Lecture 8

Maxwell's Equations Sept 19, 2018

Maxwell's equations were completed in 1865, and they have a tremendous impact in our modern world.

We first write down their integral forms:

$$\oint_C \bar{E}\cdot d\bar{l} = -\frac{\partial}{\partial t}\iint_S \bar{B}\cdot d\bar{S} \qquad \text{Faraday's Law} \quad (1)$$

$$\oint_C \bar{H}\cdot d\bar{l} = \frac{\partial}{\partial t}\iint_S \bar{D}\cdot d\bar{S} + I \qquad \text{Ampere's Law} \quad (2)$$

$$\oiint_S \bar{D}\cdot d\bar{S} = Q \qquad \text{Gauss' Law - Electric Flux} \quad (3)$$

$$\oiint_S \bar{B}\cdot d\bar{S} = 0 \qquad \text{Gauss' Law - Magnetic Flux} \quad (4)$$

The basic quantities are

$\bar{E}$: V/m, $\bar{H}$: A/m

$\bar{D}$: C/m^2, $\bar{B}$: Webers/m^2

I : Amperes, Q : Coulombs

We can convert the above integral forms into partial differential equation forms by using Stoke's theorem and Gauss' theorem as we did before. For example, using Stokes' theorem, we can write

$$\oint_C \bar{E}\cdot d\bar{l} = \iint_S (\nabla\times\bar{E})\cdot d\bar{S} \qquad (5)$$

Therefore, Faraday's law becomes

$$\iint_S (\nabla\times\bar{E})\cdot d\bar{S} = -\frac{\partial}{\partial t}\iint_S \bar{B}\cdot d\bar{S} = \iint_S \left[-\frac{\partial\bar{B}}{\partial t}\right]\cdot d\bar{S} \qquad (6)$$

In the limit when the area $S \to 0$, the above becomes a point-wise relationship. In other words,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad (7)$$

which is Faraday's law in its full glory. This equation was experimentally derived in 1831, during the age of telegraphy. We can apply the same treatment to Ampere's law to get

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \qquad (8)$$

The above is sometimes called the generalized Ampere's law as the original Ampere's law does not have the second term. The term $\frac{\partial \bar{D}}{\partial t}$ was the contribution of James Clerk Maxwell in 1865.

We can apply Gauss' divergence theorem to get

$$\nabla \cdot \bar{D} = \rho \qquad (9)$$

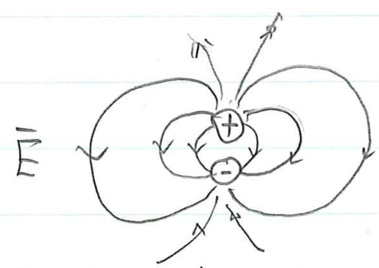
$$\nabla \cdot \bar{B} = 0 \qquad (10)$$

Equations (7) to (10) constitute the four fundamental equations of electromagnetic theory, now known as Maxwell's equations. Maxwell, in addition to contributing the extra term $\frac{\partial \bar{D}}{\partial t}$, known as displacement current, to generalized Ampere's law, was the first to write these equations down luckily.

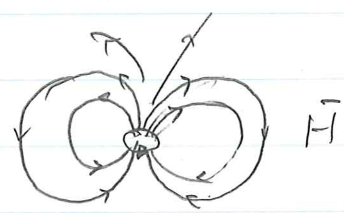
Also the present form of what text books call Maxwell's equations, (8)-(10), were actually written down by his admirer, Oliver Heaviside. Maxwell himself first wrote down electromagnetic theory using vector potential $\bar{A}$ and scalar potential Φ. We will learn more above vector and scalar potential formulation of electromagnetic theory later.

Fictitious Magnetic Currents

Even though magnetic charges or monopoles do not exist, magnetic dipoles do exist. For instance, a magnet can be regarded as magnetic dipoles. Also, it is believed that electrons have spins, and these spins make electrons behave like magnetic dipoles in the presence of magnetic field. Also, if we form current into a loop, it produces a magnetic field, that looks like the electric field of an electric dipole.



$\bar{E}$ field due to an electric dipole



$\bar{H}$-field due to a electric current loop.

Because of these similarities, it is common to introduce fictitious magnetic charges and magnetic currents into Maxwell's equations. One can think that these magnetic charges always occur in pair and together. Hence, Maxwell's equations can be alternatively written as

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} - \bar{M} \qquad (1)$$

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J} \qquad (2)$$

$$\nabla \cdot \bar{D} = \rho \qquad (3)$$

$$\nabla \cdot \bar{B} = \rho_m \qquad (4)$$

By so doing, Maxwell's equations also become more mathematically symmetrical.

When we take the divergence of (2), one gets

$$\nabla \cdot (\nabla \times \bar{H}) = \frac{\partial}{\partial t} \nabla \cdot \bar{D} + \nabla \cdot \bar{J} \tag{5}$$

Since $\nabla \cdot (\nabla \times \bar{H}) = 0$, and that $\nabla \cdot \bar{D} = \rho$, the above is

$$0 = \frac{\partial}{\partial t} \rho + \nabla \cdot \bar{J} \tag{6}$$

Which is the current continuity equation. In other words, one says that Maxwell's equations are consistent with the charge conservation since the current continuity equation is a statement of charge conservation.

Alternatively, we can say that Gauss' law, $\nabla \cdot \bar{D} = \rho$, is derivable from Ampere's law, $\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} + \bar{J}$ if we invoke charge conservation. In this view point, one can say that the third and the fourth Maxwell's equations are derivable from the first two; unless $\frac{\partial}{\partial t} = 0$, or for electrostatics.

By converting the current continuity equation (6) into integral form, we have

$$\iiint_V \nabla \cdot \bar{J} \, dV = - \iiint_V \frac{\partial \rho}{\partial t} dV \tag{7}$$

Using Gauss' divergence theorem, we can write

$$\oiint_S dS \, \hat{n} \cdot \bar{J} = - \frac{d}{dt} Q \tag{8}$$

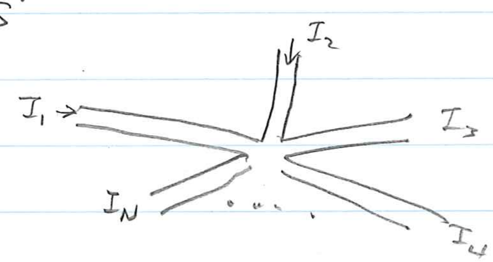


Fig. 2

When we apply the above to many wires going into a junction, and assuming that there is no charge accumulation at the junction such that $\frac{dQ}{dt} = 0$, then we have

$$\sum_{i=1}^{N} I_i = 0 \tag{9}$$

which is Kirchhoff current laws.

Also, if we take Faraday's law, assuming that $\bar{M} = 0$, and integrating it over a loop, we have

$$\oint_C \bar{E} \cdot d\bar{l} = -\frac{d}{dt} \int_S \bar{B} \cdot d\bar{S} \tag{10}$$

when the frequency is low such that $\frac{d}{dt} \simeq 0$, or $\int_S \bar{B} \cdot d\bar{S}$ is small, or the flux linkage is small, we have

$$\oint_C \bar{E} \cdot d\bar{l} = 0 \tag{11}$$

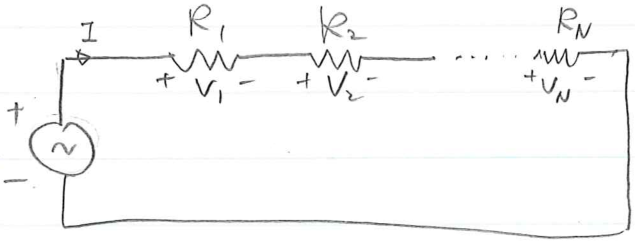


Fig. 3

If we integrate Faraday's law around a circuital loop as shown in Fig 3, and nothing that $\int_a^b \bar{E} \cdot d\bar{l} = -V_{ba}$, we get that

$$-V_s + V_1 + \cdots + V_N = -V_s + \sum_{i=1}^{N} V_i = 0 \tag{12}$$

or

$$V_s = \sum_{i=1}^{N} V_i \tag{13}$$

The above is just Kirchhoff voltage law.

Boundary Conditions

Given our experience in deriving boundary conditions before, from (1), assuming a magnetic current sheet $\bar{M}_s$ at an interface, we have from Faraday's law that

$$\hat{n} \times (\bar{E}_2 - \bar{E}_1) = -\bar{M}_s \tag{14}$$

For $\bar{M}_s = 0$, we retrieve

$$\hat{n} \times (\bar{E}_1 - \bar{E}_2) = 0 \tag{15}$$

From Ampere's law that

$$\hat{n} \times (\bar{H}_2 - \bar{H}_1) = \bar{J}_s \tag{16}$$

For electric flux $\bar{D}$, we have

$$\hat{n} \cdot (\bar{D}_2 - \bar{D}_1) = \rho_s \tag{17}$$

$$\hat{n} \cdot (\bar{B}_2 - \bar{B}_1) = \rho_{ms} \tag{18}$$

For $\rho_{ms} = 0$, $\hat{n} \cdot (\bar{B}_2 - \bar{B}_1) = 0$

We have seen that for electrostatics, a conductive medium alone can expel or screen the electric field away from a conductive region by re-orientation of the electric charges.

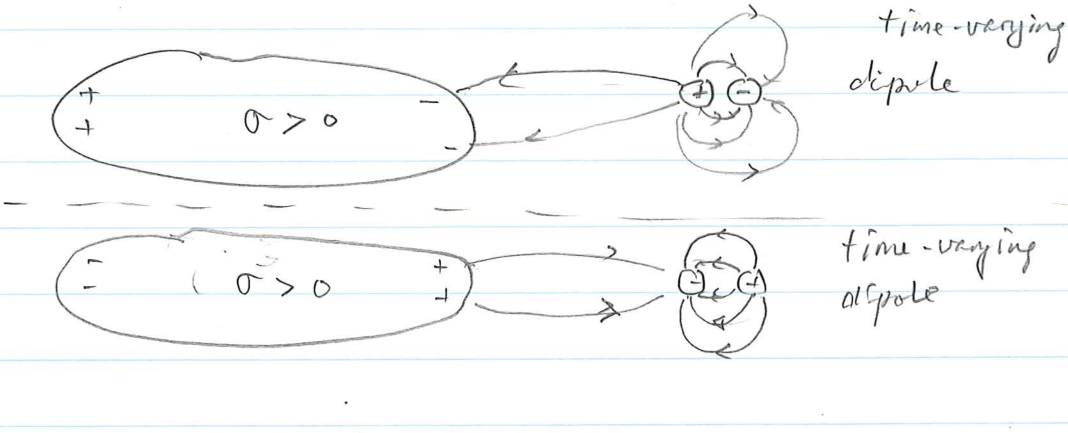
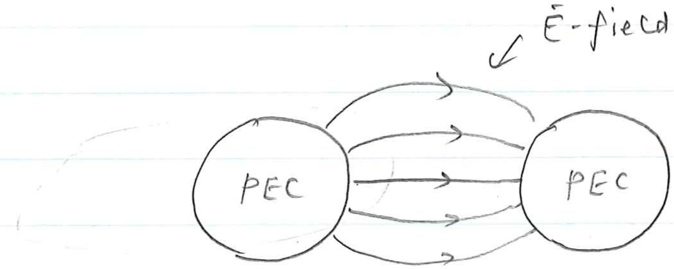
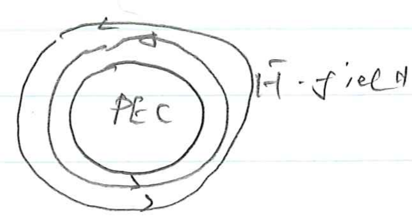


Fig. 4

We have seen that in magnetostatics, one needs a perfect conductor or super-conductor to expel the magnetic field. It turns out that for time-varying problems, we also need a perfect conductor to completely expel the electric field.



On a PEC surface, $\hat{n} \times \bar{E} = 0$, but $\hat{n} \cdot \bar{B} = 0$



Inductance

Inductance is a concept that follows from time-varying Faraday's law which is

$$\oint_C \bar{E}\cdot d\bar{l} = -\frac{d}{dt}\int_S \bar{B}\cdot d\bar{S} \tag{1}$$

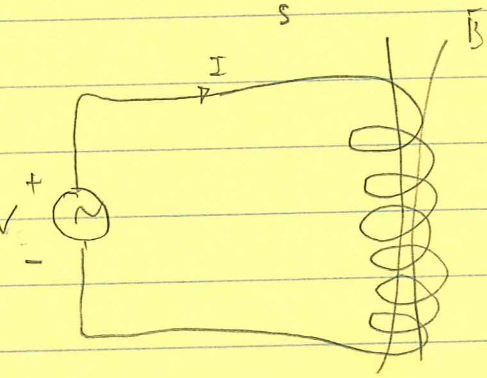


Fig. 4.

If we take C to be a closed integral along the conducting wire, then $\bar{E}=0$ inside the wire, and

$$\oint_C \bar{E}\cdot d\bar{l} = -V \tag{2}$$

The flux linkage due to the $\bar{B}$ field field is

$$\Psi = \int_S \bar{B}\cdot dS \tag{3}$$

where S is a rather convoluted surface made by C. Then (1) becomes

$$V = \frac{d}{dt}\Psi \tag{4}$$

But Ψ is linearly proportional to I: The stronger the current, the stronger is the $\bar{B}$ field generated by I. Hence $\Psi = LI$, and the constant of proportionality is L, the inductance.

Finding L for a Solenoid

For a solenoid, $H_z = \frac{NI}{l}$. Hence,

$$\int \bar{B} \cdot d\bar{s} = \mu H_z A N = \frac{\mu N^2 A I}{l} \qquad (5)$$

Hence,

$$L = \frac{\mu N^2 A}{l}. \qquad (6)$$