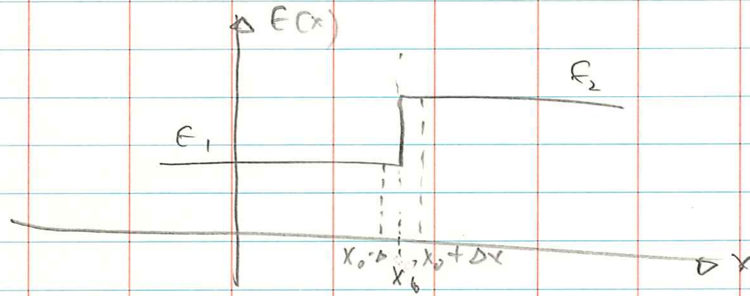
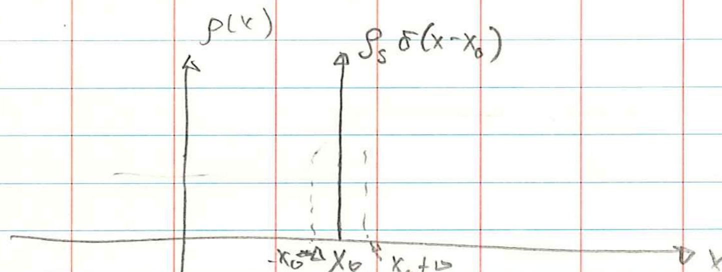
Lecture 3 W. C. Chew. 8/28/2018

B.C. are manifestation of the differential form of M.E. Take for example an one dimensional Poisson's equation that

$$\frac{d}{dx}\epsilon(x)\frac{d}{dx}\phi(x) = -\rho(x) \tag{1}$$

where $\epsilon(x)$ represents material property that has the form





where $\rho(x)$ represents a charge distribution that is everywhere finite. To find the B.C. of the potential $\phi(x)$ at x_0, we integrate (1) over an infinitesimal width around x_0, namely

$$\int_{x_0-\Delta}^{x_0+\Delta} dx \frac{d}{dx}\epsilon(x)\frac{d}{dx}\phi(x) = -\int_{x_0-\Delta}^{x_0+\Delta} dx\, \rho(x) \tag{2}$$

or

$$\epsilon(x)\frac{d}{dx}\phi(x)\Big|_{x_0-\Delta}^{x_0+\Delta} \cong -\rho_s \tag{3}$$

$$\lim_{\Delta\to 0} \epsilon(x^+)\frac{d}{dx}\phi(x^+) - \epsilon(x^-)\frac{d}{dx}\phi(x^-) \cong -\rho_s, \tag{4}$$

Since

$$E_x(x) = -\frac{d}{dx}\phi(x), \tag{5}$$

The above implies that

$$\epsilon(x^+)E_x(x^+) - \epsilon(x^-)E_x(x^-) = \rho_s \quad (6)$$

or

$$D_x(x^+) - D_x(x^-) = \rho_s \quad (7)$$

where

$$D_x(x) = \epsilon(x)E_x(x) \quad (8)$$

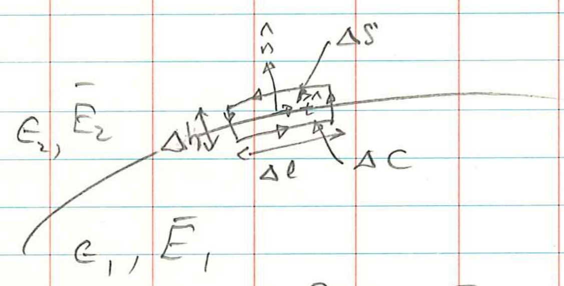
The lesson learned from above is that B.C. is obtained by integrating differential forms of an equation over an infinitesimal small segment.

We have already derived the differential forms of Faraday's Law when $\frac{d}{dt} = 0$, and Gauss's law, namely

$$\nabla \times \bar{E} = 0, \quad (9)$$

$$\nabla \cdot \bar{D} = \rho \quad (10)$$

We will start with Faraday's law, and integrating it over a small cross-section straddling two media interface



$$\int_{\Delta S} (\nabla \times \bar{E}) \cdot d\bar{S} = \oint_{\Delta C} \bar{E} \cdot d\bar{l} \quad (11)$$

When $\Delta h \to 0$, $\Delta l \to 0$, we can approximate the line integral as

$$\oint_{\Delta C} \bar{E} \cdot d\bar{l} \simeq \bar{E}_1 \cdot \hat{t}\Delta l + \bar{E}_1 \cdot \hat{n}\frac{\Delta h}{2} + \bar{E}_2 \cdot \hat{n}\frac{\Delta h}{2} - \bar{E}_2 \cdot \hat{t}\Delta l$$

$$- \bar{E}_2 \cdot \hat{n}\frac{\Delta h}{2} - \bar{E}_1 \cdot \hat{n}\frac{\Delta h}{2} = 0$$

Let $\Delta h \to 0$, $\Delta h \ll \Delta l$, then

$$\bar{E}_1 \cdot \hat{t}\Delta l - \bar{E}_2 \cdot \hat{t}\Delta l = 0$$

or

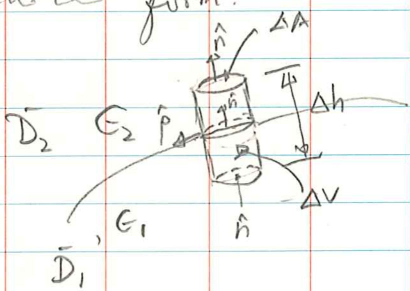
$$E_{1t} = E_{2t}.$$

Tangential component of $\bar{E}$ is always continuous from F.L.

Gauss's law tells us that

$$\nabla \cdot \bar{D}(\bar{r}) = \rho(\bar{r}) \tag{12}$$

in differential form.



$\rho(\bar{r})$ above volume charge density. However, if there is a surface charge density residing at the media interface, the volume charge density is infinitely large at the interface. Integrating (12) over ΔV, we have

$$\iiint_{\Delta V} \nabla \cdot \bar{D}\, dV = \iiint_{\Delta V} \rho \, dV \tag{13}$$

By using Gauss's divergence theorem, the LHS becomes

$$\oiint_{\Delta S} \bar{D} \cdot d\bar{S} = \oiint_{\Delta S} \bar{D} \cdot \hat{n}\, dS \tag{14}$$

$$= \bar{D}_2 \cdot \hat{n}\,\Delta A - \bar{D}_1 \cdot \hat{n}\,\Delta A + \bar{D} \cdot \hat{\rho}\, 2\pi a \Delta h \tag{15}$$

But

$$\iiint_{\Delta V} \rho\, dV = Q = \rho_s \Delta A \tag{16}$$

Since

$$\oiint_{\Delta S} \bar{D} \cdot d\bar{S} = \iiint_{\Delta V} \rho\, dV = Q = \rho_s \Delta A \tag{17}$$

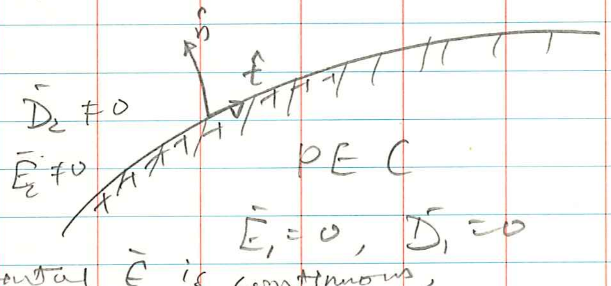
we have, when $\Delta h \to 0$

$$\bar{D}_2 \cdot \hat{n}\,\Delta A - \bar{D}_1 \cdot \hat{n}\,\Delta A = \rho_s \Delta A \tag{18}$$

or

$$\hat{n} \cdot (\bar{D}_2 - \bar{D}_1) = \rho_s \tag{19}$$

If medium 1 is a perfect conductor, then $\bar{E}_1 = 0$, $\bar{D}_1 = 0$, because $\bar{J}_1 = \sigma \bar{E}_1$, $\Rightarrow \bar{E}_1 \to 0$, $\sigma \to \infty$ .. or $\bar{D}_1 = 0$ as well.



Since tangential $\bar{E}$ is continuous,

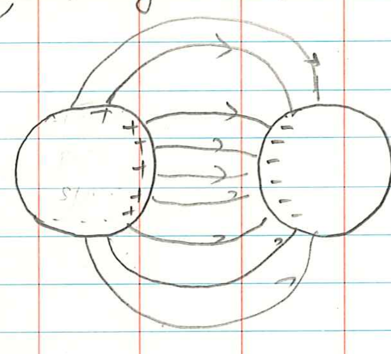
$$\bar{E}_{2t} = \bar{E}_{1t} = 0, \tag{20}$$

But since

$$\hat{n} \cdot (\bar{D}_2 - \bar{D}_1) = \rho_s \tag{21}$$

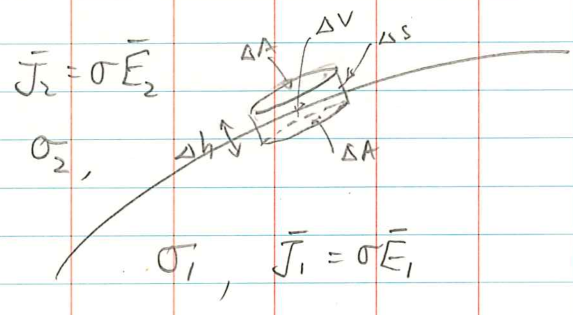
$$\hat{n} \cdot \bar{D}_2 = \rho_s \tag{22}$$

Normal $\bar{D} \neq 0$, tangential $\bar{E} = 0$.



Field plot around two charged spheres.

What about a Conductive Media Interface?



The continuity equation is

$$\nabla \cdot \bar{J} = -\frac{\partial}{\partial t} \rho \qquad (23)$$

It is a statement of charge conservation. If $\frac{\partial}{\partial t} = 0$, then

$$\nabla \cdot \bar{J} = 0 \qquad (24)$$

The above is KCL. The B.C. can be obtained by integrating (24) over a small pill-box, or

$$\iiint_{\Delta V} dV \, \nabla \cdot \bar{J} = 0 \qquad (24)$$

or

$$\oiint_{\Delta S} dS \, \hat{n} \cdot \bar{J} = 0 \qquad (25)$$

Consequently, one obtains

$$\bar{J}_2 \cdot \hat{n} \Delta A + \bar{J}_1 \cdot (-\hat{n}) \Delta A + \bar{J} \cdot \hat{\rho} \, 2\pi a \Delta h = 0 \qquad (26)$$

Letting $\Delta h \to 0$, then

$$\hat{n} \cdot (\bar{J}_2 - \bar{J}_1) = 0 \qquad (27)$$

or

$$\sigma_2 E_{2n} - \sigma_1 E_{1n} = 0 \qquad (28)$$

But Ampere's law still implies that

$$E_{2t} - E_{1t} = 0 \qquad (29)$$

And Gauss's law implies that

$$\epsilon_2 E_{2n} - \epsilon_1 E_{1n} = \rho_s \qquad (30)$$

In order for (28) to be consistent with (30), $\rho_s \neq 0$. Hence, at a conductor interface, surface charge has to accumulate.

From Poisson's Equation

$$\nabla \cdot \epsilon \nabla \Phi = -\rho \tag{31}$$

one can integrate the above over a small pill box to obtain

$$\epsilon_2 \hat{n} \nabla \Phi_2 - \epsilon_1 \hat{n} \nabla \Phi_1 = -\rho_s \tag{32}$$

$$\Phi_1 = \Phi_2 \tag{33}$$

Electric Energy:

For a point charge

$$\bar{E} = \frac{q}{4\pi\epsilon r^2}\hat{r}, \quad \Phi = \frac{q}{4\pi\epsilon r} \tag{34}$$

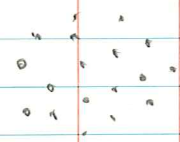
because $\bar{E} = -\nabla\Phi$, $\Phi \to 0$, when $r \to \infty$.

$$\Phi_{P_2} - \Phi_{P_1} = -\int_{P_1}^{P_2} \bar{E} \cdot d\bar{l} \tag{35}$$

$$\Phi = -\int_r^\infty \bar{E} \cdot d\bar{l} \tag{36}$$

The energy needed to bring two charges q_1 and q_2 to each other is

$$U_{12} = \frac{q_1 q_2}{4\pi\epsilon R_{12}} \tag{37}$$



N charges

The energy contained in N charges, by summing over their pairwise energy, is

$$U_E = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{q_i q_j}{4\pi\epsilon R_{ij}}, \quad i \neq j \qquad (38)$$

The factor of $\frac{1}{2}$ is needed because the double summation double counts. By using the fact that the potential at charge i due to the other charges is

$$\Phi_i = \sum_{j=1}^{N}\frac{q_j}{4\pi\epsilon R_{ij}} \qquad j \neq i \qquad (39)$$

implies that

$$U_E = \frac{1}{2}\sum_{i=1}^{N} q_i \Phi_i \qquad (40)$$

When the point charges are replaced by charge density then

$$U_E = \frac{1}{2}\int_V \rho\,\Phi\,dV \qquad (41)$$

Using Gauss's law that $\nabla\cdot\vec{D} = \rho$,

$$U_E = \frac{1}{2}\int_V \left(\nabla\cdot\vec{D}\right)\Phi\,dV \qquad (42)$$

The above ∇ operator can be made to operate on Φ by performing integration by parts in 3D. Finally

$$U_E = -\frac{1}{2}\int_V \vec{D}\cdot\nabla\Phi\,dV = \frac{1}{2}\int_V \vec{D}\cdot\vec{E}\,dV \qquad (43)$$

Capacitor $\quad U_E = \frac{1}{2}\,\mathrm{Vol}\;D\,E = \frac{1}{2}(Ad)\,\frac{\epsilon V}{d}\,\frac{V}{d} = \frac{1}{2}\left(\frac{\epsilon A}{d}\right)V^2 = \frac{1}{2}CV^2$