{E}$$

and similarly for the real part.

Finally, after taking the complex conjugate of the scalar quantity (29.1.12) and adding it to itself, we have

$$\iiint_V [-j\omega\delta\mathbf{H}^*\cdot(\overline{\boldsymbol{\mu}} - \overline{\boldsymbol{\mu}}^\dagger)\cdot\delta\mathbf{H} - j\omega\delta\mathbf{E}^*\cdot(\overline{\varepsilon} - \overline{\varepsilon}^\dagger)\cdot\delta\mathbf{E}]dV = 0 \tag{29.1.17}$$

For lossy media, $-j(\overline{\boldsymbol{\mu}} - \overline{\boldsymbol{\mu}}^\dagger)$ and $-j(\overline{\varepsilon} - \overline{\varepsilon}^\dagger)$ are hermitian negative matrices. Hence the integrand is always negative definite, and the above equation cannot be satisfied unless $\delta\mathbf{H} = \delta\mathbf{E} = 0$ everywhere in V. Thus, uniqueness is guaranteed in a lossy anisotropic medium.

Similar statement can be made as the isotropic case if the medium is active. Then the integrand is positive definite, and the above equation cannot be satisfied unless $\delta\mathbf{H} = \delta\mathbf{E} = 0$ everywhere in V and hence, uniqueness is satisfied.

29.1.3 Hind Sight

The proof of uniqueness for Maxwell's equations is very similar to the proof of uniqueness for a matrix equation [69]

$$\overline{\mathbf{A}}\cdot\mathbf{x} = \mathbf{b} \tag{29.1.18}$$

If a solution to a matrix equation exists without excitation, namely, when $\mathbf{b} = 0$, then the solution is the null space solution [69], namely, $\mathbf{x} = \mathbf{x}_N$. In other words,

$$\overline{\mathbf{A}}\cdot\mathbf{x}_N = 0 \tag{29.1.19}$$

These null space solutions exist without a "driving term" $\mathbf{b}$ on the right-hand side. For Maxwell's Equations, $\mathbf{b}$ corresponds to the source terms. They are like the homogeneous solution of an ordinary differential equation or a partial differential equation [81]. In an enclosed region of volume V bounded by a surface S, homogeneous solutions are the resonant solutions of this Maxwellian system. When these solutions exist, they give rise to non-uniqueness.

Also, notice that (29.1.7) and (29.1.8) are Maxwell's equations without the source terms. In a closed region V bounded by a surface S, only resonance solutions for $\delta\mathbf{E}$ and $\delta\mathbf{H}$ with the relevant boundary conditions can exist when there are no source terms.

As previoulsy mentioned, one way to ensure that these resonant solutions are eliminated is to put in loss or gain. When loss or gain is present, then the resonant solutions are decaying sinusoids or growing sinusoids. Since we are looking for solutions in the frequency domain, or time harmonic solutions, we are only looking for the solution on the real ω axis on the complex ω plane. These non-sinusoidal solutions are outside the solution space: They are not part of the time-harmonic solutions we are looking for. Therefore, there are no resonant null-space solutions.

29.1.4 Connection to Poles of a Linear System

The output to input of a linear system can be represented by a transfer function $H(\omega)$ [45,153]. If $H(\omega)$ has poles, and if the system is lossless, the poles are on the real axis. Therefore, when $\omega = \omega_{\text{pole}}$, the function $H(\omega)$ becomes undefined. This also gives rise to non-uniqueness of the output with respect to the input. Poles usually correspond to resonant solutions, and hence, the non-uniqueness of the solution is intimately related to the non-uniqueness of Maxwell's equations at the resonant frequencies of a structure. This is illustrated in the upper part of Figure 29.2.

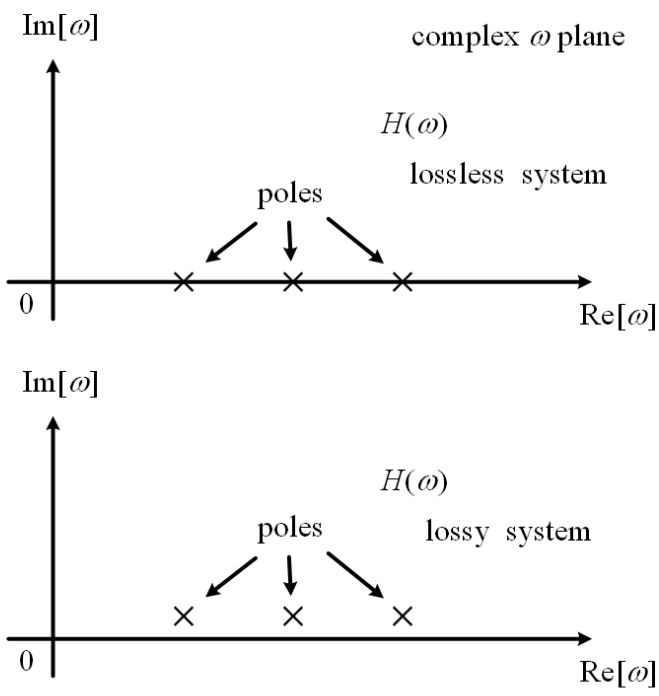

Figure 29.2: The non-uniqueness problem is intimately related to the locations of the poles of a transfer function being on the real axis.

If the input function is $f(t)$, with Fourier transform $F(\omega)$, then the output $y(t)$ is given by the following Fourier integral, viz.,

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{j\omega t} H(\omega) F(\omega) \tag{29.1.20}$$

where the Fourier inversion integral path is on the real axis on the complex ω plane. The Fourier inversion integral is undefined or non-unique.

However, if loss is introduced, these poles will move away from the real axis as shown in the lower part of Figure 29.2. Then the transfer function is uniquely determined for all frequencies on the real axis. In this way, the Fourier inversion integral in (29.1.20) is well defined, and uniqueness of the solution is guaranteed.

29.1.5 Radiation from Antenna Sources

The above uniqueness theorem guarantees that if we have some antennas with prescribed current sources on them, the radiated field from these antennas are unique. To see how this can come about, we first study the radiation of sources into a region V bounded by a large surface S_{inf} as shown in Figure 29.3 [34].

Even when $\hat{n} \times \mathbf{E}$ or $\hat{n} \times \mathbf{H}$ are specified on the surface at S_{inf}, the solution is nonunique because the volume V bounded by S_{inf}, can have many resonant solutions. In fact, the region will be replete with resonant solutions as one makes S_{inf} become very large. The way to remove these resonant solutions is to introduce an infinitesimal amount of loss in region V. Then these resonant solutions will disappear. Now we can take S_{inf} to infinity, and the solution will always be unique.

Notice that if $S_{\text{inf}} \to \infty$, the waves that leave the sources will never be reflected back because of the small amount of loss. The radiated field will just disappear into infinity. This is just what radiation loss is: power that propagate to infinity, but never to return. In fact, one way of guaranteeing the uniqueness of the solution in region V when S_{inf} is infinitely large, or that V is infinitely large is to impose the radiation condition: the waves that radiate to infinity are outgoing waves only, and never do they return. This is also called the Sommerfeld radiation condition [154]. Uniqueness of the field outside the sources is always guaranteed if we assume that the field radiates to infinity and never to return. This is equivalent to solving the cavity solutions with an infinitesimal loss, and then letting the size of the cavity become infinitely large.

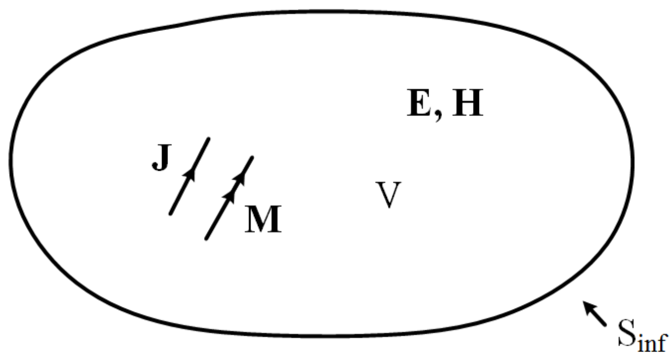

Figure 29.3: The solution for antenna radiation is unique because we impose the Sommerfeld radiation condition when seeking the solution. This is equivalent to assuming an infinitesimal loss when seeking the solution in V.

Bibliography

[1] J. A. Kong, *Theory of electromagnetic waves.* New York, Wiley-Interscience, 1975.

[2] A. Einstein *et al.*, "On the electrodynamics of moving bodies," *Annalen der Physik*, vol. 17, no. 891, p. 50, 1905.

[3] P. A. M. Dirac, "The quantum theory of the emission and absorption of radiation," *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 114, no. 767, pp. 243–265, 1927.

[4] R. J. Glauber, "Coherent and incoherent states of the radiation field," *Physical Review*, vol. 131, no. 6, p. 2766, 1963.

[5] C.-N. Yang and R. L. Mills, "Conservation of isotopic spin and isotopic gauge invariance," *Physical review*, vol. 96, no. 1, p. 191, 1954.

[6] G. t'Hooft, *50 years of Yang-Mills theory.* World Scientific, 2005.

[7] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* Princeton University Press, 2017.

[8] F. Teixeira and W. C. Chew, "Differential forms, metrics, and the reflectionless absorption of electromagnetic waves," *Journal of Electromagnetic Waves and Applications*, vol. 13, no. 5, pp. 665–686, 1999.

[9] W. C. Chew, E. Michielssen, J.-M. Jin, and J. Song, *Fast and efficient algorithms in computational electromagnetics.* Artech House, Inc., 2001.

[10] A. Volta, "On the electricity excited by the mere contact of conducting substances of different kinds. in a letter from Mr. Alexander Volta, FRS Professor of Natural Philosophy in the University of Pavia, to the Rt. Hon. Sir Joseph Banks, Bart. KBPR S," *Philosophical transactions of the Royal Society of London*, no. 90, pp. 403–431, 1800.

[11] A.-M. Ampère, *Exposé méthodique des phénomènes électro-dynamiques, et des lois de ces phénomènes.* Bachelier, 1823.

[12] ——, *Mémoire sur la théorie mathématique des phénomènes électro-dynamiques uniquement déduite de l'expérience: dans lequel se trouvent réunis les Mémoires que M. Ampère a communiqués à l'Académie royale des Sciences, dans les séances des 4 et*

26 décembre 1820, 10 juin 1822, 22 décembre 1823, 12 septembre et 21 novembre 1825. Bachelier, 1825.

[13] B. Jones and M. Faraday, *The life and letters of Faraday.* Cambridge University Press, 2010, vol. 2.

[14] G. Kirchhoff, "Ueber die auflösung der gleichungen, auf welche man bei der untersuchung der linearen vertheilung galvanischer ströme geführt wird," *Annalen der Physik*, vol. 148, no. 12, pp. 497–508, 1847.

[15] L. Weinberg, "Kirchhoff's' third and fourth laws'," *IRE Transactions on Circuit Theory*, vol. 5, no. 1, pp. 8–30, 1958.

[16] T. Standage, *The Victorian Internet: The remarkable story of the telegraph and the nineteenth century's online pioneers.* Phoenix, 1998.

[17] J. C. Maxwell, "A dynamical theory of the electromagnetic field," *Philosophical transactions of the Royal Society of London*, no. 155, pp. 459–512, 1865.

[18] H. Hertz, "On the finite velocity of propagation of electromagnetic actions," *Electric Waves*, vol. 110, 1888.

[19] M. Romer and I. B. Cohen, "Roemer and the first determination of the velocity of light (1676)," *Isis*, vol. 31, no. 2, pp. 327–379, 1940.

[20] A. Arons and M. Peppard, "Einstein's proposal of the photon concept–a translation of the Annalen der Physik paper of 1905," *American Journal of Physics*, vol. 33, no. 5, pp. 367–374, 1965.

[21] A. Pais, "Einstein and the quantum theory," *Reviews of Modern Physics*, vol. 51, no. 4, p. 863, 1979.

[22] M. Planck, "On the law of distribution of energy in the normal spectrum," *Annalen der physik*, vol. 4, no. 553, p. 1, 1901.

[23] Z. Peng, S. De Graaf, J. Tsai, and O. Astafiev, "Tuneable on-demand single-photon source in the microwave range," *Nature communications*, vol. 7, p. 12588, 2016.

[24] B. D. Gates, Q. Xu, M. Stewart, D. Ryan, C. G. Willson, and G. M. Whitesides, "New approaches to nanofabrication: molding, printing, and other techniques," *Chemical reviews*, vol. 105, no. 4, pp. 1171–1196, 2005.

[25] J. S. Bell, "The debate on the significance of his contributions to the foundations of quantum mechanics, Bell's Theorem and the Foundations of Modern Physics (A. van der Merwe, F. Selleri, and G. Tarozzi, eds.)," 1992.

[26] D. J. Griffiths and D. F. Schroeter, *Introduction to quantum mechanics.* Cambridge University Press, 2018.

[27] C. Pickover, *Archimedes to Hawking: Laws of science and the great minds behind them.* Oxford University Press, 2008.

[28] R. Resnick, J. Walker, and D. Halliday, *Fundamentals of physics.* John Wiley, 1988.

[29] S. Ramo, J. R. Whinnery, and T. Duzer van, *Fields and waves in communication electronics, Third Edition.* John Wiley & Sons, Inc., 1995, also 1965, 1984.

[30] J. L. De Lagrange, "Recherches d'arithmétique," *Nouveaux Mémoires de l'Académie de Berlin*, 1773.

[31] J. A. Kong, *Electromagnetic Wave Theory.* EMW Publishing, 2008.

[32] H. M. Schey, *Div, grad, curl, and all that: an informal text on vector calculus.* WW Norton New York, 2005.

[33] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman lectures on physics, Vols. I, II, & III: The new millennium edition.* Basic books, 2011, vol. 1,2,3.

[34] W. C. Chew, *Waves and fields in inhomogeneous media.* IEEE Press, 1995, also 1990.

[35] V. J. Katz, "The history of Stokes' theorem," *Mathematics Magazine*, vol. 52, no. 3, pp. 146–156, 1979.

[36] W. K. Panofsky and M. Phillips, *Classical electricity and magnetism.* Courier Corporation, 2005.

[37] T. Lancaster and S. J. Blundell, *Quantum field theory for the gifted amateur.* OUP Oxford, 2014.

[38] W. C. Chew, "Fields and waves: Lecture notes for ECE 350 at UIUC," *https://engineering.purdue.edu/wcchew/ece350.html*, 1990.

[39] C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers I: Asymptotic methods and perturbation theory.* Springer Science & Business Media, 2013.

[40] J. M. Crowley, *Fundamentals of applied electrostatics.* Krieger Publishing Company, 1986.

[41] C. Balanis, *Advanced Engineering Electromagnetics.* Hoboken, NJ, USA: Wiley, 2012.

[42] J. D. Jackson, *Classical electrodynamics.* John Wiley & Sons, 1999.

[43] R. Courant and D. Hilbert, *Methods of Mathematical Physics: Partial Differential Equations.* John Wiley & Sons, 2008.

[44] L. Esaki and R. Tsu, "Superlattice and negative differential conductivity in semiconductors," *IBM Journal of Research and Development*, vol. 14, no. 1, pp. 61–65, 1970.

[45] E. Kudeki and D. C. Munson, *Analog Signals and Systems.* Upper Saddle River, NJ, USA: Pearson Prentice Hall, 2009.

[46] A. V. Oppenheim and R. W. Schafer, *Discrete-time signal processing.* Pearson Education, 2014.

[47] R. F. Harrington, *Time-harmonic electromagnetic fields.* McGraw-Hill, 1961.

[48] E. C. Jordan and K. G. Balmain, *Electromagnetic waves and radiating systems.* Prentice-Hall, 1968.

[49] G. Agarwal, D. Pattanayak, and E. Wolf, "Electromagnetic fields in spatially dispersive media," *Physical Review B*, vol. 10, no. 4, p. 1447, 1974.

[50] S. L. Chuang, *Physics of photonic devices.* John Wiley & Sons, 2012, vol. 80.

[51] B. E. Saleh and M. C. Teich, *Fundamentals of photonics.* John Wiley & Sons, 2019.

[52] M. Born and E. Wolf, *Principles of optics: electromagnetic theory of propagation, interference and diffraction of light.* Elsevier, 2013.

[53] R. W. Boyd, *Nonlinear optics.* Elsevier, 2003.

[54] Y.-R. Shen, *The principles of nonlinear optics.* New York, Wiley-Interscience, 1984.

[55] N. Bloembergen, *Nonlinear optics.* World Scientific, 1996.

[56] P. C. Krause, O. Wasynczuk, and S. D. Sudhoff, *Analysis of electric machinery.* McGraw-Hill New York, 1986.

[57] A. E. Fitzgerald, C. Kingsley, S. D. Umans, and B. James, *Electric machinery.* McGraw-Hill New York, 2003, vol. 5.

[58] M. A. Brown and R. C. Semelka, *MRI.: Basic Principles and Applications.* John Wiley & Sons, 2011.

[59] C. A. Balanis, *Advanced engineering electromagnetics.* John Wiley & Sons, 1999, also 1989.

[60] Wikipedia, "Lorentz force," https://en.wikipedia.org/wiki/Lorentz_force/, accessed: 2019-09-06.

[61] R. O. Dendy, *Plasma physics: an introductory course.* Cambridge University Press, 1995.

[62] P. Sen and W. C. Chew, "The frequency dependent dielectric and conductivity response of sedimentary rocks," *Journal of microwave power*, vol. 18, no. 1, pp. 95–105, 1983.

[63] D. A. Miller, *Quantum Mechanics for Scientists and Engineers.* Cambridge, UK: Cambridge University Press, 2008.

[64] W. C. Chew, "Quantum mechanics made simple: Lecture notes for ECE 487 at UIUC," *http://wcchew.ece.illinois.edu/chew/course/QMAll20161206.pdf*, 2016.

[65] B. G. Streetman and S. Banerjee, *Solid state electronic devices.* Prentice hall Englewood Cliffs, NJ, 1995.

[66] Smithsonian, "This 1600-year-old goblet shows that the romans were nanotechnology pioneers," https://www.smithsonianmag.com/history/this-1600-year-old-goblet-shows-that-the-romans-were-nanotechnology-pioneers-787224/, accessed: 2019-09-06.

[67] K. G. Budden, *Radio waves in the ionosphere.* Cambridge University Press, 2009.

[68] R. Fitzpatrick, *Plasma physics: an introduction.* CRC Press, 2014.

[69] G. Strang, *Introduction to linear algebra.* Wellesley-Cambridge Press Wellesley, MA, 1993, vol. 3.

[70] K. C. Yeh and C.-H. Liu, "Radio wave scintillations in the ionosphere," *Proceedings of the IEEE*, vol. 70, no. 4, pp. 324–360, 1982.

[71] J. Kraus, *Electromagnetics.* McGraw-Hill, 1984, also 1953, 1973, 1981.

[72] Wikipedia, "Circular polarization," https://en.wikipedia.org/wiki/Circular_polarization.

[73] Q. Zhan, "Cylindrical vector beams: from mathematical concepts to applications," *Advances in Optics and Photonics*, vol. 1, no. 1, pp. 1–57, 2009.

[74] H. Haus, *Electromagnetic Noise and Quantum Optical Measurements*, ser. Advanced Texts in Physics. Springer Berlin Heidelberg, 2000.

[75] W. C. Chew, "Lectures on theory of microwave and optical waveguides, for ECE 531 at UIUC," *https://engineering.purdue.edu/wcchew/course/tgwAll20160215.pdf*, 2016.

[76] L. Brillouin, *Wave propagation and group velocity.* Academic Press, 1960.

[77] R. Plonsey and R. E. Collin, *Principles and applications of electromagnetic fields.* McGraw-Hill, 1961.

[78] M. N. Sadiku, *Elements of electromagnetics.* Oxford University Press, 2014.

[79] A. Wadhwa, A. L. Dal, and N. Malhotra, "Transmission media," https://www.slideshare.net/abhishekwadhwa786/transmission-media-9416228.

[80] P. H. Smith, "Transmission line calculator," *Electronics*, vol. 12, no. 1, pp. 29–31, 1939.

[81] F. B. Hildebrand, *Advanced calculus for applications.* Prentice-Hall, 1962.

[82] J. Schutt-Aine, "Experiment02-coaxial transmission line measurement using slotted line," http://emlab.uiuc.edu/ece451/ECE451Lab02.pdf.

[83] D. M. Pozar, E. J. K. Knapp, and J. B. Mead, "ECE 584 microwave engineering laboratory notebook," http://www.ecs.umass.edu/ece/ece584/ECE584_lab_manual.pdf, 2004.

[84] R. E. Collin, *Field theory of guided waves.* McGraw-Hill, 1960.

[85] Q. S. Liu, S. Sun, and W. C. Chew, "A potential-based integral equation method for low-frequency electromagnetic problems," *IEEE Transactions on Antennas and Propagation*, vol. 66, no. 3, pp. 1413–1426, 2018.

[86] M. Born and E. Wolf, *Principles of optics: electromagnetic theory of propagation, interference and diffraction of light*. Pergamon, 1986, first edition 1959.

[87] Wikipedia, "Snell's law," https://en.wikipedia.org/wiki/Snell's_law.

[88] G. Tyras, *Radiation and propagation of electromagnetic waves*. Academic Press, 1969.

[89] L. Brekhovskikh, *Waves in layered media*. Academic Press, 1980.

[90] Scholarpedia, "Goos-hanchen effect," http://www.scholarpedia.org/article/Goos-Hanchen_effect.

[91] K. Kao and G. A. Hockham, "Dielectric-fibre surface waveguides for optical frequencies," in *Proceedings of the Institution of Electrical Engineers*, vol. 113, no. 7. IET, 1966, pp. 1151–1158.

[92] E. Glytsis, "Slab waveguide fundamentals," http://users.ntua.gr/eglytsis/IO/Slab_Waveguides_p.pdf, 2018.

[93] Wikipedia, "Optical fiber," https://en.wikipedia.org/wiki/Optical_fiber.

[94] Atlantic Cable, "1869 indo-european cable," https://atlantic-cable.com/Cables/1869IndoEur/index.htm.

[95] Wikipedia, "Submarine communications cable," https://en.wikipedia.org/wiki/Submarine_communications_cable.

[96] D. Brewster, "On the laws which regulate the polarisation of light by reflexion from transparent bodies," *Philosophical Transactions of the Royal Society of London*, vol. 105, pp. 125–159, 1815.

[97] Wikipedia, "Brewster's angle," https://en.wikipedia.org/wiki/Brewster's_angle.

[98] H. Raether, "Surface plasmons on smooth surfaces," in *Surface plasmons on smooth and rough surfaces and on gratings*. Springer, 1988, pp. 4–39.

[99] E. Kretschmann and H. Raether, "Radiative decay of non radiative surface plasmons excited by light," *Zeitschrift für Naturforschung A*, vol. 23, no. 12, pp. 2135–2136, 1968.

[100] Wikipedia, "Surface plasmon," https://en.wikipedia.org/wiki/Surface_plasmon.

[101] Wikimedia, "Gaussian wave packet," https://commons.wikimedia.org/wiki/File:Gaussian_wave_packet.svg.

[102] Wikipedia, "Charles K. Kao," https://en.wikipedia.org/wiki/Charles_K._Kao.

[103] H. B. Callen and T. A. Welton, "Irreversibility and generalized noise," *Physical Review*, vol. 83, no. 1, p. 34, 1951.

[104] R. Kubo, "The fluctuation-dissipation theorem," *Reports on progress in physics*, vol. 29, no. 1, p. 255, 1966.

[105] C. Lee, S. Lee, and S. Chuang, "Plot of modal field distribution in rectangular and circular waveguides," *IEEE transactions on microwave theory and techniques*, vol. 33, no. 3, pp. 271–274, 1985.

[106] W. C. Chew, *Waves and Fields in Inhomogeneous Media.* IEEE Press, 1996.

[107] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions: with formulas, graphs, and mathematical tables.* Courier Corporation, 1965, vol. 55.

[108] ——, "Handbook of mathematical functions: with formulas, graphs, and mathematical tables," http://people.math.sfu.ca/~cbm/aands/index.htm.

[109] W. C. Chew, W. Sha, and Q. I. Dai, "Green's dyadic, spectral function, local density of states, and fluctuation dissipation theorem," *arXiv preprint arXiv:1505.01586*, 2015.

[110] Wikipedia, "Very Large Array," https://en.wikipedia.org/wiki/Very_Large_Array.

[111] C. A. Balanis and E. Holzman, "Circular waveguides," *Encyclopedia of RF and Microwave Engineering*, 2005.

[112] M. Al-Hakkak and Y. Lo, "Circular waveguides with anisotropic walls," *Electronics Letters*, vol. 6, no. 24, pp. 786–789, 1970.

[113] Wikipedia, "Horn Antenna," https://en.wikipedia.org/wiki/Horn_antenna.

[114] P. Silvester and P. Benedek, "Microstrip discontinuity capacitances for right-angle bends, t junctions, and crossings," *IEEE Transactions on Microwave Theory and Techniques*, vol. 21, no. 5, pp. 341–346, 1973.

[115] R. Garg and I. Bahl, "Microstrip discontinuities," *International Journal of Electronics Theoretical and Experimental*, vol. 45, no. 1, pp. 81–87, 1978.

[116] P. Smith and E. Turner, "A bistable fabry-perot resonator," *Applied Physics Letters*, vol. 30, no. 6, pp. 280–281, 1977.

[117] A. Yariv, *Optical electronics.* Saunders College Publ., 1991.

[118] Wikipedia, "Klystron," https://en.wikipedia.org/wiki/Klystron.

[119] ——, "Magnetron," https://en.wikipedia.org/wiki/Cavity_magnetron.

[120] ——, "Absorption Wavemeter," https://en.wikipedia.org/wiki/Absorption_wavemeter.

[121] W. C. Chew, M. S. Tong, and B. Hu, "Integral equation methods for electromagnetic and elastic waves," *Synthesis Lectures on Computational Electromagnetics*, vol. 3, no. 1, pp. 1–241, 2008.

[122] A. D. Yaghjian, "Reflections on Maxwell's treatise," *Progress In Electromagnetics Research*, vol. 149, pp. 217–249, 2014.

[123] L. Nagel and D. Pederson, "Simulation program with integrated circuit emphasis," in *Midwest Symposium on Circuit Theory*, 1973.

[124] S. A. Schelkunoff and H. T. Friis, *Antennas: theory and practice.* Wiley New York, 1952, vol. 639.

[125] H. G. Schantz, "A brief history of uwb antennas," *IEEE Aerospace and Electronic Systems Magazine*, vol. 19, no. 4, pp. 22–26, 2004.

[126] E. Kudeki, "Fields and Waves," http://remote2.ece.illinois.edu/~erhan/FieldsWaves/ECE350lectures.html.

[127] Wikipedia, "Antenna Aperture," https://en.wikipedia.org/wiki/Antenna_aperture.

[128] C. A. Balanis, *Antenna theory: analysis and design.* John Wiley & Sons, 2016.

[129] R. W. P. King, G. S. Smith, M. Owens, and T. Wu, "Antennas in matter: Fundamentals, theory, and applications," *NASA STI/Recon Technical Report A*, vol. 81, 1981.

[130] H. Yagi and S. Uda, "Projector of the sharpest beam of electric waves," *Proceedings of the Imperial Academy*, vol. 2, no. 2, pp. 49–52, 1926.

[131] Wikipedia, "Yagi-Uda Antenna," https://en.wikipedia.org/wiki/Yagi-Uda_antenna.

[132] Antenna-theory.com, "Slot Antenna," http://www.antenna-theory.com/antennas/aperture/slot.php.

[133] A. D. Olver and P. J. Clarricoats, *Microwave horns and feeds.* IET, 1994, vol. 39.

[134] B. Thomas, "Design of corrugated conical horns," *IEEE Transactions on Antennas and Propagation*, vol. 26, no. 2, pp. 367–372, 1978.

[135] P. J. B. Clarricoats and A. D. Olver, *Corrugated horns for microwave antennas.* IET, 1984, no. 18.

[136] P. Gibson, "The vivaldi aerial," in *1979 9th European Microwave Conference.* IEEE, 1979, pp. 101–105.

[137] Wikipedia, "Vivaldi Antenna," https://en.wikipedia.org/wiki/Vivaldi_antenna.

[138] ——, "Cassegrain Antenna," https://en.wikipedia.org/wiki/Cassegrain_antenna.

[139] ——, "Cassegrain Reflector," https://en.wikipedia.org/wiki/Cassegrain_reflector.

[140] W. A. Imbriale, S. S. Gao, and L. Boccia, *Space antenna handbook.* John Wiley & Sons, 2012.

[141] J. A. Encinar, "Design of two-layer printed reflectarrays using patches of variable size," *IEEE Transactions on Antennas and Propagation*, vol. 49, no. 10, pp. 1403–1410, 2001.

[142] D.-C. Chang and M.-C. Huang, "Microstrip reflectarray antenna with offset feed," *Electronics Letters*, vol. 28, no. 16, pp. 1489–1491, 1992.

[143] G. Minatti, M. Faenzi, E. Martini, F. Caminita, P. De Vita, D. González-Ovejero, M. Sabbadini, and S. Maci, "Modulated metasurface antennas for space: Synthesis, analysis and realizations," *IEEE Transactions on Antennas and Propagation*, vol. 63, no. 4, pp. 1288–1300, 2014.

[144] X. Gao, X. Han, W.-P. Cao, H. O. Li, H. F. Ma, and T. J. Cui, "Ultrawideband and high-efficiency linear polarization converter based on double v-shaped metasurface," *IEEE Transactions on Antennas and Propagation*, vol. 63, no. 8, pp. 3522–3530, 2015.

[145] D. De Schweinitz and T. L. Frey Jr, "Artificial dielectric lens antenna," Nov. 13 2001, US Patent 6,317,092.

[146] K.-L. Wong, "Planar antennas for wireless communications," *Microwave Journal*, vol. 46, no. 10, pp. 144–145, 2003.

[147] H. Nakano, M. Yamazaki, and J. Yamauchi, "Electromagnetically coupled curl antenna," *Electronics Letters*, vol. 33, no. 12, pp. 1003–1004, 1997.

[148] K. Lee, K. Luk, K.-F. Tong, S. Shum, T. Huynh, and R. Lee, "Experimental and simulation studies of the coaxially fed U-slot rectangular patch antenna," *IEE Proceedings-Microwaves, Antennas and Propagation*, vol. 144, no. 5, pp. 354–358, 1997.

[149] K. Luk, C. Mak, Y. Chow, and K. Lee, "Broadband microstrip patch antenna," *Electronics letters*, vol. 34, no. 15, pp. 1442–1443, 1998.

[150] M. Bolic, D. Simplot-Ryl, and I. Stojmenovic, *RFID systems: research trends and challenges*. John Wiley & Sons, 2010.

[151] D. M. Dobkin, S. M. Weigand, and N. Iyer, "Segmented magnetic antennas for near-field UHF RFID," *Microwave Journal*, vol. 50, no. 6, p. 96, 2007.

[152] Z. N. Chen, X. Qing, and H. L. Chung, "A universal UHF RFID reader antenna," *IEEE transactions on microwave theory and techniques*, vol. 57, no. 5, pp. 1275–1282, 2009.

[153] C.-T. Chen, *Linear system theory and design*. Oxford University Press, Inc., 1998.

[154] S. H. Schot, "Eighty years of Sommerfeld's radiation condition," *Historia mathematica*, vol. 19, no. 4, pp. 385–401, 1992.

[155] A. Ishimaru, *Electromagnetic wave propagation, radiation, and scattering from fundamentals to applications*. Wiley Online Library, 2017, also 1991.

[156] A. E. H. Love, "I. the integration of the equations of propagation of electric waves," *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, vol. 197, no. 287-299, pp. 1–45, 1901.

[157] Wikipedia, "Christiaan Huygens," https://en.wikipedia.org/wiki/Christiaan_Huygens.

[158] ——, "George Green (mathematician)," https://en.wikipedia.org/wiki/George_Green_(mathematician).

[159] C.-T. Tai, *Dyadic Green's Functions in Electromagnetic Theory.* PA: International Textbook, Scranton, 1971.

[160] ——, *Dyadic Green functions in electromagnetic theory.* Institute of Electrical & Electronics Engineers (IEEE), 1994.

[161] W. Franz, "Zur formulierung des huygensschen prinzips," *Zeitschrift für Naturforschung A*, vol. 3, no. 8-11, pp. 500–506, 1948.

[162] J. A. Stratton, *Electromagnetic Theory.* McGraw-Hill Book Company, Inc., 1941.

[163] J. D. Jackson, *Classical Electrodynamics.* John Wiley & Sons, 1962.

[164] W. Meissner and R. Ochsenfeld, "Ein neuer effekt bei eintritt der supraleitfähigkeit," *Naturwissenschaften*, vol. 21, no. 44, pp. 787–788, 1933.

[165] Wikipedia, "Superconductivity," https://en.wikipedia.org/wiki/Superconductivity.

[166] D. Sievenpiper, L. Zhang, R. F. Broas, N. G. Alexopolous, and E. Yablonovitch, "High-impedance electromagnetic surfaces with a forbidden frequency band," *IEEE Transactions on Microwave Theory and techniques*, vol. 47, no. 11, pp. 2059–2074, 1999.