Lecture 9, Sept 25, 2018

Energy and Power W. C. Chew

Consider Maxwell's equations where fictitious magnetic current is included. They are

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} - \bar{M}_i = -\mu \frac{\partial \bar{H}}{\partial t} - \bar{M}_i \qquad (1)$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} = \epsilon \frac{\partial \bar{E}}{\partial t} + \bar{J}_i + \sigma \bar{E} \qquad (2)$$

where $\bar{M}_i$ and $\bar{J}_i$ are impressed current sources, while $\bar{J} = \sigma \bar{E}$ is the induced current source. $\bar{J} = \sigma \bar{E}$ is similar to Ohm's law. We can show that

$$\bar{H} \cdot \nabla \times \bar{E} = -\mu \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} - \bar{H} \cdot \bar{M}_i \qquad (3)$$

$$\bar{E} \cdot \nabla \times \bar{H} = \epsilon \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} + \bar{E} \cdot \bar{J}_i + \sigma \bar{E} \cdot \bar{E} \qquad (4)$$

Using the identity, which is the same as the product rule for derivatives, we have

$$\nabla \cdot (\bar{E} \times \bar{H}) = \bar{H} \cdot \nabla \times \bar{E} - \bar{E} \cdot (\nabla \times \bar{H}) \qquad (5)$$

Therefore, from (3), (4), and (5) we have

$$\nabla \cdot (\bar{E} \times \bar{H}) = -\left(\mu \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} + \epsilon \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} + \sigma \bar{E} \cdot \bar{E} + \bar{H} \cdot \bar{M}_i + \bar{E} \cdot \bar{J}_i \right) \qquad (6)$$

The physical meaning of the above is more lucid if we first consider $\sigma = 0$, $\bar{M}_i = \bar{J}_i = 0$, or the absence of the impressed current sources. Then the above becomes

$$\nabla \cdot (\bar{E} \times \bar{H}) = -\left(\mu \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} + \epsilon \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} \right) \qquad (7)$$

Rewriting

$$\mu \bar{H} \cdot \frac{\partial \bar{H}}{\partial t} = \frac{1}{2} \mu \frac{\partial}{\partial t} \bar{H} \cdot \bar{H} = \frac{\partial}{\partial t} \left(\frac{1}{2} \mu |\bar{H}|^2 \right) = \frac{\partial}{\partial t} W_m \qquad (8)$$

$$\epsilon \bar{E} \cdot \frac{\partial \bar{E}}{\partial t} = \frac{1}{2} \epsilon \frac{\partial}{\partial t} \bar{E} \cdot \bar{E} = \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon |\bar{E}|^2 \right) = \frac{\partial}{\partial t} W_e \qquad (9)$$

Then (7) becomes

$$\nabla \cdot (\bar{E} \times \bar{H}) = -\frac{\partial}{\partial t} (W_m + W_e) \qquad (10)$$

Eq (10) is reminiscent of the current continuity equation

$$\nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t} \qquad (11)$$

Hence $\bar{E} \times \bar{H}$ has the meaning of power flow, and W_m, W_e are the energy density stored in the magnetic field and electric field respectively. In fact, one can shows that $\bar{E} \times \bar{H}$ has the unit of $(V m^{-1})(A m^{-1}) = \text{watt } m^{-2}$ which is the unit of power density. Similarly, W_m has the unit of henry $m^{-1} (A/m)^2 = \text{joule}/m^3$ which is energy density. We can ascertain the unit of $\frac{1}{2}\mu|\bar{H}|^2$ by noticing that the energy stored in an inductor is $\frac{1}{2}LI^2$. Also, W_e has the unit of farad $m^{-1}\left(\frac{V}{m}\right)^2 = \text{joule}/m^3$ which is energy density again. We can ascertain the unit of $\frac{1}{2}\epsilon|\bar{E}|^2$ by noticing that the energy stored in a capacitor is $\frac{1}{2}CV^2$. The quantity

$$\bar{S}_p = \bar{E} \times \bar{H} \qquad (12)$$

is called the Poynting's vector, and (6) becomes

$$\nabla \cdot \bar{S}_p = -\frac{\partial}{\partial t} W_t \qquad (13)$$

where $W_t = W_e + W_m$, the total energy stored.

Now, if we let $\sigma \neq 0$, then, the term to be included $\sigma \bar{E} \cdot \bar{E}$ has the unit of siemen $m^{-1} (V/m)^2$, or w/m^3. We gather this unit by noticing that $\frac{1}{2}\frac{V^2}{R}$ is the power dissipated in a resistor of R ohms. The reciprocal unit of ohms, which used to be called mhos is now called siemens. With $\sigma \neq 0$, (13) becomes

$$\nabla \cdot \bar{S}_p = -\frac{\partial}{\partial t} W_t - \sigma|\bar{E}|^2 = -\frac{\partial}{\partial t} W_t - P_d \qquad (14)$$

$\nabla \cdot \bar{S}_p$ has the physical meaning of power density oozing out of a point, and $P_d = \sigma|\bar{E}|^2$ has the physical meaning of power density siphoned by the loss in the medium which is proportional to $\sigma|\bar{E}|^2$.

Now if we set $\bar{J}_i$ and $\bar{M}_i$ to be nonzero, (14) is augmented by the last two terms in (6), or

$$\nabla \cdot \bar{S}_p = -\frac{\partial}{\partial t} W_t - p_d - \bar{H} \cdot \bar{M}_i - \bar{E} \cdot \bar{J}_i \qquad (15)$$

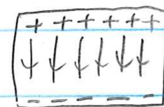
The last two terms can be interpreted as the power density supplied by the impressed currents $\bar{M}_i$ and $\bar{J}_i$. Hence, (15) becomes

$$\nabla \cdot \bar{S}_p = -\frac{\partial}{\partial t} W_t - p_d + p_s \qquad (16)$$

where

$$p_s = -\bar{H} \cdot \bar{M}_i - \bar{E} \cdot \bar{J}_i \qquad (17)$$

The last term is positive if $\bar{E}$ and $\bar{J}_i$ have opposite signs. This reminds us of a battery where the positive charges move from a region of lower potential to a region of higher potential.



A simplified battery or voltaic cell.

The positive charges move from one end of a battery to the other end of the battery. Hence, they are doing an "uphill climb" due to chemical processes within the battery.

In the above, one can easily work out that p_s has the unit of watts/m^3 which is power supplied density. One can also choose to rewrite (16) in integral form by integrating over a volume V and invoking the divergence theorem

$$\int_S d\bar{S} \cdot \bar{S}_p = -\frac{\partial}{\partial t} \int_V W_t \, dV - \int_V p_d \, dV + \int_V p_s \, dV \qquad (18)$$

The LHS is

$$\int_S d\bar{S} \cdot (\bar{E} \times \bar{H}) \qquad (19)$$

represents the power flowing out of the surface S.

Emergence of Wave Phenomenon, Triumph of M. E.

One of the major triumph of M. E. is the prediction of the wave phenomenon. Then it was realized that electromagnetic wave propagates at a tremendous velocity which is the velocity of light.

To see this, we consider M. E. in vacuum or a source free medium. They are

$$\nabla\times\bar{E} = -\mu_0 \frac{\partial \bar{H}}{\partial t} \quad (1)$$

$$\nabla\times\bar{H} = \epsilon_0 \frac{\partial \bar{E}}{\partial t} \quad (2)$$

Taking the curl of (1), we have

$$\nabla\times\nabla\times\bar{E} = -\mu_0 \frac{\partial}{\partial t}\nabla\times\bar{H} \quad (3)$$

Substituting (2) into (3), we have

$$\nabla\times\nabla\times\bar{E} = -\mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\bar{E} \quad (4)$$

Furthermore, using the identity that

$$\nabla\times\nabla\times\bar{E} = \nabla\nabla\cdot\bar{E} - \nabla^2\bar{E} \quad (5)$$

and that $\nabla\cdot\bar{E} = 0$ in a source-free medium, we have

$$\nabla^2\bar{E} - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}\bar{E} = 0 \quad (6)$$

In the above, we can let $\bar{E} = \hat{x}E_x(z,t)$ so that $\nabla\cdot\bar{E} = 0$ satisfying the source-free condition. Then (6) becomes

$$\frac{\partial^2}{\partial z^2}E_x(z,t) - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}E_x(z,t) = 0 \quad (7)$$

Eq. (7) is known mathematically as the wave equation. It can also be written as

$$\frac{\partial^2}{\partial z^2} f(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} f(z,t) = 0 \qquad (8)$$

where $c_0^2 = (\mu_0 \epsilon_0)^{-1}$! Eq (8) can also be factorized as

$$\left(\frac{\partial}{\partial z} - \frac{1}{c_0}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} + \frac{1}{c_0}\frac{\partial}{\partial t}\right) f(z,t) = 0 \qquad (9)$$

or

$$\left(\frac{\partial}{\partial z} + \frac{1}{c_0}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} - \frac{1}{c_0}\frac{\partial}{\partial t}\right) f(z,t) = 0 \qquad (10)$$

Hence we have

$$\left(\frac{\partial}{\partial z} + \frac{1}{c_0}\frac{\partial}{\partial t}\right) f_+(z,t) = 0 \qquad (11)$$

or

$$\left(\frac{\partial}{\partial z} - \frac{1}{c_0}\frac{\partial}{\partial t}\right) f_-(z,t) = 0 \qquad (12)$$

Equations (11) and (12) are known as the one-way wave equations or advective equations. Their general solutions are

$$f_+(z,t) = F_+(z - c_0 t) \qquad (13)$$

$$f_-(z,t) = F_-(z + c_0 t) \qquad (14)$$

Eq (13) constitutes a right-travelling wave function of any shape while (14) constitutes a left-travelling wave function of any shape. Eqs (13) and (14) are also solutions to (13). Hence,

$$f(z,t) = F_+(z - c_0 t) + F_-(z + c_0 t) \qquad (15)$$

This is a wonderful result since F_+ and F_- are arbitrary functions; they can be used to encode information!

Furthermore, one can calculate

$$c_0 = 299{,}792{,}458 \text{ m/s} \simeq 3 \times 10^8 \text{ m/s} \qquad (16)$$