Lecture 17, Oct 30, 2018, W. C. Chew

Duality Principle

Duality principle exploits the inherent symmetry of Maxwell's equations. Once a set of $\bar{E}, \bar{H}$ has been found to solve Maxwell's equations for a certain geometry, another set for a similar geometry can be found by invoking this principle. M.E's. in frequency domain are

$$\nabla \times \bar{E}(\bar{r},\omega) = -j\omega \bar{B}(\bar{r},\omega) - \bar{M}(\bar{r},\omega) \tag{1}$$

$$\nabla \times \bar{H}(\bar{r},\omega) = j\omega \bar{D}(\bar{r},\omega) + \bar{J}(\bar{r},\omega) \tag{2}$$

$$\nabla \cdot \bar{B}(\bar{r},\omega) = \rho_m(\bar{r},\omega) \tag{3}$$

$$\nabla \cdot \bar{D}(\bar{r},\omega) = \rho(\bar{r},\omega) \tag{4}$$

One way to make Maxwell's equations invariant is to do the following substitutions:

$$\bar{E} \to \bar{H}, \quad \bar{H} \to -\bar{E}, \quad \bar{D} \to \bar{B}, \quad \bar{B} \to -\bar{D} \tag{5}$$

$$\bar{M} \to -\bar{J}, \quad \bar{J} \to \bar{M}, \quad \rho_m \to -\rho, \quad \rho \to \rho_m \tag{6}$$

The above swaps retain the right-hand rule for plane waves. When material media is included, such that $\bar{D} = \bar{\bar{\epsilon}} \cdot \bar{E}$, $\bar{B} = \bar{\bar{\mu}} \cdot \bar{H}$, for anisotropic media, M.E's become

$$\nabla \times \bar{E} = -j\omega \bar{\bar{\mu}} \cdot \bar{H} - \bar{M} \tag{7}$$

$$\nabla \times \bar{H} = j\omega \bar{\bar{\epsilon}} \cdot \bar{E} + \bar{J} \tag{8}$$

$$\nabla \cdot \bar{\bar{\mu}} \cdot \bar{H} = \rho_m \tag{9}$$

$$\nabla \cdot \bar{\bar{\epsilon}} \cdot \bar{E} = \rho \tag{10}$$

In addition to the above swaps, one needs further to swap

$$\bar{\bar{\mu}} \to \bar{\bar{\epsilon}}, \quad \bar{\bar{\epsilon}} \to \bar{\bar{\mu}} \tag{11}$$

If one adopts swaps where the RH rule is seemingly not preserved, e.g.,

$$\bar{E} \to \bar{H}, \quad \bar{H} \to \bar{E}, \quad \bar{M} \to -\bar{J}, \quad \bar{J} \to -\bar{M}, \tag{12}$$

$$\rho_m \to -\rho, \quad \rho \to -\rho_m, \quad \bar{\bar{\mu}} \to -\bar{\bar{\epsilon}}, \quad \bar{\bar{\epsilon}} \to -\bar{\bar{\mu}} \tag{13}$$

The above swaps will leave M.E's invariant, but when applied to a plane wave, the RH rule seems violated. But in a plane wave $k = \pm\omega\sqrt{\mu\epsilon}$, and one can always choose a root where the RH rule is retained.

Reflection and Transmission — Single Interface

TE Polarization (Perpendicular, E Polarization)

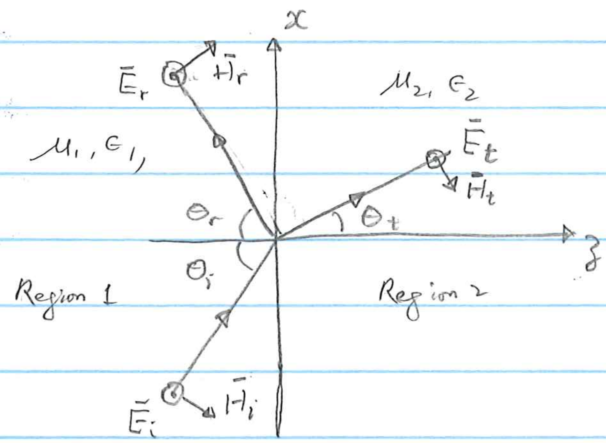

Fig. 1

To set up the above problem, the wave in Region 1 can be written as $\bar{E}_i + \bar{E}_r$. We assume plane wave polarized in the y direction where $\bar{\beta}_i = \hat{x}\beta_{ix} + \hat{z}\beta_{iz}$, $\bar{\beta}_r = \hat{x}\beta_{rx} - \hat{z}\beta_{rz}$, $\bar{\beta}_t = \hat{x}\beta_{tx} + \hat{z}\beta_{tz}$. Then

$$\bar{E}_i = \hat{y}\, E_0\, e^{-j\bar{\beta}_i\cdot\bar{r}} = \hat{y}\, E_0\, e^{-j\beta_{ix}x - j\beta_{iz}z} \tag{1}$$

$$\bar{E}_r = \hat{y}\, R^{TE} E_0\, e^{-j\bar{\beta}_r\cdot\bar{r}} = \hat{y}\, R^{TE} E_0\, e^{-j\beta_{rx}x - j\beta_{rz}z} \tag{2}$$

In Region 2, we only have transmitted wave; hence

$$\bar{E}_t = \hat{y}\, T^{TE} E_0\, e^{-j\bar{\beta}_t\cdot\bar{r}} = \hat{y}\, T^{TE} E_0\, e^{-j\beta_{tx}x - j\beta_{tz}z} \tag{3}$$

In the above, R^{TE} and T^{TE} are unknowns yet to be sought. To find them, we need two boundary conditions to yield two equations. They are $\hat{n}\times\bar{E}$ continuous and $\hat{n}\times\bar{H}$ continuous conditions at the interface. Imposing $\hat{n}\times\bar{E}$ continuous at $z=0$, we get

$$E_0\, e^{-j\beta_{ix}x} + R^{TE} E_0\, e^{-j\beta_{rx}x} = T^{TE} E_0\, e^{-j\beta_{tx}x}, \ \forall x \tag{4}$$

In order for the above to be valid for all x, it is necessary that $\beta_{ix} = \beta_{rx} = \beta_{tx}$, which is also known as the phase matching condition. Thus, the above simplifies to

$$1 + R^{TE} = T^{TE}. \tag{5}$$

To impose $\hat{n}\times\bar{H}$ continuous, one need to find the $\bar{H}$ field using $\nabla\times\bar{E} = -j\omega\mu\bar{H}$, or that $\bar{H} = -j\bar{\beta}\times\bar{E}/(-j\omega\mu)$ By so doing

$$\bar{H}_i = \frac{\bar{\beta}_i\times\bar{E}_i}{\omega\mu_1} = \frac{\bar{\beta}_i\times\hat{y}}{\omega\mu_1}E_0 e^{-j\bar{\beta}_i\cdot\bar{r}} = \frac{+\hat{z}\beta_{ix}-\hat{x}\beta_{iz}}{\omega\mu_1}E_0 e^{-j\bar{\beta}_i\cdot\bar{r}} \tag{6}$$

$$\bar{H}_r = \frac{\bar{\beta}_r\times\bar{E}_r}{\omega\mu_1} = \frac{\bar{\beta}_r\times\hat{y}}{\omega\mu_1}R^{TE}E_0 e^{-j\bar{\beta}_r\cdot\bar{r}} = \frac{+\hat{z}\beta_{rx}+\hat{x}\beta_{rz}}{\omega\mu_2}R^{TE}E_0 e^{-j\bar{\beta}_r\cdot\bar{r}} \tag{7}$$

$$\bar{H}_t = \frac{\bar{\beta}_t\times\bar{E}_t}{\omega\mu_2} = \frac{\bar{\beta}_t\times\hat{y}}{\omega\mu_2}T^{TE}E_0 e^{-j\bar{\beta}_t\cdot\bar{r}} = \frac{+\hat{z}\beta_{tx}-\hat{x}\beta_{tz}}{\omega\mu_2}T^{TE}E_0 e^{-j\bar{\beta}_{tx}\cdot\bar{r}} \tag{8}$$

Imposing $\hat{n}\times\bar{H}$ continuous or H_x continuous, we have

$$\frac{\beta_{iz}}{\omega\mu_1}E_0 e^{-j\beta_{ix}x} - \frac{\beta_{rz}}{\omega\mu_1}R^{TE}E_0 e^{-j\beta_{rx}x} = \frac{\beta_{tz}}{\omega\mu_2}T^{TE}E_0 e^{-j\beta_{tx}x} \tag{9}$$

The phase-matching condition requires that $\beta_{ix} = \beta_{rx} = \beta_{tx}$.
The dispersion relation for plane waves requires that

$$\beta_{ix}^2 + \beta_{iz}^2 = \beta_{rx}^2 + \beta_{rz}^2 = \omega^2\mu_1\epsilon_1 = \beta_1^2 \tag{10}$$

Since $\beta_{ix} = \beta_{rx} = \beta_{tx} = \beta_x$, it implies ~~that~~ $\beta_{iz} = \beta_{rz}$. Then (9) simplifies to

$$\frac{\beta_{1z}}{\mu_1}\left(1 - R^{TE}\right) = \frac{\beta_{2z}}{\mu_2}T^{TE} \tag{11}$$

where $\beta_{1z} = \sqrt{\beta_1^2 - \beta_x^2}$, $\beta_{2z} = \sqrt{\beta_2^2 - \beta_x^2}$. Solving (5) and (11) yields

$$R^{TE} = \left(\frac{\beta_{1z}}{\mu_1} - \frac{\beta_{2z}}{\mu_2}\right)\Big/\left(\frac{\beta_{1z}}{\mu_1} + \frac{\beta_{2z}}{\mu_2}\right) \tag{12}$$

$$T^{TE} = 2\left(\frac{\beta_{1z}}{\mu_1}\right)\Big/\left(\frac{\beta_{1z}}{\mu_1} + \frac{\beta_{2z}}{\mu_2}\right) \tag{13}$$

TM Polarization (Parallel, H Polarization)

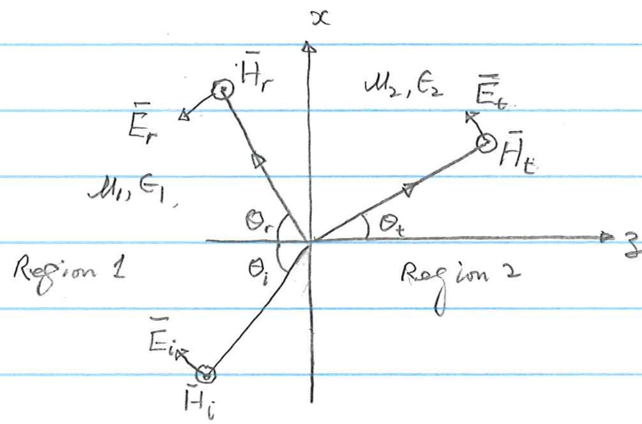

Fig. 2.

The solution to the TM polarization case can be obtained by invoking duality principle where we do the substitution $\bar{E} \to \bar{H}$, $\bar{H} \to -\bar{E}$, $\mu \to \epsilon$, $\epsilon \to \mu$. The picture for the TM polarized case is shown in Fig. 2. The reflection coefficient for the TM magnetic field is

$$R^{TM} = \left(\frac{\beta_{1z}}{\epsilon_1} - \frac{\beta_{2z}}{\epsilon_2}\right) \Big/ \left(\frac{\beta_{1z}}{\epsilon_1} + \frac{\beta_{2z}}{\epsilon_2}\right) \qquad (14)$$

$$T^{TM} = 2\left(\frac{\beta_{1z}}{\epsilon_2}\right) \Big/ \left(\frac{\beta_{1z}}{\epsilon_1} + \frac{\beta_{2z}}{\epsilon_2}\right) \qquad (15)$$

Three interesting physical phenomena emerge from the solutions of the single-interface problem. They are total-internal reflection, Brewster angle effect, and surface plasmonic resonances.

Total Internal Reflection

Total internal reflection comes about because of phase matching also called momentum matching. It turns out that because of phase matching, for certain interfaces, k_{2z} becomes pure imaginary.

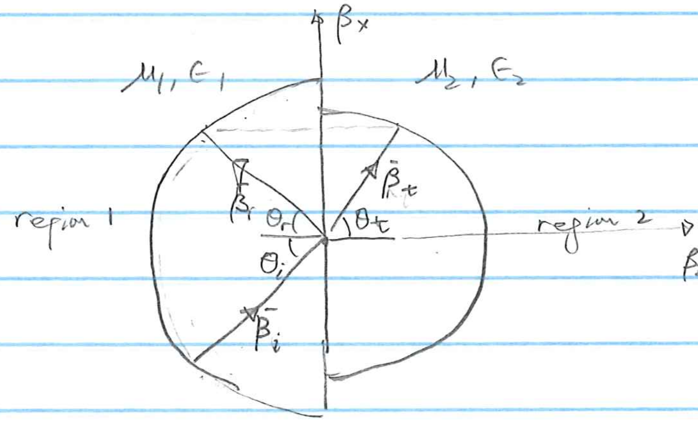

Fig. 3

As shown in Fig. 3, because of the dispersion relation that $\beta_{rx}^2 + \beta_{rz}^2 = \beta_{ix}^2 + \beta_{iz}^2 = \beta_1^2$, $\beta_{tx}^2 + \beta_{tz}^2 = \beta_2^2$, the tip of the $\bar{\beta}$ vectors for regions 1 and 2 have to be on a spherical surface in the $\beta_x, \beta_y, \beta_z$ space. Phase matching implies that the x-component of the $\bar{\beta}$ vectors are equal to each other as shown. One sees that $\theta_i = \theta_r$ in Fig. 3, and also as θ_i increases, θ_t increases. For an optically less dense medium where $\beta_2 < \beta_1$, $\bar{\beta}_t$ becomes parallel to the x axis when $\beta_{ix} = \beta_{rx} = \beta_2 = \omega\sqrt{\mu_2 \epsilon_2}$. The angle at which this happens is the critical angle θ_c. Since $\beta_{ix} = \beta_1 \sin\theta_i = \beta_{rx} = \beta_1 \sin\theta_r = \beta_2$, or

$$\sin\theta_r = \sin\theta_i = \sin\theta_c = \frac{\beta_2}{\beta_1} = \frac{\sqrt{\mu_2\epsilon_2}}{\sqrt{\mu_1\epsilon_1}} = \frac{n_2}{n_1} \qquad (16)$$

where n_i is the refractive index defined as c_0/v_i, where v_i is the phase velocity of the wave in Region i. Hence,

$$\theta_c = \sin^{-1}(n_2/n_1) \qquad (17).$$

When $\theta_i > \theta_c$, $\beta_x > \beta_2$ and $\beta_{2z} = \sqrt{\beta_2^2 - \beta_x^2}$ becomes pure imaginary. When β_{2z} becomes pure imaginary, the wave cannot propagate in region 2, or $\beta_{2z} = -j\alpha_{2z}$.

The reflection coefficient (12) becomes of the form

$$R^{TE} = (A - jB)/(A + jB) \quad (18)$$

It is clear that $|R^{TE}| = 1$ and that $R^{TE} = e^{j\theta_{TE}}$.

A phase shift corresponds to a time delay in the time domain. Such a time-delay is achieved by the wave travelling laterally in Medium 2 before being refracted back to Medium 1

TIR is also how optical fiber works as well.

Brewster Angle

Since most materials in optics have $\epsilon_2 \neq \epsilon_1$, but $\mu_2 \approx \mu_1$, then TM polarization for light behaves differently from TE polarization. For R^{TM}, it is possible that $R^{TM} = 0$ if

$$\epsilon_2 \beta_{1z} = \epsilon_1 \beta_{2z} \quad (19)$$

Squaring the above, assuming $\mu_1 = \mu_2$, one gets

$$\epsilon_2^2(\beta_1^2 - \beta_x^2) = \epsilon_1^2(\beta_2^2 - \beta_x^2) \quad (20)$$

Solving gives

$$\beta_x = \omega\sqrt{\mu}\sqrt{\frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2}} = \beta_1 \sin\theta_1 = \beta_2 \sin\theta_2 \quad (21)$$

Therefore, $\sin\theta_1 = \sqrt{\dfrac{\epsilon_2}{\epsilon_1+\epsilon_2}}$, $\sin\theta_2 = \sqrt{\dfrac{\epsilon_1}{\epsilon_1+\epsilon_2}}$ (22)

or that $\sin^2\theta_1 + \sin^2\theta_2 = 1,$ (23)

Then

$$\sin\theta_2 = \cos\theta_1 \quad (24)$$

or that

$$\theta_1 + \theta_2 = \pi/2 \quad (25)$$

This is used to explain why at Brewster, no light is reflected back to Medium 1.

Because of the Brewster angle effect, and that $\epsilon_2 \neq \epsilon_1$, $|R^{TM}| \leq |R^{TE}|$ as shown in Fig. 4. This is phenomenon is used to

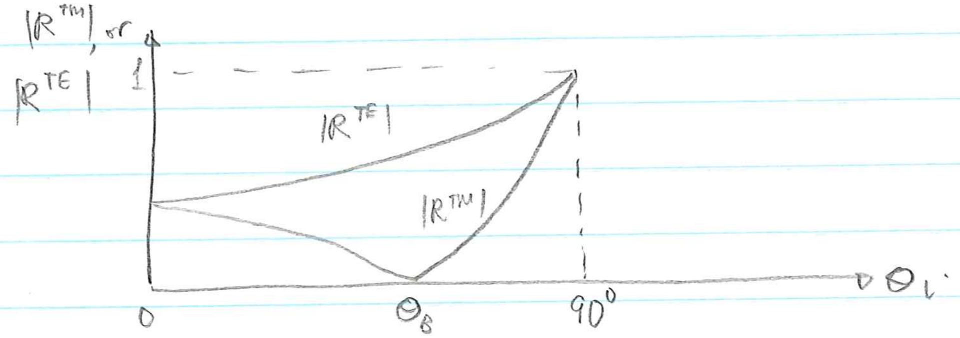

Fig. 4

design sun glasses to reduce road glare.

Surface Plasmon Polariton

Surface plasmon polariton occurs for the same mathematical reason for the Brewster angle effect. The reflection coefficient R^{TM} can become infinite if, say, $\epsilon_2 < 0$, as in a plasma medium. In this case, the criterion is

$$-\epsilon_2 \beta_{1z} = \epsilon_1 \beta_{2z} \tag{1}$$

Solving after squaring yields

$$\beta_x = \omega\sqrt{\mu}\sqrt{\frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2}} \tag{2}$$

Even if $\epsilon_2 < 0$, $\epsilon_1 + \epsilon_2 < 0$ is still possible and β_x is pure real. This corresponds to a guided wave propagating in the x direction. In this case

$$\beta_{1z} = \sqrt{\beta_1^2 - \beta_x^2} = \omega\sqrt{\mu}\left(\epsilon_1\left(1 - \frac{\epsilon_2}{\epsilon_1 + \epsilon_2}\right)\right)^{1/2}$$

Since $\epsilon_2 < 0$, $\epsilon_2/(\epsilon_1 + \epsilon_2) > 1$, and β_{1z} is pure imaginary. $\beta_{2z} = \sqrt{\beta_2^2 - \beta_x^2}$ and $\beta_2^2 < 0$ making β_{2z} also pure imaginary. This corresponds to a trapped wave at the interface. The wave decays exponentially in both directions away from the interface.