Lecture 6 Magnetostatics

Lorentz Force Law

The Lorentz force law is given by

$$\bar{F} = q\bar{E} + q\bar{v}\times\bar{B} \tag{1}$$

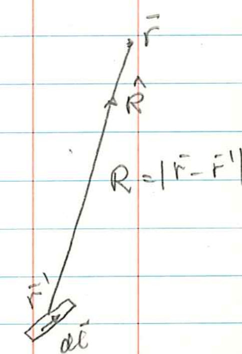
The first term is $\underbrace{\text{from}}^{\text{electric force,}}$ Coulomb's law while the second term is the magnetic force also called the $\bar{v}\times\bar{B}$ force. The magnetic force can also be written for an incremental current flowing in the wire of length $d\bar{l}$, or

$$d\bar{f} = I\,d\bar{l}\times\bar{B} \tag{2}$$

Biot-Savart Law

The incremental magnetic field due to an incremental current is

$$d\bar{H} = \frac{I\,d\bar{l}\times\hat{R}}{4\pi R^2} \tag{3}$$



$$R = |\bar{r}-\bar{r}'|$$

It is Experimentally derived

Ampere's Law

It says that in integral form

$$\oint_C \bar{H}\cdot d\bar{l} = I \tag{4}$$

Using Stokes's theorem, one gets

$$\oint_C \bar{H}\cdot d\bar{l} = \iint_S (\nabla\times\bar{H})\cdot d\bar{S} \tag{5}$$

But

$$I = \iint_S \bar{J}\cdot d\bar{S} \tag{6}$$

Therefore

$$\iint_S (\nabla\times\bar{H})\cdot d\bar{S} = \iint_S \bar{J}\cdot d\bar{S} \tag{7}$$

When $S\to 0$, it implies that

$$\nabla\times\bar{H} = \bar{J} \tag{8}$$

O-1475

Gauss's Law - Magnetic

It says that

$$\oint\!\!\oint_S \bar{B} \cdot d\bar{S} = 0 \qquad (9)$$

But from Gauss's divergence theorem,

$$\oint\!\!\oint_S \bar{B} \cdot d\bar{S} = \iiint_V \nabla \cdot \bar{B} \, dV \qquad (10)$$

Therefore

$$\iiint_V \nabla \cdot \bar{B} \, dV = 0 \qquad (11)$$

When $V \to 0$, we have $\nabla \cdot \bar{B} = 0$

Constitutive Relation

$$\bar{B} = \mu \bar{H} \qquad (12)$$

μ : permeability $\quad$ H/m

In free space,

$$\mu = \mu_0 = 4\pi \times 10^{-7} \quad \text{H/m} \qquad (13)$$

In other materials,

$$\mu = \mu_0 \mu_r \qquad (14)$$

relative permeabity

Similarly,

$$\epsilon = \epsilon_0 \epsilon_r \qquad (15)$$

relative permittivity

Magnetic Vector Potential $\bar{A}$

$$\nabla \cdot \bar{B} = 0 \tag{16}$$

Hence,

$$\bar{B} = \nabla \times \bar{A} \tag{17}$$

because $\nabla \cdot \nabla \times \bar{A} = 0$ (18)

This is similar to

$$\nabla \times \nabla \Phi = 0 \tag{19}$$

Derivation of the Vector Poisson's Equation

From

$$\nabla \times \bar{H} = \bar{J} \tag{20}$$

we have

$$\nabla \times \left(\frac{\bar{B}}{\mu}\right) = \bar{J} \tag{21}$$

Then using (17)

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \bar{A}\right) = \bar{J} \tag{22}$$

In homogeneous media, μ is a constant and hence

$$\nabla \times (\nabla \times \bar{A}) = \mu \bar{J} \tag{23}$$

Vector Identity

We use the vector identity that

$$\nabla \times (\nabla \times \bar{A}) = \nabla(\nabla \cdot \bar{A}) - (\nabla \cdot \nabla)\bar{A}$$

$$= \nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} \tag{24}$$

$$\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A} = \mu \bar{J} \tag{25}$$

However, $\bar{A}$ in (17) is not unique. One can define

$$\bar{A}' = \bar{A} - \nabla\psi \qquad (26)$$

Then

$$\nabla\times\bar{A}' = \nabla\times(\bar{A} - \nabla\psi) = \bar{B}.$$

Hence, the $\nabla\times$ of both $\bar{A}$ and $\bar{A}'$ produce the same $\bar{B}$. To fix $\bar{A}$ properly, we have to define the divergence of $\bar{A}$ or provide a gauge condition. One way is to define

$$\nabla\cdot\bar{A} = 0 \qquad (27)$$

Then

$$\nabla\cdot\bar{A}' = \nabla\cdot\bar{A} - \nabla^2\psi \neq \nabla\cdot\bar{A} \qquad (28)$$

if $\nabla^2\psi \neq 0$. If we stipulate $\nabla\cdot\bar{A}' = \nabla\cdot\bar{A} = 0$, then $-\nabla^2\psi = 0$. By so doing, we obtain

$$\nabla^2\bar{A} = -\mu\bar{J} \qquad (29)$$

In free space

$$\bar{A}(\bar{r}) = \frac{\mu}{4\pi}\iiint_V \frac{\bar{J}(\bar{r}')}{R}\,dV'$$

$$R = |\bar{r} - \bar{r}'|$$

Derivation of Biot-Savart Law

For a current element

$$\frac{\mu}{4\pi}\iiint_V \frac{\bar{J}(\bar{r}')\,dV'}{R} \approx \frac{\mu}{4\pi}\frac{I\,d\bar{l}'\left(\frac{1}{\Delta V'}\right)\Delta V'}{R}$$

A current element can be described by

$$\bar{J}(\bar{r}') = I\,d\bar{l}\,\delta(\bar{r}-\bar{r}')$$

$$\approx I\,d\bar{l}\;\frac{1}{\Delta_x'}\,\frac{1}{\Delta_y'}\,\frac{1}{\Delta_z'}$$

$$\delta(x) \approx \frac{1}{\Delta_x}, \qquad \Delta x \to 0$$

$$\delta(y) \approx \frac{1}{\Delta_y}, \qquad \Delta y \to 0$$

$$\delta(z) \approx \frac{1}{\Delta_z}, \qquad \Delta z \to 0$$

$$\bar{A}(\bar{r}) = \frac{\mu}{4\pi}\iiint_V \frac{\bar{J}(\bar{r}')\,dV'}{R}, \Longrightarrow d\bar{A}(\bar{r}) = \frac{\mu}{4\pi}\frac{I\,d\bar{l}'}{R}$$

Then

$$d\bar{B} = \nabla \times d\bar{A}(\bar{r}) \cong \frac{\mu I}{4\pi}\nabla \times \frac{d\bar{l}'}{R} = -\frac{\mu I}{4\pi}\,d\bar{l}' \times \nabla\frac{1}{R}$$

$$= +\frac{\mu I}{4\pi}\,d\bar{l}' \times \frac{1}{R^2}\hat{R}$$

$$= \frac{\mu I\,d\bar{l}' \times \hat{R}}{4\pi R^2}$$

Hence

$$d\bar{H} = \frac{I\,d\bar{l}' \times \hat{R}}{4\pi R^2}$$

or

$$\bar{H}(\bar{r}) = \int \frac{I(\bar{r}')\,d\bar{l}' \times \hat{R}}{4\pi R^2},$$

which is Biot-Savart Law.