Lecture 5 W.C. Chew, Sept 4, 2018

Green's Function

A Green's function is a point source response to a partial differential equation (PDE). We can take the Poisson's equation as an illustration,

$$\nabla^2 \Phi(\bar{r}) = -\frac{\rho(\bar{r})}{\epsilon} \tag{1}$$

To find the Green's function, we let the RHS be a point source, and seek the solution of

$$\nabla^2 g(\bar{r}, \bar{r}') = -\delta(\bar{r}-\bar{r}') \tag{2}$$

Equations (1) and (2) are very similar except for the RHS. The solution to (2) is the solution due to a point source, and hence, is called the Green's function $g(\bar{r}, \bar{r}')$. We know the solution to (1) when the RHS is a point charge located at $r = \bar{r}'$, namely,

$$\rho(\bar{r}) = q\,\delta(\bar{r}-\bar{r}') \tag{3}$$

We know what $\Phi(\bar{r})$ is in this case from Coulomb's law, that is

$$\Phi(\bar{r}) = \frac{q}{4\pi\epsilon|\bar{r}-\bar{r}'|} \tag{4}$$

By comparing (1), (2), (3), and (4), we conclude that the Green's function is

$$g(\bar{r}, \bar{r}') = \frac{1}{4\pi|\bar{r}-\bar{r}'|} = g(\bar{r}-\bar{r}') \tag{5}$$

Since $g(\bar{r}, \bar{r}')$ only depends on $|\bar{r}-\bar{r}'|$, we can rewrite it as $g(\bar{r}-\bar{r}')$. Eq. (5) is known as the Green's function of the PDE given by (1). Mathematicians also call the Green's function the fundamental solution, as it is of fundamental importance to the PDE.

We can write a source $\rho(\bar{r})$ as a linear superposition of point sources mathematically as

$$\rho(\bar{r}) = \int_V dV' \rho(\bar{r}') \delta(\bar{r}-\bar{r}') \qquad (6)$$

The RHS is a linear superposition of point sources. Can we also write $\Phi(\bar{r})$ as a linear superposition of point source response? Or

$$\Phi(\bar{r}) = \int_V dV' \frac{\rho(\bar{r}')}{4\pi\epsilon|\bar{r}-\bar{r}'|} \qquad (7)$$

In other words,

$$\Phi(\bar{r}) = \frac{1}{\epsilon}\int_V dV' \rho(\bar{r}')\, g(\bar{r}-\bar{r}') \qquad (8)$$

To confirm that (8) is a solution of (1), we can substitute it back into the LHS of (1), and exchange the order of differentiation and integration. That is

$$\nabla^2 \Phi(\bar{r}) = \frac{1}{\epsilon}\int_V dV' \rho(\bar{r}')\, \nabla^2 g(\bar{r}-\bar{r}') \qquad (9)$$

Using (2) for the definition of $g(\bar{r}-\bar{r}')$, we in fact see that the RHS of (9) is the RHS of (2), confirming that (8) is the solution of (1) for a general source $\rho(\bar{r})$. In other words, the general solution is obtained by convolving a general source with the point source response $g(\bar{r}-\bar{r}')$, or the Green's function.

Instead of relying on Coulomb's law to determine the Green's function, it can be also derived mathematically. To this end, we put the point source at the origin such that (2) becomes

$$\nabla^2 g(\bar{r}) = -\delta(\bar{r}) \qquad (10)$$

where we have set $\bar{r}' = 0$, calling $g(\bar{r})$ the solution to (10). Due to symmetry, $g(\bar{r})$ has to be spherically symmetric; hence $g(\bar{r}) = g(r)$. The Laplacian operator ∇^2 becomes

$$\nabla^2 = \nabla \cdot \nabla = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \qquad (11)$$

Away from $r \neq 0$, $\nabla^2 g(r) = 0$, and possible solution for $g(r)$ is

$$g(r) = \frac{C}{r} + D \qquad (12)$$

However, we are seeking solutions such that $g(r) \to 0$, $r \to \infty$; therefore $D = 0$, and

$$g(r) = \frac{C}{r} \qquad (13)$$

But

$$\nabla \cdot \nabla \frac{C}{r} = -\delta(x)\delta(y)\delta(z) \qquad (14)$$

Integrating the above about a small sphere of radius a around the origin, one gets, after invoking Gauss's divergence theorem that

$$\int_{\Delta S} dS\, \hat{n} \cdot \nabla \frac{C}{r} = -1 \qquad (15)$$

where ΔS is the surface of a small sphere of radius a. Moreover, $\hat{n} \cdot \nabla \frac{C}{r} = \frac{\partial}{\partial r}\frac{C}{r} = -\frac{C}{r^2}$. For a small sphere of radius a, LHS of the above evaluates to

$$-4\pi a^2 \left(\frac{C}{a^2}\right) = -1 \qquad (16)$$

or that $C = \frac{1}{4\pi}$. Therefore,

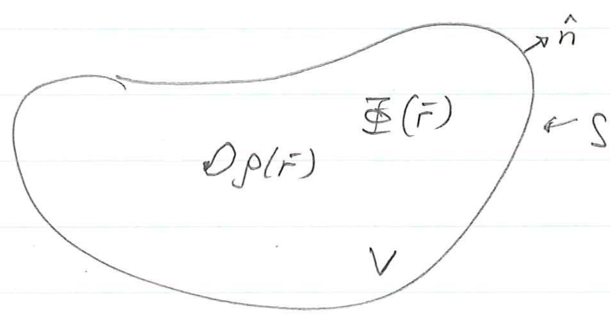
$$g(r) = \frac{1}{4\pi r} \qquad (17)$$

In general, $g(\bar{r} - \bar{r}') = \dfrac{1}{4\pi|\bar{r} - \bar{r}'|}$ (18)

Uniqueness Theorem

This theorem says that if the boundary condition for a certain boundary value problem (BVP) is stipulated, the solution is unique. For example, we wish to solve for $\Phi(\bar{r})$ of Poisson's equation with a source term $\rho(\bar{r})$ in a certain volume bounded by a surface S,

Fig. 1



Then, we know that for the generalized Poisson's equation,

$$\nabla \cdot \epsilon \nabla \Phi = -\rho(\bar{r}) \quad (1)$$

with stipulated boundary condition. The question to ask is if there could be two different $\Phi(\bar{r})$, namely $\Phi_1(\bar{r})$ and $\Phi_2(\bar{r})$ that satisfy the above. In other words,

$$\nabla \cdot \epsilon \nabla \Phi_1(\bar{r}) = -\rho(\bar{r}) \quad (2)$$

$$\nabla \cdot \epsilon \nabla \Phi_2(\bar{r}) = -\rho(\bar{r}) \quad (3)$$

If this is really the case, then taking the difference of the above, we have

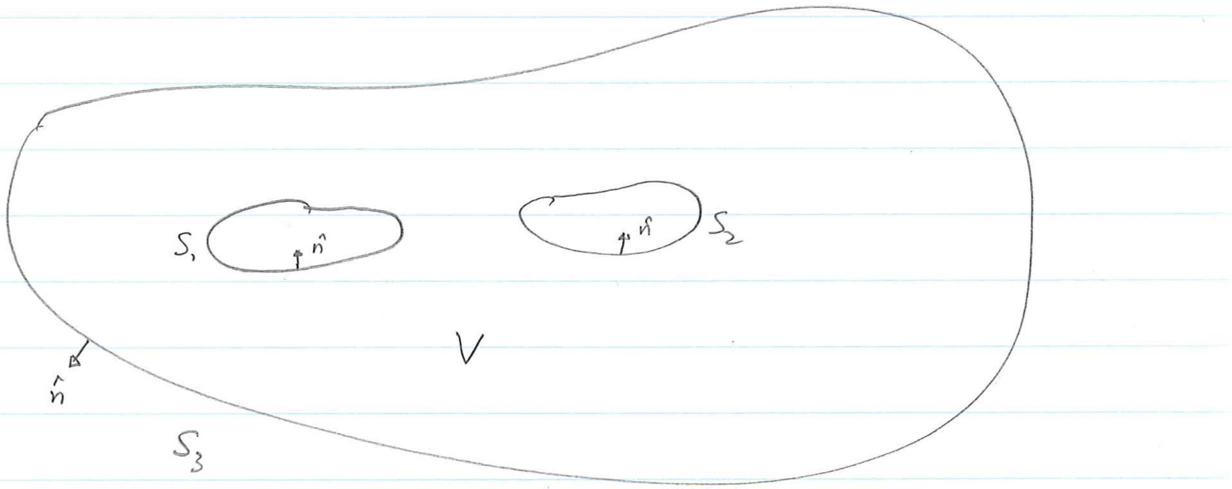
$$\nabla \cdot \epsilon \nabla \delta \Phi(\bar{r}) = 0 \quad (4)$$

where

$$\delta \Phi(\bar{r}) = \Phi_1(\bar{r}) - \Phi_2(\bar{r}) \quad (5)$$

The above does not imply that $\delta\Phi = 0$ yet. Hence, more work is needed!

Fig. 2



In this proof, we allow for the possibility of a not simply connected surface as shown in Fig. 2. The total surface $S = S_1 \cup S_2 \cup S_3$, and V is enclosed by S.

To prove the theorem, we multiply (4) by $\delta\Phi(\mathbf{r})$, and integrate over V, or

$$\int_V dV\, \delta\Phi\, \nabla\cdot\epsilon\nabla\delta\Phi = 0 \qquad (6)$$

We invoke something called the Green's theorem that allows us to convert the volume integral into a surface integral. One notices that

$$\nabla\cdot(\delta\Phi\,\epsilon\nabla\delta\Phi) = \nabla\delta\Phi\cdot\epsilon\nabla\delta\Phi + \delta\Phi\,\nabla\cdot\epsilon\nabla\delta\Phi \qquad (7)$$

Integrating the above, Gauss's divergence theorem allows one to convert the LHS into a surface integral, namely

$$\int_S dS\, \hat{n}\cdot(\delta\Phi\,\epsilon\nabla\delta\Phi) = \int_V dV \epsilon(\nabla\delta\Phi)^2 + \int_V dV\, \delta\Phi\, \underbrace{\nabla\cdot\epsilon\nabla\delta\Phi}_{=0} \qquad (8)$$

Therefore, if

(i) $\delta\Phi = 0$, or $\hat{n}\cdot\nabla\delta\Phi = 0$ on S, or

(ii) $\delta\Phi = 0$ on part of S and $\hat{n}\cdot\nabla\delta\Phi = 0$, on the rest of S,

Therefore, if $\Phi_1 = \Phi_2$ on S, or, $\hat{n}\cdot\nabla\Phi_1 = \hat{n}\cdot\nabla\Phi_2$ on S, or mixture thereof, then, the LHS of (8) is zero.

Then, we have

$$\int_V dV \epsilon (\nabla \delta \Phi)^2 = 0 \qquad (9)$$

Since $(\nabla \delta \Phi)^2 > 0$ always if $\delta \Phi \neq 0$, then the above is possible only if $\delta \Phi = 0$.

We further elaborate on the boundary conditions.

The above is equivalent to

(i) $\Phi_1 = \Phi_2$ on S, or $\hat{n} \cdot \nabla \Phi_1 = \hat{n} \cdot \nabla \Phi_2$ on S

(ii) $\Phi_1 = \Phi_2$ on part of S, and $\hat{n} \cdot \nabla \Phi_1 = \hat{n} \cdot \nabla \Phi_2$ on the rest of S.

The specification of Φ on S is known as the Dirichlet boundary condition, while the specification of $\hat{n} \cdot \nabla \Phi$ or $\frac{\partial}{\partial n} \Phi$ on S is or its normal derivative on S is known as the Neumann boundary condition. With these boundary conditions, there could only be a unique solution to the equation

$$\nabla \cdot \epsilon \nabla \Phi = - \rho \qquad (10)$$