Lecture 14, Oct 18, 2018 W. C. Chew

Multi-Junction Transmission Lines

By concatenating sections of transmission lines of different characteristic impedances, a large variety of devices such as resonators, filters, and matching networks can be formed. We will start with a single junction transmission line.

Single - Junction Transmission Line

Consider two transmission lines connected at a single junction as shown in Fig. 1. For simplicity, we assume that the transmission line to the right is infinitely long so that there is no reflected wave. And that the two transmission lines have different characteristic impedances, Z_{01} and Z_{02}

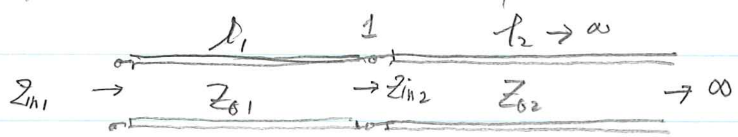

Fig. 1

The impedance of the transmission line at 1 looking to the right is

$$Z_{in2} = Z_{02} \tag{1}$$

since no reflected wave exists. The reflection coefficient at 1 between line 1 and line 2 is Γ_{12}, and is given by

$$\Gamma_{12} = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} \tag{2}$$

The reflection is due to the missmatch of the characteristic impedances of the two lines.

Now, we look at when line 2 is terminated by a load Z_L as shown in Fig. 2

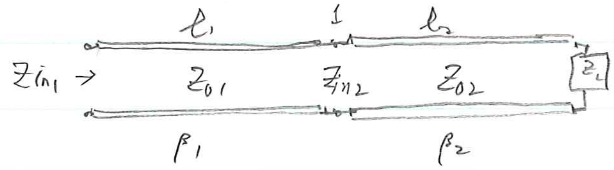

Fig. 2.

Then,

$$Z_{in_2} = Z_{02} \frac{1+\Gamma(-l_2)}{1-\Gamma(-l_2)} = Z_{02} \frac{1+\Gamma_L e^{2j\beta_2 l_2}}{1-\Gamma_L e^{2j\beta_2 l_2}} \quad (3).$$

The (generalized) reflection coefficient at junction 1 is now

$$\tilde{\Gamma}_{12} = \frac{Z_{in_2} - Z_{01}}{Z_{in_2} + Z_{01}} \quad (4)$$

Substituting (2) into (1), we have

$$\tilde{\Gamma}_{12} = \frac{Z_{02}\left(\frac{1+\Gamma}{1-\Gamma}\right) - Z_{01}}{Z_{02}\left(\frac{1+\Gamma}{1-\Gamma}\right) + Z_{01}} \quad (5)$$

where $\Gamma = \Gamma_L e^{2j\beta_2 l_2}$. The above can be rearranged to give

$$\tilde{\Gamma}_{12} = \frac{Z_{02}(1+\Gamma) - Z_{01}(1-\Gamma)}{Z_{02}(1+\Gamma) + Z_{01}(1-\Gamma)} \quad (6)$$

Finally, it can be shown that

$$\tilde{\Gamma}_{12} = \frac{\Gamma_{12} + \Gamma}{1 + \Gamma_{12}\Gamma} = \frac{\Gamma_{12} + \Gamma_L e^{2j\beta_2 l_2}}{1 + \Gamma_{12}\Gamma_L e^{2j\beta_2 l_2}} \quad (7)$$

where Γ_{12}, the local reflection coefficient, is given by (2), and $\Gamma = \Gamma_L e^{2j\beta_2 l_2}$ is the reflection coefficient due to the load Z_L. Eq. (6) is a powerful formula for multi-junction transmission line. Imagine now that we add another section of transmission line as shown in Fig. 3.

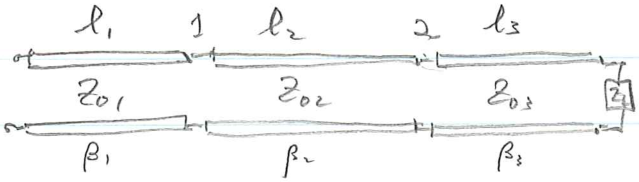

Fig 3

We can use the aforementioned method to first find $\tilde{\Gamma}_{23}$, the generalized reflection coefficient at junction 2. Using formula (6), it is given by

$$\tilde{\Gamma}_{23} = \frac{\Gamma_{23} + \Gamma_L e^{2j\beta_3 l_3}}{1 + \Gamma_{23}\Gamma_L e^{2j\beta_3 l_3}} \tag{8}$$

Given the knowledge of $\tilde{\Gamma}_{23}$, we can use (7) again to find the new $\tilde{\Gamma}_{12}$ at 1. It is now

$$\tilde{\Gamma}_{12} = \frac{\Gamma_{12} + \tilde{\Gamma}_{23} e^{2j\beta_2 l_2}}{1 + \Gamma_{12}\tilde{\Gamma}_{23} e^{2j\beta l_2}} \tag{9}$$

Therefore, we can use (7) recursively to find the generalized reflection coefficient for a multi-junction transmission line. Once the reflection coefficient is known, the impedance at that location can also be found. For instance, at junction 1, the impedance is given by

$$Z_{in2} = Z_{02}\frac{1 + \tilde{\Gamma}_{12}}{1 - \tilde{\Gamma}_{12}} \tag{10}$$

instead of Eq. (3).

Uniform Plane Waves

The vector wave equation for a source-free, homogeneous medium is given by

$$\nabla \times \nabla \times \bar{E} - \omega^2 \mu \epsilon \bar{E} = 0 \qquad (1)$$

Taking the divergence of the above equation, we have

$$\nabla \cdot (\nabla \times \nabla \times \bar{E}) - \omega^2 \mu \epsilon \nabla \cdot \bar{E} = 0$$

Since the first term is zero because $\nabla \cdot (\nabla \times \bar{A}) = 0$, and if $\omega \neq 0$, then $\nabla \cdot \bar{E} = 0$. Hence, the solution to (1) is consistent with $\nabla \cdot \bar{E} = 0$. The general solution to (1) is

$$\bar{E} = \bar{E}_0 e^{-jk_x x - jk_y y - jk_z z} = \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}} \qquad (2)$$

where $\bar{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$, $\bar{r} = \hat{x}x + \hat{y}y + \hat{z}z$. This vector function represents a uniform plane wave propagating in the $\bar{k}$ direction. As can be seen, when $\bar{k}\cdot\bar{r} = \text{constant}$, it is represented by all points of $\bar{r}$ that represents a plane

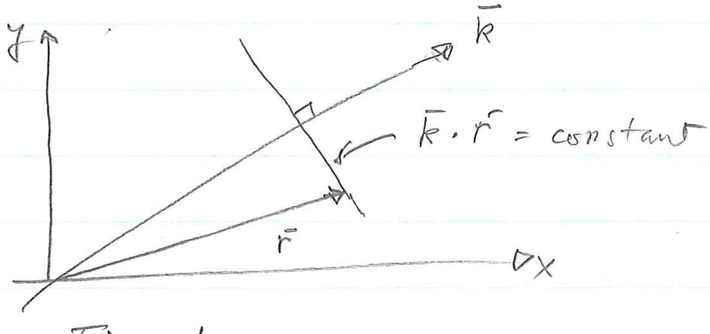

Fig. 1.

Further, since $\nabla \cdot \bar{E} = 0$, we have

$$\nabla \cdot \bar{E} = \nabla \cdot \bar{E}_0 e^{-jk_x x - jk_y y - jk_z z} = \nabla \cdot \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}}$$

$$= (-\hat{x}jk_x - \hat{y}jk_y - \hat{z}jk_z) \cdot \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}} = 0 \qquad (3)$$

or

$$\bar{k} \cdot \bar{E}_0 = \bar{k} \cdot \bar{E} = 0 \qquad (4)$$

In a source-free homogeneous medium,

$$\nabla \times \bar{E} = -j\omega\mu \bar{H} \qquad (5)$$

But $\bar{E}$ is a plane wave, $\nabla \to -j\bar{k}$, and the above equation is

$$-j\bar{k} \times \bar{E} = -j\omega\mu \bar{H} \qquad (6)$$

or that

$$\bar{H} = \frac{\bar{k} \times \bar{E}}{\omega\mu} \qquad (7)$$

Also, from

$$\nabla \times \bar{H} = j\omega\epsilon \bar{E} \qquad (8)$$

we get that

$$\bar{E} = -\frac{\bar{k} \times \bar{H}}{\omega\epsilon} = -\frac{\bar{k} \times \bar{k} \times \bar{E}}{\omega^2\mu\epsilon} \qquad (9)$$

But the above simplifies to

$$\bar{E} = -\frac{\bar{k}(\bar{k}\cdot\bar{E}) - (\bar{k}\cdot\bar{k})\bar{E}}{\omega^2\mu\epsilon} = \frac{\bar{k}\cdot\bar{k}}{\omega^2\mu\epsilon}\bar{E} \qquad (10)$$

But it can be shown that the dispersion relation

? $$\bar{k}\cdot\bar{k} = k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\epsilon \qquad (13)$$

So equation (10) is consistent.

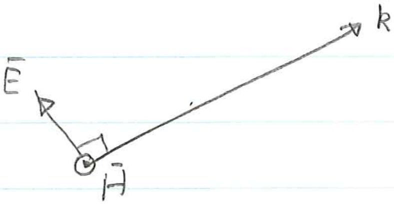

Fig. 2.

The above figure shows that $\bar{k}\cdot\bar{E} = 0$, and that $\bar{k}\times\bar{E}$ points in the direction of $\bar{H}$. The above figure also shows that

$$|\bar{H}| = \frac{|\bar{k}||\bar{E}|}{\omega\mu} = \sqrt{\frac{\epsilon}{\mu}}\,|E| \qquad (14)$$

The quantity

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \qquad (15)$$

is call the intrinsic impedance. For vacuum or free-space, it is about $377\,\Omega$.

Lossy Conductive Media

The above can be generalized to a lossy conductive medium by invoking mathematical homomorphism. When conductive loss is present, $\sigma \neq 0$, and $\bar{J} = \sigma \bar{E}$. Then Ampere's law becomes

$$\nabla \times \bar{H} = j\omega\epsilon \bar{E} + \sigma \bar{E} = j\omega\left(\epsilon + \frac{\sigma}{j\omega}\right)\bar{E} \quad (1)$$

A complex permittivity can be defined as $\underset{\sim}{\epsilon} = \epsilon - j\frac{\sigma}{\omega}$

Eq. (1) can be rewritten as

$$\nabla \times \bar{H} = j\omega \underset{\sim}{\epsilon} \bar{E} \quad (2)$$

A plane-wave solution $\bar{E} = \bar{E}_0 e^{-j\bar{k}\cdot\bar{r}}$ will have the dispersion relation is now

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \underset{\sim}{\epsilon} \quad (3)$$

Since $\underset{\sim}{\epsilon}$ is complex now, k_x, k_y, and k_z need not be all real.

Equation (14) is derived by assuming that $\bar{k}$ is a real vector. When $\bar{k}$ is a complex vector, the derivation that leads to (14) may not be correct. But we can look at the simplified case where

$$\bar{E} = \hat{x} E_x(z) \quad (4)$$

so that $\nabla \cdot \bar{E} = \partial_x E_x(z) = 0$. And let $\bar{k} = \hat{z} k = \hat{z}\omega\sqrt{\mu \underset{\sim}{\epsilon}}$.

Then (7) gives rise to

$$\bar{H} = -\hat{y}\frac{k E_x(z)}{\omega\mu} = -\hat{y}\sqrt{\frac{\underset{\sim}{\epsilon}}{\mu}}\, E_x \quad (5)$$

or

$$-E_x/H_y = \sqrt{\frac{\mu}{\underset{\sim}{\epsilon}}} \quad (6)$$

For such a simple plane wave,

$$\bar{E} = \hat{x} E_x(z) = \hat{x} E_0 e^{-jkz} \tag{7}$$

where $k = \omega\sqrt{\mu \underset{\sim}{\epsilon}}$, since $\bar{k}\cdot\bar{k} = k^2 = \omega^2 \mu \underset{\sim}{\epsilon}$ is still true. When the medium is highly conductive, $\sigma \to \infty$,

$$k = \omega\sqrt{\mu \underset{\sim}{\epsilon}} \simeq \omega\sqrt{-\mu \frac{j\sigma}{\omega}} = \sqrt{-j\omega\mu\sigma} \tag{8}$$

Taking $\sqrt{-j} = \frac{1}{\sqrt{2}}(1-j)$, we have

$$k = (1-j)\sqrt{\frac{\omega\mu\sigma}{2}} = k' - jk'' \tag{9}$$

For a plane wave, e^{-jkz}, it becomes

$$e^{-jkz} = e^{-jk'z - k''z} \tag{10}$$

This plane wave decays exponentially in the z direction. The penetration depth of this wave is

$$\delta = \frac{1}{k''} = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{11}$$

This distance δ is called the skin depth of a plane wave propagating in a highly lossy conductive medium. This happen for radio wave propagating in the saline solution of the ocean, the earth, or wave propagating in highly conductive metals.

When the conductivity is low, then, $\frac{\sigma}{\omega\epsilon} \ll 1$, we have

$$k = \omega\sqrt{\mu\left(\epsilon - j\frac{\sigma}{\omega}\right)} = \omega\sqrt{\mu\epsilon\left(1 - \frac{j\sigma}{\omega\epsilon}\right)}$$

$$\simeq \omega\sqrt{\mu\epsilon}\left(1 - \frac{1}{2}j\frac{\sigma}{\omega\epsilon}\right) = k' - jk'' \tag{12}$$

The term $\frac{\sigma}{\omega\epsilon}$ is called the loss tangent of a lossy medium.