Lecture 4 Aug 30, 2018, W.C. Chew

Reviews

B. C. – Unconventional View

From Faraday's Law, that

$$\nabla \times \bar{E} = 0, \tag{1}$$

we have

$$E_{1t} = E_{2t} \tag{2}$$

or tangential $\bar{E}$ is continuous.

$$\bar{E} = \hat{t} E_t + \hat{n} E_n = \hat{x} E_x + \hat{y} E_y.$$

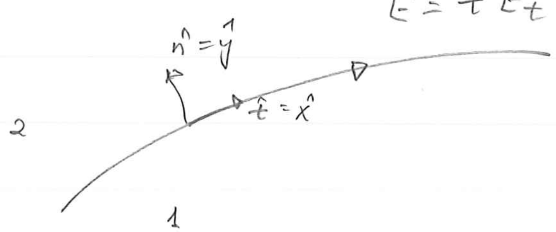

Fig. 1

Think of $\hat{t}$ and $\hat{n}$ as local $\hat{x}$ and $\hat{y}$ coordinates,

then $\nabla \times \bar{E} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right) \times \left(\hat{x} E_x + \hat{y} E_y\right)$

$$= \hat{z}\frac{\partial}{\partial x} E_y - \hat{z}\frac{\partial}{\partial y} E_x \tag{3}$$

Since $\nabla \times \bar{E}$ is finite, the above implies that $\frac{\partial}{\partial x} E_y$ and $\frac{\partial}{\partial y} E_x$ have to be finite. In other words, E_x is continuous in the y direction and E_y is continuous in the x direction. Since $E_x = E_t$, E_t is continuous across the boundary. The above implies that

$$E_{1t} = E_{2t}$$

From Gauss's law, that

$$\nabla \cdot \bar{D} = \rho \tag{4}$$

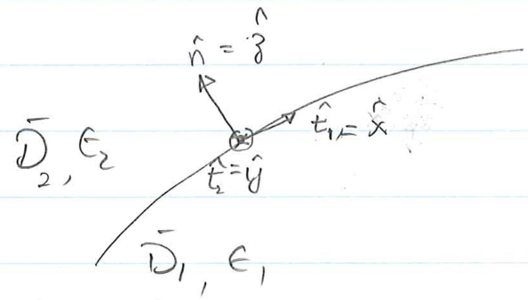

Fig. 2

Using local coord.

$$\nabla \cdot \bar{D} = \frac{\partial}{\partial x} D_x + \frac{\partial}{\partial y} D_y + \frac{\partial}{\partial z} D_z = \rho \tag{5}$$

If $\rho = \rho_s \, \delta(z)$, The only term that can produce $\delta(z)$ is from $\frac{\partial}{\partial z} D_z$. In other words, D_z has a jump discontinuity at $z = 0$. The other terms do not. Then

$$\frac{\partial}{\partial z} D_z = \rho_s \, \delta(z) \tag{6}$$

Integrating the above from $z - \Delta$ to $z + \Delta$, we get

$$D_z(z) \Big|_{z-\Delta}^{z+\Delta} = \rho_s \tag{7}$$

or

$$D_z(z^+) - D_z(z^-) = \rho_s \tag{8}$$

where $z^+ = \lim_{\Delta \to 0} z + \Delta$, $z^- = \lim_{\Delta \to 0} z - \Delta$.

Since $D_z(z^+) = D_{2n}$, $D_z(z^-) = D_{1n}$, the above becomes

$$D_{2n} - D_{1n} = \rho_s \tag{9}$$

or

$$\hat{n} \cdot (\bar{D}_2 - \bar{D}_1) = \rho_s \tag{10}$$

$\bar{E}$ Field at a Dielectric Interface

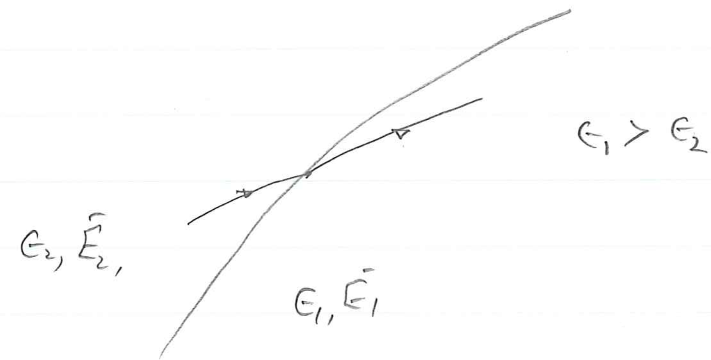

Fig. 3

Since we have, from Faraday's law and Gauss's law that

$$E_{1t} = E_{2t}$$

$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

if $\epsilon_1 > \epsilon_2$, then $E_{1n} < E_{2n}$.

The electric field line will appear to be as shown.

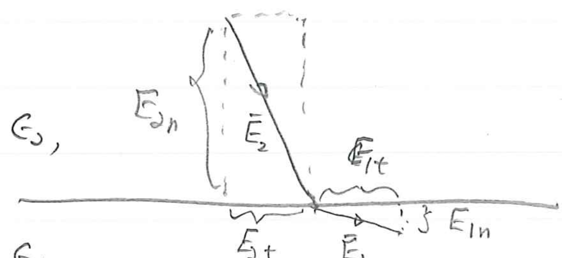

Fig. 4

When $\epsilon_1 \to \infty$, $\epsilon_1 E_{1n}$ becomes very large, and $\epsilon_2 E_{2n}$ is finite. Therefore, $E_{1n} \to 0$, The B.C. compensates by making $\bar{E}_1$ small so that E_{1n} & E_{2t} becomes small, but $\epsilon_1 E_{1n}$ is finite. Then $\bar{E}_2$ becomes almost normal to the dielectric interface

This is similar to when region 1 becomes a PEC. Then, $E_{1t} = E_{2t} = 0$, and E_{2n} is finite.

Conductive Media Case

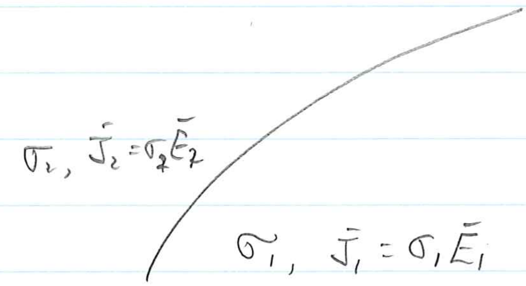

Fig. 5

$$\nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial}{\partial t} = 0, \text{ static} \tag{1}$$

$$\nabla \cdot \bar{J} = 0. \tag{2}$$

The above implies that $\frac{\partial}{\partial n} J_n = 0$ or that J_n is continuous, or that $J_{1n} = J_{2n}$, or

$$\hat{n} \cdot (\bar{J}_2 - \bar{J}_1) = 0 \tag{3}$$

Hence

$$\sigma_2 E_2 - \sigma_1 E_1 = 0 \tag{4}$$

Ampere's law implies that

$$E_{2t} - E_{1t} = 0$$

But Gauss's law implies that

$$\epsilon_2 E_{2n} - \epsilon_1 E_{1n} = \rho_s .$$

Hence, surface charge accumulation is necessary

Electric Energy

Pairwise energy stored is

$$U_{12} = \frac{q_1 q_2}{4\pi \epsilon R_{12}} \tag{1}$$

because

$$\Phi = \frac{q}{4\pi \epsilon r} . \tag{2}$$

Fig. 6

Total pairwise energy stored is

$$U_E = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{q_i q_j}{4\pi \epsilon R_{ij}} \tag{3}$$

But since

$$\Phi_i = \sum_{j=1}^{N} \frac{q_j}{4\pi \epsilon R_{ij}} \tag{4}$$

then (3) becomes

$$U_E = \frac{1}{2} \sum_{i=1}^{N} q_i \Phi_i \tag{5}$$

Replacing the above by charge density, we have

$$U_E = \frac{1}{2} \int_V \rho(\bar{r}) \Phi(\bar{r}) dV \tag{6}$$

$$\rho(\bar{r}) = \sum_{i=1}^{N} q_i \delta(\bar{r} - \bar{r}_i) \tag{7}$$

Since $\nabla \cdot \bar{D} = \rho$, we have

$$U_E = \frac{1}{2} \int_V (\nabla \cdot \bar{D}) \Phi \, dV \qquad (8)$$

Using integration by parts in 3D, we have

$$U_E = -\frac{1}{2} \int_V \bar{D} \cdot \nabla \Phi \, dV = \frac{1}{2} \int_V \bar{D} \cdot \bar{E} \, dV \qquad (9)$$

If $\bar{D} = \epsilon \bar{E}$, then the above become

$$U_E = \frac{1}{2} \int_V \epsilon \bar{E} \cdot \bar{E} \, dV = \frac{1}{2} \int_V \epsilon |\bar{E}|^2 dV \qquad (10)$$

<u>Capacitor</u>

$$U_E = \frac{1}{2} (\text{Vol}) D E = \frac{1}{2} (A d) \frac{\epsilon V}{d} \frac{V}{d}$$

$$= \frac{1}{2} \left(\frac{\epsilon A}{d} \right) V^2 = \frac{1}{2} C V^2 \qquad (11)$$

Integration by Parts in 3D

$$U_E = \frac{1}{2}\int_V (\nabla\cdot\bar{D})\,\Phi\, dV = -\frac{1}{2}\int_V \bar{D}\cdot(\nabla\Phi)\, dV \qquad (1)$$

$$\nabla\cdot(\Phi\bar{D}) = (\nabla\Phi)\cdot\bar{D} + \Phi\,\nabla\cdot\bar{D} \qquad (2)$$

$$\int_V \nabla\cdot(\Phi\bar{D})\, dV = \int_V (\nabla\Phi)\cdot\bar{D}\, dV + \int_V \Phi(\nabla\cdot\bar{D})\, dV \qquad (3)$$

$$\oint_S (\Phi\bar{D})\cdot d\bar{s} = \int_V (\nabla\Phi)\cdot\bar{D}\, dV + \int_V \Phi(\nabla\cdot\bar{D})\, dV \qquad (4)$$

We let $V\to\infty$, $S\to\infty$, then

$$(\Phi\bar{D})\to 0,\quad r\to\infty,\ \to LHS \Rightarrow 0 \qquad (5)$$

Then

$$\int_V (\nabla\Phi)\cdot\bar{D}\, dV = -\int_V \Phi(\nabla\cdot\bar{D})\, dV \qquad (6)$$