Lecture 20, Nov 8, 2018, W. C. CHEW

Radiation of Electromagnetic Fields

Electromagnetic fields are used for communication, sensing, wireless power transfer applications. Hence, it is imperative to understand how electromagnetic fields radiate.

To this end, we will start with frequency domain M. E.'s with a source $\bar{J}$ included, and see how this source $\bar{J}$ can radiate electromagnetic fields. Maxwell's equations are

$$\nabla\times\bar{E} = -j\omega\mu\bar{H} \tag{1}$$

$$\nabla\times\bar{H} = j\omega\epsilon\bar{E} + \bar{J} \tag{2}$$

$$\nabla\cdot\mu\bar{H} = 0 \tag{3}$$

$$\nabla\cdot\epsilon\bar{E} = \rho \tag{4}$$

In order to satisfy the third M.E., we let

$$\mu\bar{H} = \nabla\times\bar{A} \tag{5}$$

Since $\nabla\cdot(\nabla\times\bar{A}) = 0$, and it will be automatically satisfied. Now, using (5) in (1), we have

$$\nabla\times(\bar{E}+j\omega\bar{A}) = 0 \tag{6}$$

Since $\nabla\times(\nabla\Phi) = 0$, the above implies that

$$\bar{E} = -j\omega\bar{A} - \nabla\Phi \tag{7}$$

Hence, the knowledge of $\bar{A}$ and Φ uniquely determines $\bar{E}$ and $\bar{H}$. To this end, we derive a formula for $\bar{A}$ and Φ in terms of the sources $\bar{J}$ and ρ. Substituting (5) and (7) into (2) gives

$$\nabla\times\nabla\times\bar{A} = j\omega\mu\epsilon(-j\omega\bar{A} - \nabla\Phi) + \mu\bar{J} \tag{8}$$

or upon rearrangement,

$$\nabla^2\bar{A} + \omega^2\mu\epsilon\bar{A} = -\mu\bar{J} + j\omega\mu\epsilon\nabla\Phi + \nabla\nabla\cdot\bar{A} \tag{9}$$

Using (7) in (4), we have

$$\nabla\cdot(j\omega\bar{A} + \nabla\Phi) = -\frac{\rho}{\epsilon} \tag{10}$$

In (5), we can add a gradient term to $\bar{A}$ making $\bar{H}$ invariant, namely, $\bar{A}' = \bar{A} + \nabla\Psi$. To make $\bar{E}$ invariant, we let

$$\Phi' = \Phi - j\omega\Psi \tag{11}$$

Because $\bar{A}$ and Φ are not unique, in addition to specifying what $\nabla\times\bar{A}$ should be in (5), we need to specify its divergence. One way is to specify

$$\nabla\cdot\bar{A} = -j\omega\mu\epsilon\Phi \tag{12}$$

Then (9) and (10) become

$$\nabla^2\bar{A} + \omega^2\mu\epsilon\bar{A} = -\mu\bar{J} \tag{13}$$

$$\nabla^2\Phi + \omega^2\mu\epsilon\Phi = -\frac{\rho}{\epsilon} \tag{14}$$

Equation (12) is known as the Lorenz gauge.

Equations (13) and (14) can be solved using the Green's function method. They together constitute four scalar equations similar to each other. Hence, we need to solve the point-source response, or the Green's function of these equations by solving

$$\nabla^2 g(\bar{r},\bar{r}') + \beta^2 g(\bar{r},\bar{r}') = -\delta(\bar{r}-\bar{r}') \tag{15}$$

In Lecture 5, we have shown that when $\beta = 0$, $g(\bar{r},\bar{r}') = g(\bar{r}-\bar{r}') = 1/(4\pi|\bar{r}-\bar{r}'|)$. When $\beta \neq 0$, the correct solution is

$$g(\bar{r},\bar{r}') = g(\bar{r}-\bar{r}') = \frac{e^{-j\beta|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} \tag{16}$$

which can be verified by back substitution.

By using the principle of linear superposition, or convolution, the solutions to (13) and (14) are then

$$\bar{A}(\bar{r}) = \frac{\mu}{4\pi}\iiint d\bar{r}'\,\frac{\bar{J}(\bar{r}')}{|\bar{r}-\bar{r}'|}\,e^{-j\beta|\bar{r}-\bar{r}'|} \tag{17}$$

$$\Phi(\bar{r}) = \frac{1}{4\pi\epsilon}\iiint d\bar{r}'\,\frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}\,e^{-j\beta|\bar{r}-\bar{r}'|} \tag{18}$$

Radiation Field or Far-Field Approximation

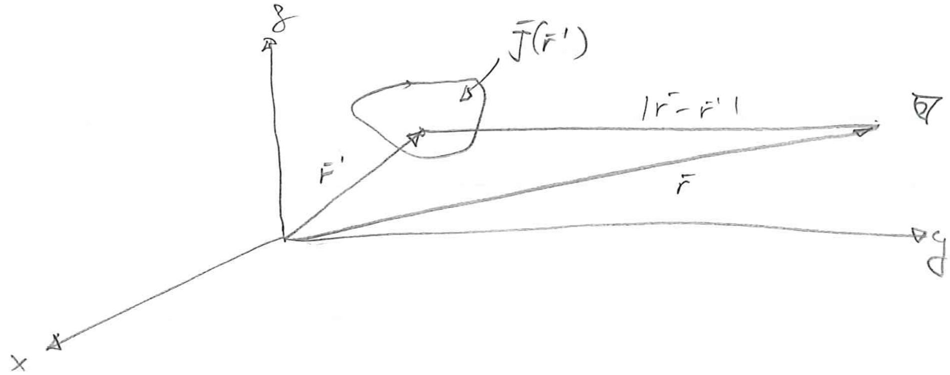


Fig 1.

The integrals in (17) and (18) are normally untenable, but when the observation point is far from the source, approximation to the integral can be made giving it a nice physical interpretation. When $|\bar r| \gg |\bar r'|$, then $|\bar r - \bar r'| \approx r - \bar r' \cdot \hat r$, and (17) previously becomes

$$\bar A(\bar r) \simeq \iiint_V d\bar r' \, \frac{\mu \bar J(\bar r')}{r - \bar r' \cdot \hat r} \, e^{-j\beta r + j\beta \bar r' \cdot \hat r}$$

$$\approx \frac{\mu e^{-j\beta r}}{4\pi r} \iiint_V d\bar r' \, \bar J(\bar r') e^{j\beta \bar r' \cdot \hat r} \qquad (1)$$

In the above we have made used of that $1/(1-\Delta) \approx 1$ when Δ is small, but $e^{j\beta\Delta} \neq 1$, unless $j\beta\Delta \ll 1$. Hence, we keep the exponential term in (1). If we let $\bar\beta = \beta \hat r$, and $\bar r' = \hat x x' + \hat y y' + \hat z z'$, then

$$e^{j\beta \bar r' \cdot \hat r} = e^{j\bar\beta \cdot \bar r'} = e^{j\beta_x x' + j\beta_y y' + j\beta_z z'} \qquad (2)$$

Therefore (1) resembles a Fourier transform integral, and (1) can be rewritten as

$$\bar A(\bar r) \cong \frac{\mu e^{-j\beta r}}{4\pi r} \bar F(\bar\beta) \qquad (3)$$

where $\bar F(\bar\beta)$ is the Fourier transform of $\bar J(\bar r')$.

This is not a normal Fourier transform, because $|\bar{\beta}|^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 = \beta^2$. It is the Fourier transform of the current source $\bar{J}(\bar{r}')$ with Fourier variables, $\beta_x, \beta_y, \beta_z$ lying on a sphere of radius β, and $\bar{\beta} = \beta\hat{r}$. We can write $\hat{r}$ or $\bar{\beta}$ in terms of direction cosines in spherical coordinates so that

$$\hat{r} = \hat{x}\cos\phi\sin\theta + \hat{y}\sin\phi\sin\theta + \hat{z}\cos\theta \qquad (4)$$

Hence

$$\bar{F}(\bar{\beta}) = \bar{F}(\beta\hat{r}) = \bar{F}(\beta, \theta, \phi)$$

Also in (3), when $r \gg \bar{r}'\cdot\hat{r}$, $e^{j\beta r}$ is rapidly varying function of r while, $\bar{F}(\bar{\beta})$ is only a function of θ and ϕ, the observation angles. Hence, we can write $e^{j\beta r} = e^{j\bar{\beta}\cdot\bar{r}}$ where $\bar{\beta} = \hat{r}\beta$. Then, it is clear that $\nabla\times \rightarrow j\bar{\beta} = -j\beta\hat{r}$, and

$$\bar{H} = \frac{1}{\mu}\nabla\times\bar{A} \simeq -\frac{j\beta}{\mu}\hat{r}\times(\hat{\theta}A_\theta + \hat{\phi}A_\phi)$$

$$= j\frac{\beta}{\mu}(\hat{\theta}A_\phi - \hat{\phi}A_\theta) \qquad (5).$$

Similarly,

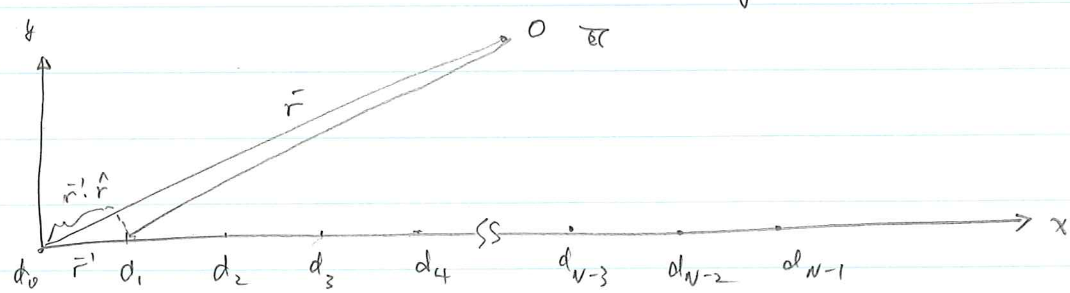
$$\bar{E} = \frac{1}{j\omega\epsilon}\nabla\times\bar{H} \simeq -j\omega[\hat{\theta}A_\theta + \hat{\phi}A_\phi] \qquad (6)$$

Power flow & intrinsic impedance.

Linear Array of Dipole Antennas

First, we assume that

$$\bar{J}(\bar{r}') = \hat{z}\,\mathcal{U}[A_0\delta(x') + A_1\delta(x'-d_1) + A_2\delta(x'-d_2) + \cdots + A_{N-1}\delta(x'-d_{N-1})]\delta(y')\delta(z')$$



The the vector potential on the xy-plane is derived to be

$$\bar{A}(\bar{r}) \cong \hat{z}\frac{\mu I l}{4\pi r} e^{-j\beta r} \iiint d\bar{r}' \left[A_0\delta(x') + A_1\delta(x'-d_1) + \dots\right] \delta(y')\delta(z') e^{j\beta \bar{r}'\cdot\hat{r}}$$

$$= \hat{z}\frac{\mu I l}{4\pi r} e^{-j\beta r}\left[A_0 + A_1 e^{j\beta d_1 \cos\phi} + A_2 e^{j\beta d_2\cos\phi} + \dots + A_{N-1} e^{j\beta d_{N-1}\cos\phi}\right] \quad (7)$$

If $d_n = nd$, and $A_n = e^{jn\psi}$, then (7) becomes

$$\bar{A}(\bar{r}) \cong \hat{z}\frac{\mu I l}{4\pi r} e^{-j\beta r}\left[1 + e^{j(\beta d\cos\phi + \psi)} + e^{2j(\beta d\cos\phi+\psi)} + \dots + e^{j(N-1)(\beta d\cos\phi+\psi)}\right] \quad (8)$$

which can be summed because

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x} \quad (9)$$

Then,

$$\bar{A}(\bar{r}) = \hat{z}\frac{\mu I l}{4\pi r} e^{-j\beta r} \frac{1 - e^{jN(\beta d\cos\phi+\psi)}}{1 - e^{j(\beta d\cos\phi+\psi)}} \quad (10)$$

Since on the xy plane, $E_\theta = -j\omega A_\theta = j\omega A_z$. Then,

$$|E_\theta| = |E_0| \left|\frac{1 - e^{jN(\beta d\cos\phi+\psi)}}{1 - e^{j(\beta d\cos\phi+\psi)}}\right|$$

$$= |E_0| \left|\frac{\sin\frac{N}{2}(\beta d\cos\phi+\psi)}{\sin\frac{1}{2}(\beta d\cos\phi+\psi)}\right| \quad (11)$$

<u>When is Far-Field Approximation Valid?</u>

In (1) in page 3, it will be interesting to ponder when the far-field approximation is valid? That is, when we can approximate

$$e^{-j\beta|\bar{r}-\bar{r}'|} \approx e^{-j\beta r + j\beta\bar{r}'\cdot\hat{r}} \quad (1)$$

This is especially important because when we integrate over $\bar{r}'$, it can range over large values. We need to study (1) more carefully.

$$|\bar{r}-\bar{r}'|^2 = (\bar{r}-\bar{r}')\cdot(\bar{r}-\bar{r}') = r^2 - 2\bar{r}\cdot\bar{r}' + r'^2 \quad (2)$$

We can take the square root of the above to get

$$|\bar{r}-\bar{r}'| = r\left(1 - \frac{2\bar{r}\cdot\bar{r}'}{r^2} + \frac{r'^2}{r^2}\right)^{1/2}$$

$$\approx r\left[1 - \frac{\bar{r}\cdot\bar{r}'}{r^2} + \frac{1}{2}\frac{r'^2}{r^2} - \frac{1}{4}\left(\frac{\bar{r}\cdot\bar{r}'}{r^2}\right)^2 + \cdots\right]$$

$$= r - \frac{\bar{r}\cdot\bar{r}'}{r} + \frac{1}{2}\frac{r'^2}{r} - \frac{1}{4}\frac{(\bar{r}\cdot\bar{r}')^2}{r^3} +$$

$$= r - \hat{r}\cdot\bar{r}' + \frac{1}{2}\frac{r'^2}{r} - \frac{1}{4r}(\hat{r}\cdot\bar{r}')^2 + \cdots \qquad (3)$$

The last two terms are of the same order. Hence, the far field approximation is valid if

$$\beta\frac{r'^2}{r} << 1 \qquad (4)$$

or that

$$r >> \frac{2\pi}{\lambda} r'^2 \qquad (5)$$

If the aperture of the antenna is of diameter D, then $r'_{max} \cong D$ and the far field approximation is valid if

$$r >> \frac{2\pi}{\lambda} D^2 = r_R \qquad (6).$$

If r is larger than this distance, then a focus antenna beam behaves like a spherical wave and starts to diverge.

This distance is also known as the Rayleigh distance.