Hollow Waveguides

Many waveguide problems can be solved in closed form. An example is the coaxial waveguide. But there are many other waveguide problems that have closed form solutions. Closed form solutions to Laplace and Helmholtz equations are obtained by the separation of variables. These separation of variables methods work only for separable coordinate systems. There are 11 separable coordinates for Helmholtz equations, but 13 for Laplace equation. Some examples of separable coordinate systems are cartesian, cylindrical, and spherical coordinates.

When a waveguide has a center conductor (or two conductors) like a coaxial cable, it can support a TEM wave where both the electric field and the magnetic field are orthogonal to the direction of propagation. The uniform plane wave is a TEM wave, for instance. However, when the waveguide is hollow or is filled completely with a homogeneous medium, it can only support a TE_z or TM_z wave like the case of a layered medium. For a TE_z wave (or TE wave), $E_z = 0$, $H_z \neq 0$, while for a TM_z wave (or TM wave), $H_z = 0$, $E_z \neq 0$. These classes of problems can be decomposed into two scalar problems like the layered medium, by using the pilot potential method.

TE Case ($E_z = 0$, $H_z \neq 0$)

In this case, the field inside the waveguide is TE to z. We can write the TE $\bar{E}$ field as

$$\bar{E}(\bar{r}) = \nabla \times \hat{z}\, \psi_h(\bar{r}) \qquad (1)$$

Equation (1) will guarantee that $E_z = 0$ due to its construction. Here, $\psi_h(\bar{r})$ is a scalar potential and $\hat{z}$ is the pilot vector.

The waveguide is assumed source free, and hence,

$$\left(\nabla^2+\beta^2\right)\bar{E}(\bar{r})=0 \qquad (2)$$

Substituting (1) into (2), we get

$$\nabla\times\hat{z}\left(\nabla^2+\beta^2\right)\psi_h(\bar{r})=0 \qquad (3)$$

If

$$\left(\nabla^2+\beta^2\right)\psi_h(\bar{r})=0 \qquad (4)$$

where $\beta^2=\omega^2\mu\epsilon$, then (2) is satisfied. Clearly, from (1), $\nabla\cdot\bar{E}=0$ as well. Hence, the $\bar{E}$ field constructed with (1), where $\psi_h(\bar{r})$ satisfies (4) satisfies Maxwell's equations.

Next, we look at the boundary condition for $\psi_h(\bar{r})$. The boundary condition for $\bar{E}$ is that $\hat{n}\times\bar{E}=0$ on C, the wall of the waveguide.

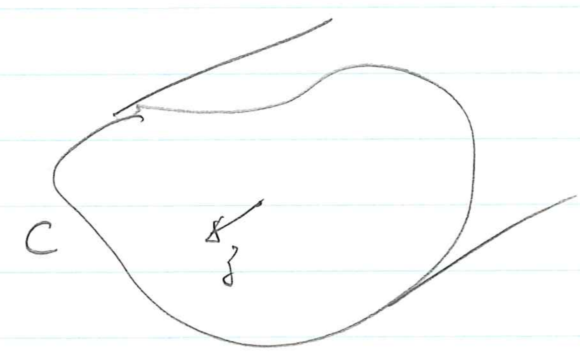

Fig. 1

But from (1), using the back-of-the-cab formula,

$$\hat{n}\times\bar{E}=\hat{n}\times\left(\nabla\times\hat{z}\psi_h\right)=-\hat{n}\cdot\nabla\psi_h=0 \qquad (5)$$

In the above $\hat{n}\cdot\nabla=\hat{n}\cdot\nabla_s$ where $\nabla_s=\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}$ since $\hat{n}$ has no z component. The B.C. (5) becomes

$$\hat{n}\cdot\nabla_s\psi_h=\partial_n\psi_h=0 \quad \text{on } C \qquad (6)$$

which is also known as the homogeneous Neumann B.C..

Furthermore, in a waveguide, just as in a transmission line, we are looking for solutions of the form for (4), or

$$\psi_h(\bar{r}) = \psi_{hs}(\bar{r}_s)e^{\mp j\beta_z z} \qquad (7)$$

where $\bar{r}_s = \hat{x}x + \hat{y}y$, or in short, $\psi_{hs}(\bar{r}_s) = \psi_{hs}(x,y)$. With this assumption, $\frac{\partial^2}{\partial z^2} \to -\beta_z^2$, and (4) becomes

$$(\nabla_s^2 + \beta^2 - \beta_z^2)\psi_{hs}(\bar{r}_s) = 0, \quad \partial_n \psi_{hs}(\bar{r}_s) = 0, \text{ on } C \qquad (8)$$

which is a BVP for a 2D waveguide problem.

<u>TM Case</u> $(E_z \neq 0, H_z = 0)$

Repeating the treatment for TM waves, the TM magnetic field is

$$\bar{H} = \nabla \times \hat{z}\psi_e(\bar{r}) \qquad (9)$$

where

$$(\nabla^2 + \beta^2)\psi_e(\bar{r}) = 0 \qquad (10)$$

The corresponding $\bar{E}$ field is proportional to

$$\bar{E} \sim \nabla \times \nabla \times \hat{z}\psi_e(\bar{r}) = \nabla\nabla\cdot(\hat{z}\psi_e) - \nabla^2\hat{z}\psi_e$$
$$= \nabla\frac{\partial}{\partial z}\psi_e + \hat{z}\beta^2\psi_e \qquad (11)$$

Taking the z component of the above, we get

$$E_z \sim \frac{\partial^2}{\partial z^2}\psi_e + \beta^2\psi_e \qquad (12)$$

Assuming that

$$\psi_e \sim e^{\pm j\beta_z z} \qquad (13)$$

then

$$E_z \sim (\beta^2 - \beta_z^2)\psi_e \qquad (14)$$

Therefore, if

$$\psi_e(\bar{r}) = 0 \quad \text{on} \quad C, \qquad (15)$$

then,

$$E_z(\bar{r}) = 0 \quad \text{on} \quad C \qquad (16)$$

With some manipulation, the BVP related to (10) is

$$\left(\nabla_s^2 + \beta^2 - \beta_z^2\right)\psi_{es}(\bar{r}_s) = 0 \qquad (17)$$

$$\psi_{es}(\bar{r}_s) = 0, \qquad \bar{r}_s \in C \qquad (18)$$

the homogeneous Dirichlet B.C.
where we have assumed that

$$\psi_e(\bar{r}) = \psi_{es}(\bar{r}_s)e^{\mp j\beta_z z} \qquad (19)$$

We can solve some simple waveguides as illustrations.

<u>Rectangular Waveguides</u>

<u>TE Modes (H modes)</u>

The scalar potential $\psi_{hs}(\bar{r}_s)$ satisfies

$$\left(\nabla_s^2 + \beta_s^2\right)\psi_{hs}(\bar{r}_s) = 0, \quad \frac{\partial}{\partial n}\psi_{hs}(\bar{r}_s) = 0 \text{ on } C \qquad (1)$$

where $\beta_s^2 = \beta^2 - \beta_z^2$. An solution for $\psi_{hs}(x,y)$ is

$$\psi_{hs}(x,y) = A\cos(\beta_x x)\cos(\beta_y y) \qquad (2)$$

where $\beta_x^2 + \beta_y^2 = \beta_s^2$.
It is quite clear that $\psi_{hs}(x,y)$ satisfies equation (1). Furthermore, it satisfies the homogeneous Neumann B.C. at $x=0$, and $y=0$ surfaces.

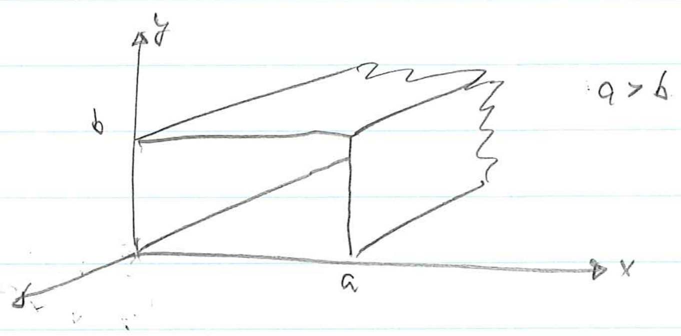

Fig. 2

To further satisfy the BC at $x=a$, and $y=b$ surfaces, it is necessary that B.C. for eq. (6) is satisfied or that

$$\partial_x \psi_{hs}(x,y) \sim \sin(\beta_x a)\cos(\beta_y y) = 0, \qquad (3)$$

$$\partial_y \psi_{hs}(x,y) \sim \cos(\beta_x x)\sin(\beta_y b) = 0, \qquad (4)$$

The above implies that $\beta_x a = m\pi$, $\beta_y b = n\pi$ where m and n are integers. Hence, (2) becomes

$$\psi_{hs}(x,y) = A \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \tag{5}$$

where
$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \beta^2 - \beta_z^2 \tag{6}$$

or
$$\beta_z = \sqrt{\beta^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{7}$$

When $m = n = 0$, then $\bar{E}(\bar{r}) = \nabla \times \hat{z}\,\psi_h(\bar{r}) = \nabla_s \times \hat{z}\,\psi_h(\bar{r}) = 0$. It turns out that $\bar{H}(\bar{r}) = \hat{z} H_0$, a static field in the waveguide. This is not a very interesting mode, and TE_{00} propagating mode is assumed not to exist. So the TE_{mn} modes cannot have both $m = n = 0$.

Furthermore, when
$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 > \beta^2 = \omega^2\mu\epsilon \tag{8}$$

β_z becomes pure imaginary, and the mode cannot propagate or become evanescent in the z direction. For a fixed m and n, a and b, the frequency at which the above happens is called the cut-off frequency of the TE_{mn} mode. It is given by

$$\omega > \omega_{mn,c} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{9}$$

A corresponding cutoff wavelength is

$$\lambda < \lambda_{mn,c} = \frac{2}{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^{1/2}} \tag{10}$$

For the TE_{10} mode, the above reduces to

$$\lambda < \lambda_{10,c} = 2a \tag{11}$$

The above have the physical meaning that the wavelength has to be smaller than $2a$ in order for the mode to fit into the waveguide. The TE_{10} mode, when $a > b$, is also the mode with the lowest cutoff frequency or longest cutoff wavelength.

TM Modes (E Modes)

The scalar wave function for the TM modes is

$$\psi_{es}(x,y) = A \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \quad (12)$$

In this case, both $m>0$, and $n>0$ are needed. Hence, the lowest TM mode is the TM_{11} mode. The corresponding cutoff frequencies and cutoff wavelengths are the same as the TE case. Therefore, TM_{11} mode has a higher cutoff frequency than TE_{10} mode. Also, TE_{11} and TM_{11} modes have the same cutoff frequency. The modes are degenerate in this case.

Circular Waveguides

For

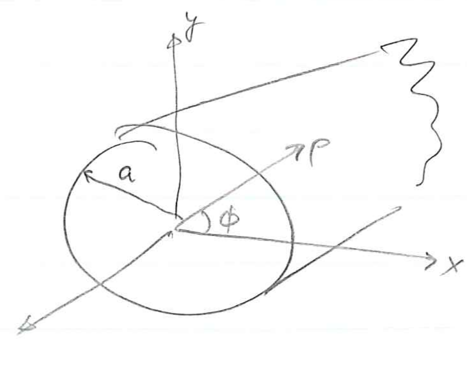

Fig 3.

TE Case

For a circular waveguide, it is best to express the Laplacian operator $\nabla_s^2 = \nabla_s \cdot \nabla_s$ in cylindrical coordinates. Doing a table lookup, $\nabla_s \psi = \hat{\rho}\frac{\partial}{\partial\rho}\psi + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}$, $\nabla_s\cdot\bar{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho A_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}A_\phi$. Then

$$\left(\nabla_s^2 + \beta_s^2\right)\psi_{hs} = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \beta_s^2\right)\psi_{hs}(\rho,\phi) = 0 \quad (1)$$

By separation of variables, we let

$$\psi_{hs}(\rho,\phi) = B_n(\beta_s\rho)e^{\pm jn\phi} \quad (2)$$

Then $\frac{\partial^2}{\partial\phi^2} \rightarrow -n^2$, and (1) becomes an ODE, which is

$$\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{n^2}{\rho^2} + \beta_s^2\right)B_n(\beta_s\rho) \quad (3)$$

The above is known as the Bessel equation whose solutions are special functions

Bessel, $J_n(\beta_s\rho) = \frac{1}{2}[H_n^{(1)}(\beta_s\rho) + H_n^{(2)}(\beta_s\rho)]$ (4)

Neumann, $Y_n(\beta_s\rho) = \frac{1}{2j}[H_n^{(1)}(\beta_s\rho) - H_n^{(2)}(\beta_s\rho)]$ (5)

Hankel-First kind, $H_n^{(1)}(\beta_s\rho) = J_n(\beta_s\rho) + j\,Y_n(\beta_s\rho)$ (6)

Hankel-Second kind, $H_n^{(2)}(\beta_s\rho) = J_n(\beta_s\rho) - j\,Y_n(\beta_s\rho)$ (7)

It can be shown that

$$H_n^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{jx - j(n+\frac{1}{2})\frac{\pi}{2}}, \quad x \to \infty \tag{8}$$

$$H_n^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{-jx + j(n+\frac{1}{2})\frac{\pi}{2}}, \quad x \to \infty \tag{9}$$

Moreover, $Y_n(x)$, $H_n^{(1)}(x)$, and $H_n^{(2)}(x) \to \infty$ when $x \to 0$. So the only viable solution for the waveguide is that $B_n(\beta_s\rho) = A\,J_n(\beta_s\rho)$. The homogeneous Neumann B.C. on the waveguide wall translates to

$$\frac{d}{d\rho} J_n(\beta_s\rho) = 0, \quad \rho = a. \tag{10}$$

Defining $J_n'(x) = \frac{d}{dx} J_n(x)$, the above is the same as

$$J_n'(\beta_s a) = 0, \tag{11}$$

These are the zeros of Bessel function derivative and they are tabulated. The m-th zero of $J_n'(x)$ is defined to be β_{nm} in many books, and hence,

$$\beta_s = \beta_{nm}/a, \quad TE_{nm} \text{ mode} \tag{12}$$

The corresponding cutoff frequency of the TE_{nm} mode is

$$\omega_{nm,c} = \frac{1}{\sqrt{\mu\epsilon}} \frac{\beta_{nm}}{a} \tag{13}$$

with the corresponding cutoff wavelength to be

$$\lambda_{nm,c} = \frac{2\pi}{\beta_{nm}} a \tag{14}$$

TM Case

The corresponding BVP for this case is

$$\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \beta_s^2\right)\psi_{es}(\rho, \phi) = 0 \qquad (15)$$

The solution is

$$\psi_{es}(\rho,\phi) = A\, J_n(\beta_s \rho)\, e^{\pm j n \phi} \qquad (16)$$

with the B.C. that $J_n(\beta_s a) = 0$. The zeros of $J_n(x)$ are labeled as α_{nm} in many textbooks, and hence, the guidance condition is that for the TM_{nm} mode

$$\beta_s = \frac{\alpha_{nm}}{a} \qquad (17)$$

with the corresponding cutoff frequency to be

$$\omega_{nm,c} = \frac{1}{\sqrt{\mu\epsilon}}\frac{\alpha_{nm}}{a}.$$

and the cutoff wavelength to be

$$\lambda_{nm,c} = \frac{2\pi}{\alpha_{nm}}\, a \qquad (18)$$

It turns out that the lowest mode in a circular waveguide is the TE_{11} mode. It is actually a close cousin of the TE_{10} mode of a rectangular waveguide.