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<u>Homomorphism of Uniform Plane Waves and Transmission Lines</u>

For uniform plane wave, we know that

$$\bar{\beta} \times \bar{E} = \omega\mu \bar{H} \tag{1}$$

$$\bar{\beta} \times \bar{H} = -\omega\epsilon \bar{E} \tag{2}$$

Assuming a TE wave traveling in z direction, $\bar{E} = \hat{y} E_y$, then we have from (1)

$$\beta_z E_y = -\omega\mu H_x, \quad \beta_x E_y = \omega\mu H_z \tag{3}$$

From (2), we have

$$\beta_z H_x - \beta_x H_z = -\omega\epsilon E_y \tag{4}$$

Then, expressing H_z in terms of E_y from (3), we can show that

$$\beta_z H_x = -\omega\epsilon \cos^2\theta \, E_y \tag{5}$$

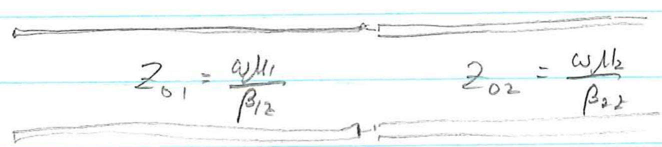
where $\beta \cos\theta = \beta_z$. Eq (3) and (5) can be written to look like the telegrapher's equation

$$\frac{\partial}{\partial z} E_y = -j\omega\mu H_x \tag{6}$$

$$\frac{\partial}{\partial z} H_x = j\omega\epsilon \cos^2\theta \, E_y \tag{7}$$

If we let $E_y \to V$, $H_x \to -I$, $\mu \to L$, $\epsilon \cos^2\theta \to C$, the above is exactly the telegrapher's equation. The characteristic impedance of this equation is then

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\epsilon}} \frac{1}{\cos\theta} = \sqrt{\frac{\mu}{\epsilon}} \frac{\beta}{\beta_z} = \frac{\omega\mu}{\beta_z} \tag{8}$$



We can use the above to find Γ_{12} as given by

$$\Gamma_{12} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} = \frac{(\mu_2/\beta_{2z}) - (\mu_1/\beta_{1z})}{(\mu_2/\beta_{2z}) + (\mu_1/\beta_{1z})} \tag{9}$$

The above is the same as the Fresnel reflection coefficient we have found earlier.

For the TM polarization, from duality principle, the corresponding equations are, from (6) and (7),

$$\frac{\partial}{\partial z} H_y = -j\omega\epsilon E_x \tag{10}$$

$$\frac{\partial}{\partial z} E_x = -j\omega\mu\cos^2\theta\, H_y \tag{11}$$

Just for consistency of units, we may choose the following map to convert the above into the telegrapher's equations, viz;

$$E_x \to V, \quad H_y \to I, \quad \mu\cos^2\theta \to L, \quad \epsilon \to C. \tag{12}$$

Then, the characteristic impedance is now

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\epsilon}}\cos\theta = \frac{\beta_z}{\omega\epsilon} \tag{13}$$

Now,

$$\Gamma_{12} = \frac{(\beta_{2z}/\epsilon_2) - (\beta_{1z}/\epsilon_1)}{(\beta_{2z}/\epsilon_2) + (\beta_{1z}/\epsilon_1)} \tag{14}$$

Notice that (14) has a sign difference from the definition of R^{TM}. The reason is that R^{TM} is for the reflection coefficient of magnetic field while Γ_{12} above is for the reflection coefficient of electric field. This difference is also seen in the definition for transmission coefficients.

Because of the above homomorphism, one can easily use multi-section transmission line formula to study electromagnetic waves in layered media.

Layered Medium Problems—Homomorphism with Transmission Line Problems

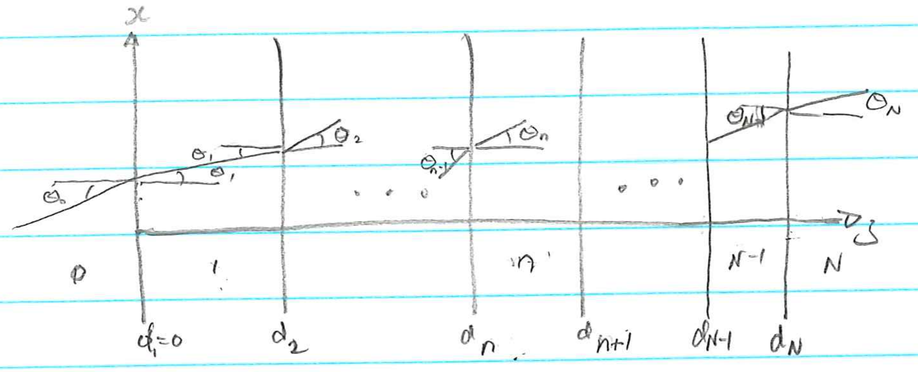


Fig. 1.

For a layered medium problem, due to phase matching condition, all the fields in the layers have the same $e^{-j\beta_x x}$ dependence. For the TE problem, we can factor out the x dependence and write the field in each region as

$$\bar{E}(\bar{r}) = \bar{e}(z)e^{-j\beta_x x} = \hat{y}\,e(z)e^{-j\beta_x x} \qquad (1)$$

The field in a region can be expressed as

$$\bar{e}_0(z) = \hat{y}\,E_0\left(e^{-j\beta_{0z} z} + \tilde{R}_{01}e^{+j\beta_{0z} z}\right) \qquad (2)$$

In general,

$$\bar{e}_n(z) = \hat{y}\,E_n\left(e^{-j\beta_{nz} z} + \tilde{R}_{n,n+1}e^{+j\beta_{nz}(z-2d_{n+1})}\right) \qquad (3)$$

The corresponding tangential H_x field is

$$H_x = \frac{1}{j\omega\mu}\frac{\partial}{\partial z}e_y(z)\,e^{-j\beta_x x} = h_x(z)\,e^{-j\beta_x x} \qquad (4)$$

Hence, we see that

$$j\omega\mu\,h_x(z) = \frac{\partial}{\partial z}e_y(z) \qquad (5)$$

Similarly, we can derive that

$$j\omega\epsilon\,e_y(z) = \frac{\partial}{\partial z}h_x(z) \qquad (6)$$

Wave Polarization

We can write a general uniform plane wave propagating in the z direction as

$$\bar{E} = \hat{x} E_x(z,t) + \hat{y} E_y(z,t) \qquad (1)$$

Clearly, $\nabla \cdot \bar{E} = 0$, and $E_x(z,t)$ and $E_y(z,t)$ are solutions to the one-D wave equation. For a time harmonic field, we have in general

$$E_x(z,t) = E_1 \cos(\omega t - \beta z) \qquad (2)$$

$$E_y(z,t) = E_2 \cos(\omega t - \beta z + \alpha) \qquad (3)$$

We shall study how this wave behaves for different α. We set $z=0$ to observe this field. Then

$$\bar{E} = \hat{x} E_1 \cos(\omega t) + \hat{y} E_2 \cos(\omega t + \alpha). \qquad (4)$$

For $\alpha = \frac{\pi}{2}$,

$$E_x = E_1 \cos(\omega t), \quad E_y = E_2 \cos(\omega t + \tfrac{\pi}{2}) \qquad (5)$$

We evaluate the above for different ωt's.

$$\omega t = 0, \quad E_x = E_1, \quad E_y = 0 \qquad (6)$$

$$\omega t = \pi/4, \quad E_x = E_1/\sqrt{2}, \quad E_y = -E_2/\sqrt{2} \qquad (7)$$

$$\omega t = \pi/2, \quad E_x = 0, \quad E_y = -E_2 \qquad (8)$$

$$\omega t = 3\pi/4, \quad E_x = -E_1/\sqrt{2}, \quad E_y = -E_2/\sqrt{2} \qquad (9)$$

$$\omega t = \pi, \quad E_x = -E_1, \quad E_y = 0 \qquad (10)$$

The tip of the vector field $\bar{E}$ traces out an ellipse as shown in Fig. 1. With the thumb pointing in the z direction, and the wave rotating in the direction of the fingers, such a wave is called left hand elliptically polarized wave.

When $E_1 = E_2$, the ellipse becomes a circle, we have a left-hand circularly polarized wave. When $\alpha = -\pi/2$, the wave rotates in the counter-clockwise direction, and the wave is either right-hand elliptically polarized, or right-hand circularly polarized wave depending on the ratio of E_1/E_2.

For the general case for arbitrary α, we let

$$E_x = E_1 \cos\alpha, \qquad E_y = E_2 \cos(\omega t + \alpha) = E_2(\cos\omega t \cos\alpha - \sin\omega t \sin\alpha) \tag{11}$$

Then

$$E_y = \frac{E_2}{E_1} E_x \cos\alpha - E_2 \left[1 - \left(\frac{E_x}{E_1}\right)^2\right]^{1/2} \sin\alpha . \tag{12}$$

Rearranging and squaring, we get

$$aE_x^2 - b E_x E_y + c E_y^2 = 1. \tag{13}$$

where

$$a = \frac{1}{E_1^2 \sin^2\alpha}, \quad b = \frac{2\cos\alpha}{E_1 E_2 \sin\alpha}, \quad c = \frac{1}{E_2^2 \sin^2\alpha} \tag{14}$$

Equation (13) is of the form

$$ax^2 - bxy + cy^2 = 1$$

The equation of an ellipse in its self coordinates is

$$\left(\frac{x'}{A}\right)^2 + \left(\frac{y'}{B}\right)^2 = 1 \tag{15}$$

We can transform the above back to the (x, y) coordinates by letting

$$x' = x\cos\theta - y\sin\theta \tag{16}$$

$$y' = x\sin\theta + y\cos\theta \tag{17}$$

to get

$$x^2\left(\frac{\cos^2\theta}{A^2} + \frac{\sin^2\theta}{B^2}\right) - xy \sin 2\theta \left(\frac{1}{A^2} - \frac{1}{B^2}\right) + y^2\left(\frac{\sin^2\theta}{A^2} + \frac{\cos^2\theta}{B^2}\right) = 1 \tag{18}$$

Comparing (13) and (18), one gets

$$\Theta = \frac{1}{2}\tan^{-1}\left(\frac{2\cos\alpha E_1 E_2}{E_2^2 - E_1^2}\right) \qquad (19)$$

$$AR = \left(\frac{1+\Delta}{1-\Delta}\right)^{1/2} > 1 \qquad (20)$$

where AR is the axial ratio where

$$\Delta = \left(1 - \frac{4E_1^2 E_2^2 \sin^2\alpha}{E_1^2 + E_2^2}\right)^{1/2} \qquad (21)$$

It is to be noted that in the phasor world, (1) becomes

$$\bar{E}(z,\omega) = \hat{x} E_1 e^{-j\beta z} + \hat{y} E_2 e^{-j\beta z + j\alpha} \qquad (22)$$

For LHEP,

$$\bar{E}(z,\omega) = e^{-j\beta z}(\hat{x} E_1 + j\hat{y} E_2) \qquad (23)$$

where for LHCP.

$$\bar{E}(z,\omega) = e^{-j\beta z} E_1 (\hat{x} + j\hat{y}) \qquad (24)$$

For RHEP, the above becomes

$$\bar{E}(z,\omega) = e^{-j\beta z}(\hat{x} E_1 - j\hat{y} E_2) \qquad (25)$$

where for RHCP,

$$\bar{E}(z,\omega) = e^{-j\beta z} E_1 (\hat{x} - j\hat{y}) \qquad (26)$$

For a linearly polarized wave,

$$\bar{S}(t) = \bar{E} \times \bar{H} = \hat{z}\frac{E_0^2}{\eta}\cos^2(\omega t - \beta z) \qquad (27)$$

For a circularly polarized wave,

$$\bar{E} = (\hat{x} \pm j\hat{y}) E_0 e^{-j\beta z} \qquad (28)$$

$$\bar{H} = (\mp\hat{x} - j\hat{y}) j \frac{E_0}{\eta} e^{-j\beta z} \qquad (29)$$

Then

$$\bar{E}(t) = \hat{x}\,E_0 \cos(\omega t-\beta z) \pm \hat{y}\,E_0 \sin(\omega t-\beta z) \qquad (30)$$

$$\bar{H}(t) = \mp \hat{x}\,\frac{E_0}{\eta}\sin(\omega t-\beta z) + \hat{y}\,\frac{E_0}{\eta}\cos(\omega t-\beta z) \qquad (31)$$

Then

$$\bar{S}(t) = \bar{E}(t)\times\bar{H}(t) = \hat{z}\,\frac{E_0^2}{\eta}\cos^2(\omega t-\beta z) + \hat{z}\,\frac{E_0^2}{\eta}\sin^2(\omega t-\beta z)$$

$$= \hat{z}\,\frac{E_0^2}{\eta} \qquad (32)$$

A CP wave delivers constant power.

It is to be noted that in cylindrical coordinates,

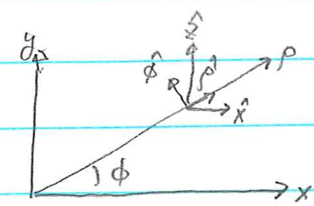


Fig. 2

$\hat{x} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$, $\hat{y} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi$, then

$$(\hat{x} \pm j\hat{y}) = \hat{\rho}\,e^{\pm j\phi} \pm j\hat{\phi}\,e^{\pm j\phi} \qquad (33)$$

Hence, the $\hat{\rho}$ and $\hat{\phi}$ of a CP is also a traveling wave in the $\hat{\phi}$ direction in addition to being a travelling wave $e^{-j\beta z}$ in the $\hat{z}$ direction. Hence, the wave possesses angular momentum called the spin angular momentum (SAM), just as a traveling wave $e^{-j\beta z}$ possesses linear angular momentum in the $\hat{z}$ direction.