Lecture 15, Oct 23, 2018

Drude-Lorentz-Sommerfeld Model   W.C. Chew

We have seen how loss can be introduced by having a conduction current flowing in a medium. Now that we have learnt the frequency domain method, other loss mechanism can be introduced.

First let us look at the constitutive relation where

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P} \tag{1}$$

We have a simple model where

$$\bar{P} = \epsilon_0 \chi_e \bar{E} \tag{2}$$

where χ_e is the electric susceptibility. When used in the generalized Ampere's law, $\bar{P}$, the polarization density, plays an important role for the flow of the displacement current. We can think of displacement current flow as capacitive coupling of polarization current flow through space. Namely,

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} = \epsilon_0 \frac{\partial \bar{E}}{\partial t} + \frac{\partial \bar{P}}{\partial t} \tag{3}$$

for a source-free medium.

Fig. 1.

For example, for a sinusoidal oscillating field, the dipoles will flip back and forth giving rise to flow of displacement current just as how electric current can flow through a capacitor.

The relationship between $\bar{P}$ and $\bar{E}$ can be written more generally as

$$\bar{P} = \epsilon_0 \bar{\chi}_e(\bar{E}) \tag{4}$$

where the relationship can appear in many different forms. For nonlinear media, the relationship can be non-linear. Eq (2), however, represents a linear medium. This linear relationship can be generalized to that of a linear time-invariant system, or that at any given $\bar{r}$,

$$\bar{P}(t) = \epsilon_0 \chi_e(t) \circledast \bar{E}(t) \qquad (5)$$

where $\circledast$ here implies a convolution. In the frequency domain or the Fourier space, the above relationship becomes

$$\bar{P}(\omega) = \epsilon_0 \chi_e(\omega) \bar{E}(\omega), \quad \bar{D} = \epsilon_0 (1 + \chi_e(\omega)) \bar{E} = \epsilon(\omega) \bar{E} \qquad (6)$$

at any point $\bar{r}$ in space. There is a rich variety of ways at which $\chi_e(\omega)$ can manifest itself.

To see how $\chi_e(\omega)$ can be derived, we will study the Drude-Lorentz-Sommerfeld model. This is usually just known as the Drude model in many textbooks although Lorentz and Sommerfeld also contributed to it. We can start with a simple electron driven by an electric field $\bar{E}$. If the electron is free to move, then the force acting on it is $-e\bar{E}$. Then from Newton's law, it follows that

$$m_e \frac{d^2 x}{dt^2} = -eE \qquad (7)$$

assuming that $\bar{E}$ points in the x-direction, and we neglect the vector nature of the electric field. Writing the above in the frequency domain, one gets

$$-\omega^2 m_e x = -eE \qquad (8)$$

From this, one gets

$$x = \frac{e}{\omega^2 m_e} E \qquad (9)$$

This for instance, can happen in a plasma medium where the atoms are ionized, and the electrons are free to roam. Here, we assume that the positive ions, we are more massive, move very little compared to the electrons when an electric field is applied.

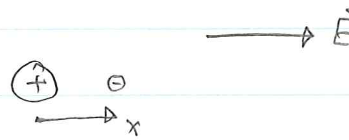

Fig. 2.

The dipole moment formed by the displaced electron is

$$p = -ex = -\frac{e^2}{\omega^2 m_e} E \qquad (10)$$

for one electron. When there are N electrons per unit volume, the dipole density is given by

$$P = Np = \frac{-Ne^2}{\omega^2 m_e} E \qquad (11)$$

In general,

$$\bar{P} = -\frac{Ne^2}{\omega^2 m_e} \bar{E} = -\frac{\omega_p^2 \epsilon_0}{\omega^2} \bar{E} \qquad (12)$$

where we have defined $\omega_p^2 = Ne^2/(m_e \epsilon_0)$. Then,

$$\bar{D} = \epsilon_0 \bar{E} + \bar{P} = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \bar{E} \qquad (13)$$

In this manner, we see that

$$\epsilon = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \qquad (14)$$

This gives the interesting result that in the frequency domain $\epsilon < 0$ if $\omega < \omega_p = \sqrt{N/(m_e \epsilon_0)}\, e$. Here, ω_p is the plasma frequency. Since, $k = \omega\sqrt{\mu\epsilon}$, if ϵ is negative, k becomes pure imaginary, and a wave such as e^{jkz} decays exponentially.

The above model can be generalized to the case where the electron is bound to the ion, and the ion provides a restoring force, namely,

$$m_e \frac{d^2x}{dt^2} + kx = -eE \qquad (15).$$

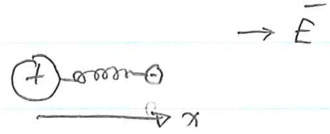

Fig. 3.

We assume that the ion provides a restoring force just like Hooke's law. Again, (15) can be solved easily in the frequency domain, and yield that

$$x = \frac{e}{(\omega^2 - \omega_0^2) m_e} \cdot E \qquad (16)$$

where $\omega_0^2 m_e = k$. Equation (15) can be generalized to when frictional forces are involved, or that

$$m_e \frac{d^2x}{dt^2} + m_e \Gamma \frac{dx}{dt} + kx = -eE \qquad (17)$$

The second term is a force that is proportional to the velocity dx/dt of the electron. This is the hall-mark of a frictional force. Writing the above in the frequency domain, one gets

$$x = \frac{e}{(\omega^2 - j\omega\Gamma - \omega_0^2) m_e} E \qquad (18)$$

Following the same procedure in arriving at (11), we get

$$P = \frac{-Ne^2}{(\omega^2 - j\omega\Gamma - \omega_0^2) m_e} E \qquad (19)$$

In this, one can identify

$$\chi_e(\omega) = -\frac{Ne^2}{(\omega^2 - j\omega\Gamma - \omega_0^2) m_e \epsilon_0}$$

$$= -\frac{\omega_p^2}{\omega^2 - j\omega\Gamma - \omega_0^2} \qquad (20)$$

where we have used ω_p as defined before. If $\Gamma = 0$, then when $\omega = \omega_0$, one sees a resonance peak exhibited by the DLS model. When Γ is small, but $\omega \approx \omega_0$, then

$$\chi_e \approx +\frac{\omega_p^2}{j\omega\Gamma} = -j\frac{\omega_p^2}{\omega\Gamma} \qquad (21)$$

χ_e exhibits a large negative imaginary part, the hallmark of a dissipative medium.

The DLS model is a wonderful model because it can capture the essence of the physics of many electromagnetic media. It can capture the resonance behavior of an atom absorbing energy from light excitation. When the light wave comes in at the correct frequency, it will excite electronic transition within an atom which can be approximately model as a resonance behavior. This electronic resonances will be radiationally damped, and the damped oscillation can be modeled by $\Gamma \neq 0$. (Molecular vibration)

In the case of plasma, $\Gamma \neq 0$ can represents the collision between the free electrons and the ions, giving rise to loss. Also, if there is no restoring force so that $\omega_0 = 0$, and for sufficiently low frequency, from (20)

$$\chi_e = -j\frac{\omega_p^2}{\omega\Gamma} \qquad (22)$$

and

$$\epsilon = \epsilon_0 (1 + \chi_e) = \epsilon_0 \left(1 - j\frac{\omega_p^2}{\omega\Gamma}\right) \tag{23}$$

We recall that for a conductive medium, we define a complex permittivity to be

$$\epsilon = \epsilon_0 \left(1 - j\frac{\sigma}{\omega\epsilon_0}\right) \tag{24}$$

Comparing (23) and (24), we see that

$$\sigma = \epsilon_0 \frac{\omega_p^2}{\Gamma} \tag{25}$$

Because the DLS is so powerful, it can be used to explain a wide range of phenomena from very low frequency to optical frequency. The fact that $\epsilon < 0$ can be used to explain many phenomena.

The ionosphere is essentially a plasma medium, described by

$$\epsilon = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \tag{26}$$

Radio wave or microwave can only penetrate through the ionosphere when $\omega > \omega_p$, so that $\epsilon > 0$.

Even at optical frequency, many metals can be modeled approximately as a plasma. A metal consists of a sea of electrons which are not tightly bound to the ions or the lattice. In this case $\omega_0 \ll \omega$ in (20), and if Γ is small, $\chi_e(\omega)$ resembles that of a plasma, and it and ϵ can be negative. When a plasmonic nano-particle made of gold is excited by light, its response is given by

$$\Phi_r = E_0 \frac{a^3 \cos\theta}{r^2} \frac{\epsilon_s - \epsilon_0}{\epsilon_s + 2\epsilon_0} \tag{27}$$

When $\epsilon_s = -2\epsilon_0$, $\Phi_r \to \infty$.