Lecture 7, Sept 17, 2018, W.C. Chew

Scalar Magnetic Potential Φ_m

In a source-free region, Ampere's law becomes

$$\nabla \times \bar{H} = 0 \qquad (1)$$

Hence, we can also use the identity that $\nabla \times \nabla \Phi_m = 0$ to suggest that

$$\bar{H} = -\nabla \Phi_m \qquad (2)$$

Φ_m, is analogous to the electric scalar potential where we let $\bar{E} = -\nabla \Phi$ so that $\nabla \times \bar{E} = 0$. Thus Φ_m is called the magnetic scalar potential.

From Gauss' law for magnetic field such that

$$\nabla \cdot \bar{B} = 0. \qquad (3)$$

and that $\bar{B} = \mu \bar{H}$ from the constitutive relation, then

$$\nabla \cdot \mu \nabla \Phi_m = 0 \qquad (4)$$

For a homogeneous medium where μ is independent of position, then $\nabla \cdot \mu \nabla \Phi_m = \mu \nabla \cdot \nabla \Phi_m = \mu \nabla^2 \Phi_m$. Hence,

$$\nabla^2 \Phi_m = 0 \qquad (5)$$

which is Laplace's equation.

The Ampere's law when a current source $\bar{J}$ is present is

$$\nabla \times \bar{H} = \bar{J} \qquad (6)$$

Next, we ask the kind of boundary conditions induced by such an equation.

Boundary Condition for $\bar{H}$

To derive the boundary condition induced by the PDE (6), we can use the conventional method described by most textbooks by integrating (6) over a small loop straddling the interface between two media μ_1 and μ_2. This is illustrated by Prof. Dan Jiao's notes.

Alternatively, we can use the unconventional method by projecting the PDE to a local orthogonal coordinate system at the interface as shown in Fig. 1.

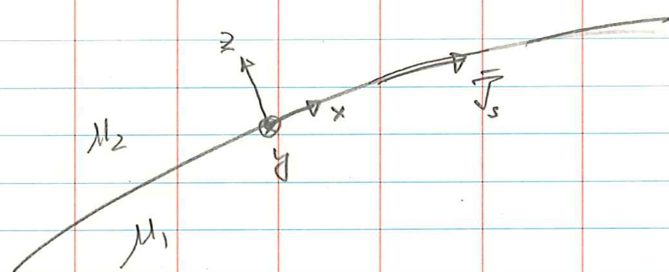

Fig. 1

To be general, we also include the presence of a current sheet at the interface. Rewriting (6) in a local coordinate system, assuming that $\bar{J} = \hat{x} J_{sx}\delta(z)$, then

$$\nabla\times\bar{H} = \hat{x}\left(\frac{\partial}{\partial y}H_z - \frac{\partial}{\partial z}H_y\right) = \hat{x}\, J_{sx}\delta(z). \qquad (7)$$

From the above, a current sheet, or a surface current density becomes a delta function singularity in a volume current density; Hence, the form of the RHS of the above equation. From the above, the only term that can produce a $\delta(z)$ singularity on the LHS is from the $-\frac{\partial}{\partial z}H_y$ term. Therefore, we conclude that

$$-\frac{\partial}{\partial z}H_y = J_{sx}\delta(z) \qquad (8)$$

In other words, H_y has to have a jump discontinuity. Or

$$H_y(z=0^+) - H_y(z=0^-) = -J_{sx} \qquad (9)$$

The above implies that

$$H_{2y} - H_{1y} = -J_{sx} \qquad (10)$$

But H_y is just the tangential component of the H field. Now if we repeat the same exercise with $\bar{J} = \hat{y} J_{sy} \delta(z)$, we have

$$H_{2x} - H_{1x} = J_{sy} \qquad (11)$$

(10) and (11) can be rewritten using

$$\hat{z} \times (\hat{y} H_{2y} - \hat{y} H_{1y}) = +\hat{x} J_{sx} \qquad (12)$$

$$\hat{z} \times (\hat{x} H_{2x} - \hat{x} H_{1x}) = \hat{y} J_{sy} \qquad (13)$$

The above two equation can be combined as one to give

$$\hat{z} \times (\bar{H}_2 - \bar{H}_1) = \bar{J}_s \qquad (14)$$

Taking $\hat{z} = \hat{n}$ in general, we have

$$\hat{n} \times (\bar{H}_2 - \bar{H}_1) = \bar{J}_s \qquad (15)$$

Boundary Condition for $\bar{B}$

$\bar{B}$ field satisfies the PDE given by (3). It induces a different boundary condition for $\bar{B}$.

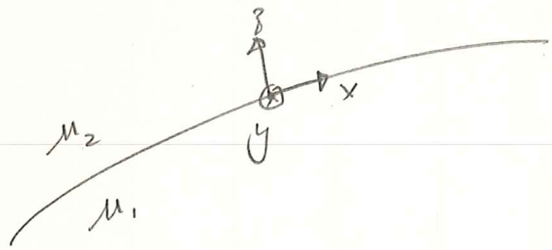

Fig. 2.

Again, writing Gauss' law using a local coordinates shown in Fig. 2, we have

$$\frac{\partial}{\partial x} B_x + \frac{\partial}{\partial y} B_y + \frac{\partial}{\partial z} B_z = 0 \qquad (16)$$

The above implies that there cannot be jump discontinuity for B_x, B_y, or B_z in their respective x, y, z directions. Focussing on B_z, it implies that

$$\frac{\partial}{\partial z} B_z = \text{finite} \qquad (17)$$

Integrating the above across the interface, over an infinitesimal length, we have

$$B_z(z=0^+) - B_z(z=0^-) = 0 \qquad (18)$$

Or

$$\hat{n} \cdot (\bar{B}_2 - \bar{B}_1) = 0 \qquad (19)$$

Perfect Conductor and Conductor

We have seen that for a finite conductor, as long as $\sigma \neq 0$, the charges will re-orient themselves until all electric field is expelled from the conductor. Otherwise, the current will keep flowing. But there are no magnetic charges nor magnetic conductor in this world. So this physical phenomenon does not happen for magnetic field: Magnetic field cannot be expelled from a conductor. However, a magnetic field is expelled from a perfect conductor or a super-conductor. You can only understand this physical phenomenon if we study Maxwell's equations in their full glory.

In a perfect conductor where $\sigma \to \infty$, it is unstable for the magnetic field $\bar{B}$ to be nonzero. As time varying magnetic field gives rise to an electric field by Faraday's law in its full form, a small time variation of the $\bar{B}$ field will give rise to infinite current flow in a perfect conductor. Therefore $\bar{B} = 0$ in a perfect conductor.

So if medium 1 is a perfect conductor, $\bar{B}_1 = \bar{H}_1 = 0$. The boundary condition becomes

$$\hat{n} \times \bar{H} = \bar{J}_s \qquad (20)$$

$$\hat{n} \cdot \bar{B} = 0 \qquad (21)$$

The $\bar{B}$ field is expelled from the perfect conductor, and there is no normal component of the $\bar{B}$ field as there cannot be magnetic charges, as shown in Fig. 3.

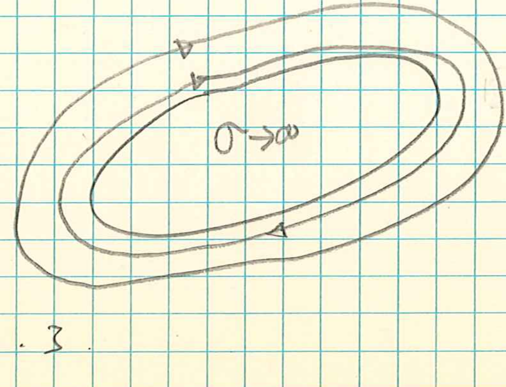

Fig. 3

Boundary Condition for Φ_m

From Gauss' law that $\nabla \cdot \bar{B} = 0$, and Ampere's law for a source-free region that $\nabla \times \bar{H} = 0$, we conclude that

$$\bar{H} = -\nabla \Phi_m . \tag{1}$$

$$\nabla \cdot \mu \nabla \Phi_m = 0 \tag{2}$$

(2) is the generalized Laplace's equation. From it, we can deduce the boundary condition that

$$\hat{n} \cdot \mu_1 \nabla \Phi_{m1} = \hat{n} \cdot \mu_2 \nabla \Phi_{m2} \tag{3}$$

across an interface. The above is similar to

$$\mu_1 \frac{\partial}{\partial n} \Phi_{m1} = \mu_2 \frac{\partial}{\partial n} \Phi_{m2} \tag{4}$$

as $\hat{n} \cdot \nabla = \frac{\partial}{\partial n}$. Also, since $\nabla \Phi_m$ cannot be infinite, because $\nabla \cdot \mu \nabla \Phi_m = 0$, Φ_m must be a continuous function. Consequently, we have

$$\Phi_{m1} = \Phi_{m2} \tag{5}$$

at a medium interface.

Magnetic Energy Density

Unlike the electric energy stored where we can do Gedanken experiment of moving an electric charge against an electric field as work done, no such Gedanken experiment exists for magnetic field. At this point, we have to derive the magnetic energy stored by analogy.

Since the energy stored in the electric field is

$$U_E = \frac{1}{2}\int_V \bar{E}\cdot\bar{D}\, dV \qquad (1)$$

by the same token, the energy stored in the magnetic field is

$$U_H = \frac{1}{2}\int_V \bar{H}\cdot\bar{B}\, dV \qquad (2)$$

Only when we study Maxwell's equations in its full glory can we prove the physical meaning of the above.

For an isotropic medium where $\bar{B} = \mu \bar{H}$, then the above becomes

$$U_H = \frac{1}{2}\int_V \mu |\bar{H}|^2\, dV \qquad (3)$$

The above is consistent with the energy stored in an inductor which is

$$W_H = \frac{1}{2} L I^2 \qquad (4)$$

The above can be derived by considering the instantaneous power in an inductor which is given by

$$P = vi = L\, i \frac{di}{dt} \qquad (5)$$

where we have used the fact that $v = L\frac{di}{dt}$. Integrating (5), we get

$$W_H = \int_0^t P\,dt = \int_0^I L\, i' di' = \frac{1}{2} L I^2 \qquad (6)$$

Notice that (6) is proved by considering a time-varying system. Hence, (2) and (3) can be only proved by considering a time-varying system, which is described by the full form of Maxwell's equations.