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Phase Velocity and Group Velocity

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Now that we know how a medium can be frequency dispersive in the Drude-Lorentz-Sommerfeld (DLS) model, we are ready to distinguish the difference between the phase velocity and the group velocity.

Phase Velocity

The phase velocity is the velocity of the phase of a wave. It is only defined for a mono-chromatic, time-harmonic, CW signal at one given frequency. A sinusoidal wave signal, e.g., the voltage signal on a transmission line, can take the form

$$V(z,t) = V_0 \cos(\omega t - kz + \alpha) \qquad (1)$$

This sinusoidal signal moves with a velocity $v_{ph} = \frac{\omega}{k}$, where $k = \omega\sqrt{\mu\epsilon}$, inside a simple coax, for example. Hence, $v_{ph} = 1/\sqrt{\mu\epsilon}$. But a dielectric medium can be frequency dispersive, or $\epsilon(\omega)$ is not a constant but a function of ω. Therefore, signals with different ω will travel with different velocity. More bizzare still, what if the coax is filled with a plasma medium where

$$\epsilon = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right) \qquad (2)$$

Then, $\epsilon < \epsilon_0$ meaning that the phase velocity can be larger than the speed of light in vacuum if $\mu = \mu_0$. Also, if $\epsilon = 0$ when $\omega = \omega_p$, then $v_{ph} = \infty$. These ludicrous observations can be justified if we can show that information can only be send by a wave packet. The same goes for energy. These wave packets can only travel at the group velocity, which is always less than the speed of light.

Group Velocity

Now, consider a narrow band wave packet as shown.

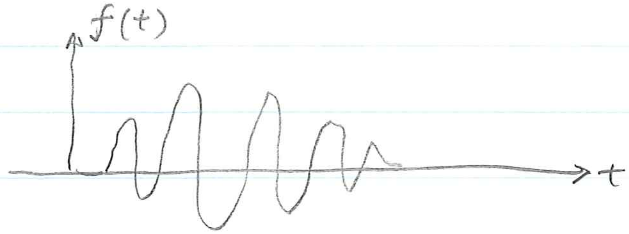

Fig. 1

It cannot be mono-chromatic, but can be written as a linear superposition of many frequencies. One way to express this is to write this wave packet as an integral summation over many frequencies, namely

$$V(z,t) = \int_{-\infty}^{\infty} d\omega\, V(z,\omega)\, e^{j\omega t} = \int_{-\infty}^{\infty} d\omega\, V_0(\omega)\, e^{j(\omega t - kz)} \qquad (3)$$

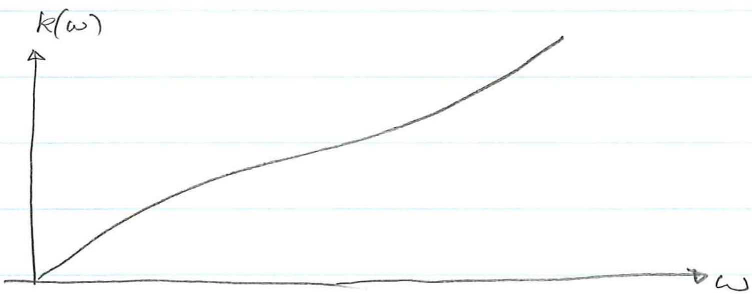

Fig. 2.

It is clear that each Fourier component is a solution to the one-D Helmholtz equation

$$\frac{d^2}{dz^2} V(z,\omega) + k^2 V(z,\omega) = 0 \qquad (4)$$

Since this is a wave packet, we assume that $V_0(\omega)$ is narrow band centered about a frequency ω_0, the carrier frequency.

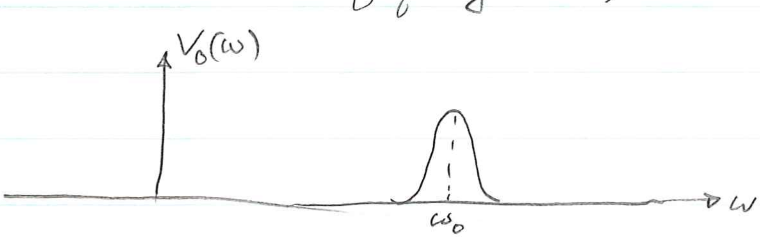

Fig. 3

Thus, we can approximate the integrand, and set

$$k(\omega) \cong k(\omega_0) + (\omega-\omega_0)\frac{dk(\omega_0)}{d\omega} + \frac{1}{2}(\omega-\omega_0)^2\frac{d^2k(\omega_0)}{d\omega^2} + \cdots \qquad (5)$$

To ensure the real-valuedness of (3), one can ensure that the $-\omega$ part of the integrand is exactly the complex conjugate of the $+\omega$ part. Another way is to sum over only the $+\omega$ part and take real part of the integral. So, for simplicity, we write (3) as

$$V(z,t) = 2\,\mathrm{Re}\int_0^\infty d\omega\, V_0(\omega)\, e^{j(\omega t - kz)} \qquad (6).$$

Now, we can substitute (5) into (6) and rewrite it as

$$V(z,t) \cong 2\,\mathrm{Re}\Big[e^{j[\omega_0 t - k(\omega_0)z]} \underbrace{\int_0^\infty d\omega\, V_0(\omega)\, e^{j(\omega-\omega_0)t}\, e^{-j(\omega-\omega_0)\frac{dk}{d\omega}z}}_{F(t-\frac{dk}{d\omega}z)}\Big] \qquad (7)$$

It can be seen that the inner integrand now involves the integral summation over a few ω in the vicity of ω_0, and hence is a slowly varying function. Moreover, this function F moves with a velocity

$$v_g = \frac{d\omega}{dk} \qquad (8).$$

$F(t - \frac{z}{v_g})$ in fact is the velocity of the envelop in Fig. 1. The envelop function $F(t-\frac{z}{v_g})$ is multiplied by the rapidly varying function

$$e^{j[\omega_0 t - k(\omega_0) z]} \qquad (9)$$

Hence, this rapidly varying part represents the rapidly varying carrier frequency shown in Fig. 1. More importantly, this carrier moves with the velocity

$$v_{ph} = \frac{\omega_0}{k(\omega_0)} \qquad (10)$$

which is its phase velocity.

Energy Density in Dispersive Media

A dispersive medium alters our understanding of phase and group velocity, but it also changes our concept of what energy density is. To this end, we assume that the field has complex ω dependence rather than real ω dependence of $e^{j\omega t}$, where $\omega = \omega' - j\omega''$. We take the divergence of the complex power for fields with such time dependence, and let $e^{j\omega t}$ be attached to the field. So $\bar{E}(t)$ and $\bar{H}(t)$ are complex field but not exactly like a phasors. Hence,

$$\nabla \cdot \left[\bar{E}(t) \times \bar{H}^*(t)\right] = \bar{H}^*(t) \cdot \nabla \times \bar{E}(t) - \bar{E}(t) \cdot \nabla \times \bar{H}^*(t)$$

$$= -\bar{H}^*(t) \cdot j\omega\mu \bar{H}(t) + \bar{E}(t) \cdot j\omega^* \epsilon^* \bar{E}^*(t) \qquad (1)$$

The space dependence of the field is understood, and we assume a source-free medium so that $J = 0$. If $\bar{E}(t) \sim e^{j\omega t}$, then $\bar{H}^*(t) \sim e^{-j\omega^* t}$, and the term like

$$\bar{E}(t) \times \bar{H}^*(t) \sim e^{j(\omega - \omega^*)t} = e^{+2\omega'' t} \qquad (2),$$

and each of the term above will have similar time dependence. Writing (1) more explicitly, we have

$$\nabla \cdot \left[\bar{E}(t) \times \bar{H}^*(t)\right] = -j(\omega' - j\omega'')\mu(\omega' - j\omega'')|\bar{H}(t)|^2 + j(\omega' + j\omega'')\epsilon^*(\omega' - j\omega'')|\bar{E}(t)|^2 \qquad (3)$$

Assuming that $\omega'' \ll \omega'$, we can let

$$\mu(\omega' - j\omega'') = \mu(\omega') - j\omega'' \frac{\partial \mu(\omega')}{\partial \omega'} \qquad (4)$$

$$\epsilon(\omega' - j\omega'') = \epsilon(\omega') - j\omega'' \frac{\partial \epsilon(\omega')}{\partial \omega'} \qquad (5)$$

Using (4) and (5) in (3), and collecting terms of the same order gives

$$\begin{aligned}\nabla \cdot \left[\bar{E}(t) \times \bar{H}^*(t)\right] = & -j\omega' \mu(\omega')|\bar{H}(t)|^2 + j\omega' \epsilon^*(\omega')|\bar{E}(t)|^2 \\ & - \omega'' \mu(\omega')|\bar{H}(t)|^2 - \omega'\omega'' \frac{\partial \mu}{\partial \omega'}|\bar{H}(t)|^2 \\ & - \omega'' \epsilon^*(\omega)|\bar{E}(t)|^2 - \omega'\omega'' \frac{\partial \epsilon^*}{\partial \omega'}|\bar{E}(t)|^2 \end{aligned} \qquad (6)$$

The above can be rewritten as

$$\begin{aligned}\nabla \cdot \left[\bar{E}(t) \times \bar{H}^*(t)\right] = & -j\omega' \left[\mu(\omega')|\bar{H}(t)|^2 + \epsilon^*(\omega')|\bar{E}(t)|^2\right] \\ & - \omega'' \left[\frac{\partial \omega' \mu(\omega')}{\partial \omega'}|\bar{H}(t)|^2 + \frac{\partial \omega' \epsilon^*(\omega')}{\partial \omega'}|\bar{E}(t)|^2\right] \end{aligned} \qquad (7)$$

For a lossless medium, $\epsilon(\omega')$ is purely real, and the first term on the RHS is purely imaginary while the second term is purely real. In the limit when $\omega'' \to 0$, when we take half the imaginary part of the above equation, we have

$$\nabla \cdot \frac{1}{2} \operatorname{Im}\left[\bar{E} \times \bar{H}^*\right] = -\omega' \left[\frac{1}{2}\mu|\bar{H}|^2 - \frac{1}{2}\epsilon|\bar{E}|^2\right] \qquad (8)$$

which has the physical interpretation of reactive power. When we take half the real part of (7), we obtain

$$\nabla \cdot \frac{1}{2} \operatorname{Re}\left[E \times \bar{H}^*\right] = -\frac{\omega''}{2}\left[\frac{\partial \omega' \mu}{\partial \omega'}|\bar{H}|^2 + \frac{\partial \omega' \epsilon}{\partial \omega'}|\bar{E}|^2\right] \qquad (9)$$

Since the RHS has time dependence of $e^{2\omega'' t}$, it can be written as

$$\nabla \cdot \frac{1}{2} \operatorname{Re}\left[\bar{E} \times \bar{H}^*\right] = -\frac{\partial}{\partial t} \frac{1}{4}\left[\frac{\partial \omega' \mu}{\partial \omega'}|\bar{H}|^2 + \frac{\partial \omega' \epsilon}{\partial \omega'}|\bar{E}|^2\right] = -\frac{\partial}{\partial t}\langle W_T \rangle$$

Therefore, the time-average stored energy density is

$$\langle W_T \rangle = \frac{1}{4}\left[\frac{\partial \omega' \mu}{\partial \omega'}|\bar{H}|^2 + \frac{\partial \omega' \epsilon}{\partial \omega'}|\bar{E}|^2\right] \qquad (10)$$

For a non-dispersive medium, the above reduces to

$$\langle W_T \rangle = \frac{1}{4}\left[\mu|\bar{H}|^2 + \epsilon|\bar{E}|^2\right] \qquad (11)$$