Lecture 22

Reciprocity Theorem, Nov 27, 2018, W. C. Chew

Reciprocity theorem is like "tit-for-tat" relationships in humans: good-will is reciprocated with good will while ill-will is reciprocated with ill-will. Not exactly as in electromagnetics, this relationship can be expressed exactly and succinctly using mathematics. We shall see how this is done.

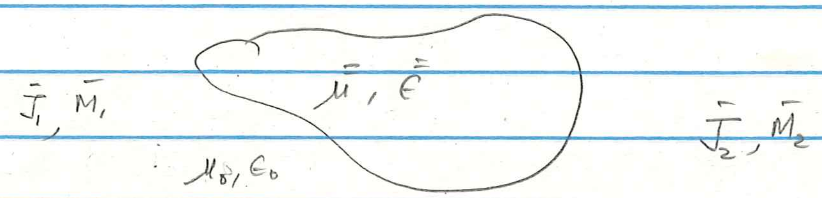


Fig 1.

Consider a general anisotropic inhomogeneous medium where both $\bar{\bar{\mu}}$ and $\bar{\bar{\epsilon}}$ are described by permeability tensor and permittivity tensor over a finite part of space as shown in Fig. 1. When $\bar{J}_1$ and $\bar{M}_1$ are turned on, they generate fields $\bar{E}_1$ and $\bar{H}_1$ in this medium. On the other hand, when only $\bar{J}_2$ and $\bar{M}_2$ are turned on, they generate $\bar{E}_2$ and $\bar{H}_2$ in this medium. Therefore, the pertinent equations for these two cases are

$$\nabla\times\bar{E}_1 = -j\omega\bar{\bar{\mu}}\cdot\bar{H}_1 - \bar{M}_1, \qquad (1)$$

$$\nabla\times\bar{H}_1 = j\omega\bar{\bar{\epsilon}}\cdot\bar{E}_1 + \bar{J}_1 \qquad (2)$$

$$\nabla\times\bar{E}_2 = -j\omega\bar{\bar{\mu}}\cdot\bar{H}_2 - \bar{M}_2 \qquad (3)$$

$$\nabla\times\bar{H}_2 = j\omega\bar{\bar{\epsilon}}\cdot\bar{E}_2 + \bar{J}_2 \qquad (4)$$

From the above, we can show that

$$\bar{H}_2\cdot\nabla\times\bar{E}_1 = -j\omega\bar{H}_2\cdot\bar{\bar{\mu}}\cdot\bar{H}_1 - \bar{H}_2\cdot\bar{M}_1 \qquad (5)$$

$$\bar{E}_1\cdot\nabla\times\bar{H}_2 = j\omega\bar{E}_1\cdot\bar{\bar{\epsilon}}\cdot\bar{E}_2 + \bar{E}_1\cdot\bar{J}_2 \qquad (6)$$

Then,

$$\begin{aligned}\nabla\cdot(\bar{E}_1\times\bar{H}_2) &= \bar{H}_2\cdot\nabla\times\bar{E}_1 - \bar{E}_1\cdot\nabla\times\bar{H}_2\\ &= -j\omega\bar{H}_2\cdot\bar{\bar{\mu}}\cdot\bar{H}_1 - j\omega\bar{E}_1\cdot\bar{\bar{\epsilon}}\cdot\bar{E}_2\\ &\quad - \bar{H}_2\cdot\bar{M}_1 - \bar{E}_1\cdot\bar{J}_2 \end{aligned} \qquad (7)$$

By the same token,

$$\nabla\cdot(\bar{E}_2\times\bar{H}_1) = -j\omega\,\bar{H}_1\cdot\bar{\bar{\mu}}\cdot\bar{H}_2 - j\omega\,\bar{E}_2\cdot\bar{\bar{\epsilon}}\cdot\bar{E}_1 - \bar{H}_1\cdot\bar{M}_2 - \bar{E}_2\cdot\bar{J}_1 \qquad (8)$$

Subtracting (7) and (8), and using the fact that $\bar{H}_1\cdot\bar{\bar{\mu}}\cdot\bar{H}_2 = \bar{H}_2\cdot\bar{\bar{\mu}}^t\cdot\bar{H}_1$, then

$$\nabla\cdot(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1) = -j\omega\,\bar{H}_1\cdot(\bar{\bar{\mu}}-\bar{\bar{\mu}}^t)\cdot\bar{H}_2 - j\omega\,\bar{E}_1\cdot(\bar{\bar{\epsilon}}-\bar{\bar{\epsilon}}^t)\cdot\bar{E}_2 - \bar{H}_2\cdot\bar{M}_1 - \bar{E}_1\cdot\bar{J}_2 + \bar{H}_1\cdot\bar{M}_2 + \bar{E}_2\cdot\bar{J}_1 \qquad (9)$$

If $\bar{\bar{\mu}} = \bar{\bar{\mu}}^t$ and $\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}^t$, or when the tensors are symmetric, then the RHS of (8) simplifies as the terms involving the permeability tensors and permeability tensors disappear. Now, integrating (9.) over a volume V bounded by a surface S, and invoking Gauss' divergence theorem, we have reciprocity theorem

$$\oiint_S d\bar{S}\cdot(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1) = -\iiint_V dV\,[\bar{H}_2\cdot\bar{M}_1 + \bar{E}_1\cdot\bar{J}_2 - \bar{H}_1\cdot\bar{M}_2 - \bar{E}_2\cdot\bar{J}_1] \qquad (10)$$

When the volume V contains no sources, the reciprocity theorem reduces to

$$\oiint_S d\bar{S}\cdot(\bar{E}_1\times\bar{H}_2 - \bar{E}_2\times\bar{H}_1) = 0 \qquad (11)$$

On the other hand, when the surface $S\to\infty$, $\bar{E}_1$ and $\bar{H}_2$ become spherical waves sharing the same $\bar{\beta}$ vector. Moreover $\omega\mu_0\bar{H}_2 = \bar{\beta}\times\bar{E}_2$, $\omega\mu_0\bar{H}_1 = \bar{\beta}\times\bar{E}_1$, then

$$\bar{E}_1\times\bar{H}_2 \sim \bar{E}_1\times(\bar{\beta}\times\bar{E}_2) = \bar{E}_1(\bar{\beta}\cdot\bar{E}_2) - \bar{\beta}(\bar{E}_1\cdot\bar{E}_2) \qquad (12)$$

$$\bar{E}_2\times\bar{H}_1 \sim \bar{E}_2\times(\bar{\beta}\times\bar{E}_1) = \bar{E}_2(\bar{\beta}\cdot\bar{E}_1) - \bar{\beta}(\bar{E}_1\cdot\bar{E}_2) \qquad (13)$$

But $\bar{\beta}\cdot\bar{E}_2 = \bar{\beta}\cdot\bar{E}_1 = 0$ in the far field. Therefore, the LHS of (10) vanishes when $S\to\infty$, and (10) can be rewritten as

$$\int_V dV\,[\bar{E}_2\cdot\bar{J}_1 - \bar{H}_2\cdot\bar{M}_1] = \int_V dV\,[\bar{E}_1\cdot\bar{J}_2 - \bar{H}_1\cdot\bar{M}_2] \qquad (14)$$

The inner product symbol is often used to rewrite the above as

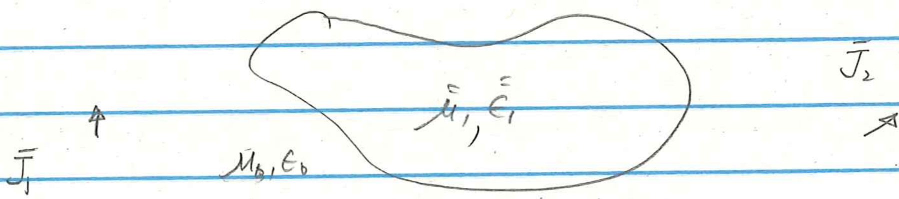
$$\langle \bar{E}_2, \bar{J}_1 \rangle - \langle \bar{H}_2, \bar{M}_1 \rangle = \langle \bar{E}_1, \bar{J}_2 \rangle - \langle \bar{H}_1, \bar{M}_2 \rangle \quad (15)$$

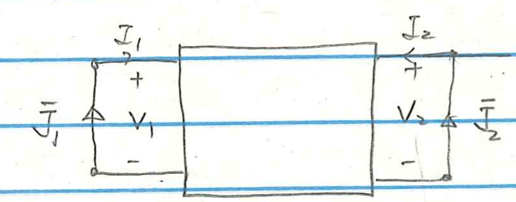
The above inner product is also called reaction, and the above is also called Rumsey's reaction theorem. The above is rewritten as

$$\langle 2, 1 \rangle = \langle 1, 2 \rangle \quad (16)$$

where

$$\langle 2, 1 \rangle = \langle \bar{E}_2, \bar{J}_1 \rangle - \langle \bar{H}_2, \bar{M}_1 \rangle \quad (17)$$





In the long wavelength limit, we can model a current source as a hertzian dipole, or that

$$\bar{J}_1 = I_1 l_1 \hat{l}_1 \delta(\bar{r} - \bar{r}_1), \qquad \bar{J}_2 = I_2 l_2 \hat{l}_2 \delta(\bar{r} - \bar{r}_2) \quad (18)$$

Applying the reciprocity theorem, we have

$$\langle \bar{E}_2, \bar{J}_1 \rangle = I_1 l_1 (\bar{E}_2 \cdot \hat{l}_1) = \langle \bar{E}_1, \bar{J}_2 \rangle = I_2 l_2 (\bar{E}_1 \cdot \hat{l}_2) \quad (19)$$

A general impedance relationship between V_i and I_j is

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad (20)$$

Focussing on a two-port network, we have

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{18}$$

Then

$$\langle \bar{E}_2, \bar{J}_1 \rangle = I_1 \int_{\text{Port 1}} \bar{E}_2 \cdot d\bar{l} = -I_1 V_1^{oc} \tag{19}$$

$$\langle \bar{E}_1, \bar{J}_2 \rangle = I_2 \int_{\text{Port 2}} \bar{E}_1 \cdot d\bar{l} = -I_2 V_2^{oc} \tag{20}$$

But $V_1^{oc} = Z_{12} I_2$, $V_2^{oc} = Z_{21} I_1$. Since $I_1 V_1^{oc} = I_2 V_2^{oc}$ by the reaction concept, then $Z_{12} = Z_{21}$. In the above, we assume that $\bar{J}_1$ is constant in the input port 1 when it is turned on, so is $\bar{J}_2$ when it is in the input port 2.

Paraxial Wave Equation

We have seen previously that in a source-free space

$$\nabla^2 \bar{A} + \omega^2 \mu\epsilon \bar{A} = 0 \quad (1)$$

$$\nabla^2 \Phi + \omega^2 \mu\epsilon \Phi = 0 \quad (2)$$

The above are four scalar wave equations with the Lorenz gauge

$$\nabla \cdot \bar{A} = -j\omega \mu_0 \epsilon_0 \Phi \quad (3)$$

connecting $\bar{A}$ and Φ. We can examine the solution of $\bar{A}$ such that

$$\bar{A}(\bar{r}) = \bar{A}_0(\bar{r}) e^{-j\beta z} \quad (4)$$

where $\bar{A}_0(\bar{r})$ is a slowly varying function while $e^{-j\beta z}$ is rapidly varying in the z direction. This is primarily a quasi-plane wave propagating in the z-direction. We know to be the case in the far field of a source, but let us assume that this form persists less than the far field.

Taking the x component of (4), we have

$$A_x(\bar{r}) = \psi(\bar{r}) e^{-j\beta z} \quad (5)$$

where $\psi(\bar{r}) = \psi(x, y, z)$ is a slowly varying function of $x, y,$ and z. Substituting (5) into (1), we require that

$$\frac{\partial^2}{\partial z^2}\left[\psi(x,y,z) e^{-j\beta z}\right] = \left[\frac{\partial^2}{\partial z^2}\psi(x,y,z) - 2j\beta\frac{\partial}{\partial z}\psi(x,y,z) - \beta^2\psi(x,y,z)\right] e^{-j\beta z} \quad (6)$$

Consequently, we obtain an equation for $\psi(\bar{r})$, the slowly varying envelope as

$$\frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi - 2j\beta\frac{\partial}{\partial z}\psi + \frac{\partial^2}{\partial z^2}\psi = 0 \quad (7)$$

When $\beta \to \infty$, or in the high frequency limit,

$$\left|2j\beta\frac{\partial}{\partial z}\psi\right| >> \left|\frac{\partial^2}{\partial z^2}\psi\right| \quad (8)$$

and then (7) can be approximated by

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-2j\beta\frac{\partial\psi}{\partial z}=0 \qquad (9)$$

The above is called the paraxial wave equation. It implies that the β vector of the wave is approximately parallel to the z axis, and hence, the name.

A closed form solution to the paraxial wave equation can be obtained by a simple trick. It is known that

$$A_x(\bar r)=\frac{e^{-j\beta|\bar r-\bar r'|}}{4\pi|\bar r-\bar r'|} \qquad (10)$$

is the solution to

$$\nabla^2 A_x+\beta^2 A_x=0 \qquad (11)$$

if $\bar r\neq\bar r'$. If we make $\bar r'=-\hat z jb$, a complex number, then (10) is always a solution to (11) for all $\bar r$, because $|\bar r-\bar r'|\neq 0$ always. Then

$$|\bar r-\bar r'|=\sqrt{x^2+y^2+(z+jb)^2}$$

$$\approx(z+jb)\left[1+\frac{x^2+y^2}{(z+jb)^2}+\cdots\right]^{1/2}$$

$$\approx(z+jb)+\frac{x^2+y^2}{2(z+jb)}+\cdots \qquad (12)$$

Then

$$A_x(\bar r)\approx\frac{e^{-j\beta(z+jb)}}{4\pi(z+jb)}\,e^{-j\beta\frac{x^2+y^2}{2(z+jb)}} \qquad (13)$$

We hence identify

$$\psi(x,y,z)=A_0\frac{+jb}{(z+jb)}\,e^{-j\beta\frac{x^2+y^2}{2(z+jb)}} \qquad (14)$$

By separating the exponential part into the real part and the imaginary part, we have

$$\psi(x,y,z)=\frac{A_0}{\sqrt{1+z^2/b^2}}\,e^{+j\tan^{-1}\left(\frac{z}{b}\right)}\,e^{-j\beta\frac{x^2+y^2}{2(z^2+b^2)}z}\,e^{-b\beta\frac{x^2+y^2}{2(z^2+b^2)}} \qquad (15)$$

The above can be rewritten as

$$\psi(x, y, z) = \frac{A_0}{\sqrt{1+\frac{z^2}{b^2}}} e^{-j\beta \frac{x^2+y^2}{2R}} e^{-\frac{x^2+y^2}{w^2}} e^{j\phi} \qquad (16)$$

where $w^2 = \frac{2b}{\beta}\left(1+\frac{z^2}{b^2}\right)$, $R = \frac{z^2+b^2}{z}$, $\phi = \tan^{-1}\left(\frac{z}{b}\right)$. (17)

For a fixed z, w, R, and ϕ are constants. Here, w is the beam waist and it is smallest when $z=0$, or $w = w_0 = \sqrt{\frac{2b}{\beta}}$.

In general, the paraxial wave equation has solution of the form

$$\psi_{mn}(x, y, z) = \frac{w_0}{w} \phi_m\left(\frac{\sqrt{2}x}{w}\right) \phi_n\left(\frac{\sqrt{2}y}{w}\right) e^{-j\frac{\beta}{2R}(x^2+y^2)} e^{j(m+n+1)\phi} \qquad (18)$$

where $\phi_m(\xi)$ and $\phi_n(\xi)$ are Hermite Gaussian function given by

$$\phi_m(\xi) = H_m(\xi) e^{-\xi^2/2} \qquad (19)$$

and $H_m(\xi)$ is a Hermite polynomial.