Lecture 13, Oct 16, 2018, W.C. CHEW

Terminated Transmission Lines

For an infinitely long transmission line, the solution consists of the linear superposition of a wave travelling to the right plus a wave travelling to the left. If the transmission line is terminated by a load as shown in the figure, a right-travelling wave will be reflected by the load, and in general, the wave on the transmission line will be

$$V(z) = a_+ e^{-\gamma z} + a_- e^{\gamma z} = V_+(z) + \underbrace{V_-(z)}_{\text{left-going}} \quad (1)$$

At $z=0$, we can define the amplitude of the reflected wave a_- to be related to the amplitude of the right-going or incident wave. In other words, at $z=0$,

$$a_- = \Gamma_L a_+ \quad (2)$$

where Γ_L is the reflection coefficient. Hence,

$$V(z) = a_+ e^{-\gamma z} + \Gamma_L a_+ e^{\gamma z} = a_+ \left(e^{-\gamma z} + \Gamma_L e^{\gamma z}\right) \quad (3).$$

The corresponding current $I(z)$ on the transmission line is given by

$$I(z) = -\frac{1}{Z}\frac{dV}{dz} = \frac{a_+}{Z}\gamma\left(e^{-\gamma z} - \Gamma_L e^{\gamma z}\right) \quad (4)$$

where $\gamma = \sqrt{(j\omega L + R)(j\omega C + G)}$, and $Z = j\omega L + R$.

Hence, $Z/\gamma = \sqrt{Z/Y} = Z_0$, and

$$I(z) = \frac{a_+}{Z_0}\left(e^{-\gamma z} - \Gamma_L e^{\gamma z}\right) \quad (5)$$

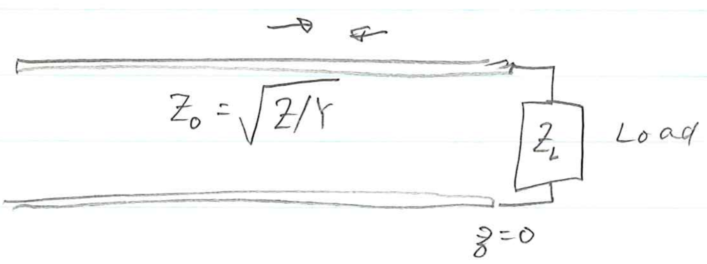


Fig. 1.

A generalized reflection coefficient $\overset{\text{at } z}{\smile}$ can be derived such that

$$\Gamma(z) = \frac{V_-(z)}{V_+(z)} = \frac{a_- e^{\gamma z}}{a_+ e^{-\gamma z}} = \Gamma_L e^{2\gamma z} \qquad (6)$$

Of course, $\Gamma(z=0) = \Gamma_L$. Furthermore, we must have

$$\frac{V(z=0)}{I(z=0)} = Z_L \qquad (7)$$

or that using (3) and (4) with $z=0$, we have

$$\frac{1+\Gamma_L}{1-\Gamma_L} \cdot Z_0 = Z_L \qquad (8)$$

From the above,

$$\Gamma_L = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad (9)$$

$\Gamma_L = 0$ if $Z_L = Z_0$.
Hence, given the termination load Z_L, the reflection coefficient Γ_L can be found, or vice versa.

In general, we can define the generalized impedance at z to be

$$Z(z) = \frac{V(z)}{I(z)} = \frac{a_+\left(e^{-\gamma z} + \Gamma_L e^{\gamma z}\right)}{\frac{1}{Z_0} a_+ \left(e^{-\gamma z} - \Gamma_L e^{\gamma z}\right)}$$

$$= Z_0 \left(\frac{1 + \Gamma_L e^{2\gamma z}}{1 - \Gamma_L e^{2\gamma z}}\right) = Z_0 \frac{1+\Gamma(z)}{1-\Gamma(z)} \qquad (10)$$

Conversely, one can write

$$\Gamma(z) = \frac{Z(z) - Z_0}{Z(z) + Z_0} \qquad (11)$$

Usually, a transmission is lossless, and $\gamma = j\beta$. In this case,

$$Z(z) = Z_0 \frac{1 + \Gamma_L e^{2j\beta z}}{1 - \Gamma_L e^{2j\beta z}} \qquad (12)$$

From the above, one can show that

$$Z(-l) = Z_0 \frac{Z_L + j Z_0 \tan \beta l}{Z_0 + j Z_L \tan \beta l} \qquad (13)$$

Shorted Terminations

From the above, when we have a short such that $Z_L = 0$, then

$$Z(-l) = jZ_0 \tan(\beta l) = jX \qquad (14)$$

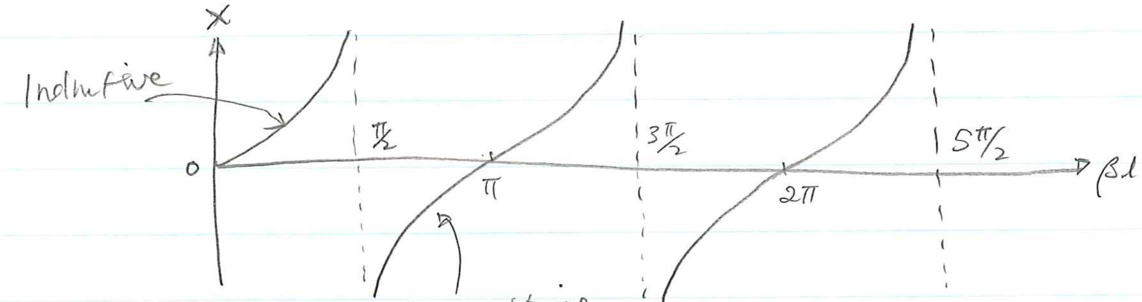


Fig. 2

When $\beta l \ll 1$, then $\tan\beta l \approx \beta l$, and (14) becomes, after using $\beta = \omega\sqrt{LC}$, $Z_0 = \sqrt{L/C}$,

$$Z(-l) \cong jZ_0\beta l = j\omega L l \qquad (15)$$

The above implies that a short length of transmission line connected to a short looks like an inductor as much current will pass through this short producing a strong magnetic field.

Open Terminations

When we have an open circuit such that $Z_L = \infty$, then,

$$Z(-l) = -jZ_0 \cot(\beta l) \qquad (15)$$

when $\beta l \ll 1$, $\cot(\beta l) \approx 1/\beta l$, and using $\beta = \omega\sqrt{LC}$, $Z_0 = \sqrt{L/C}$

$$Z(-l) \approx -j\frac{Z_0}{\beta l} = \frac{1}{j\omega C l} \qquad (16)$$

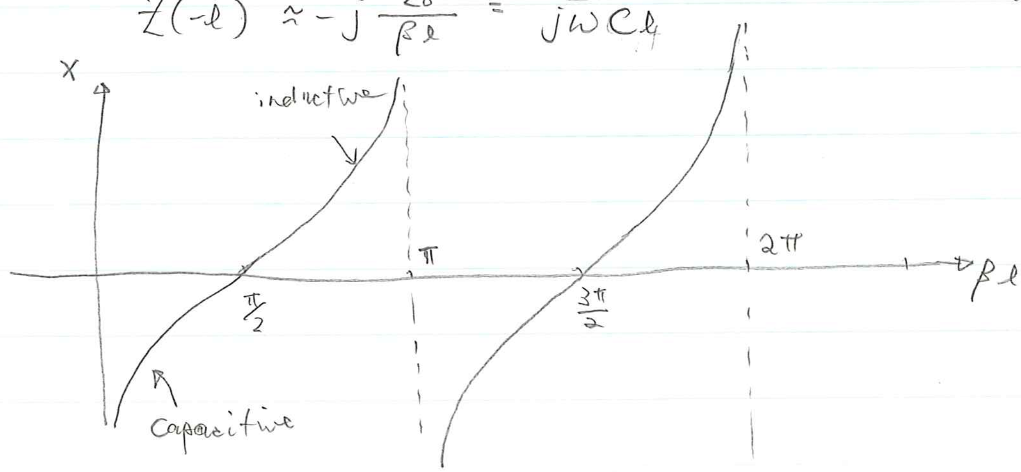


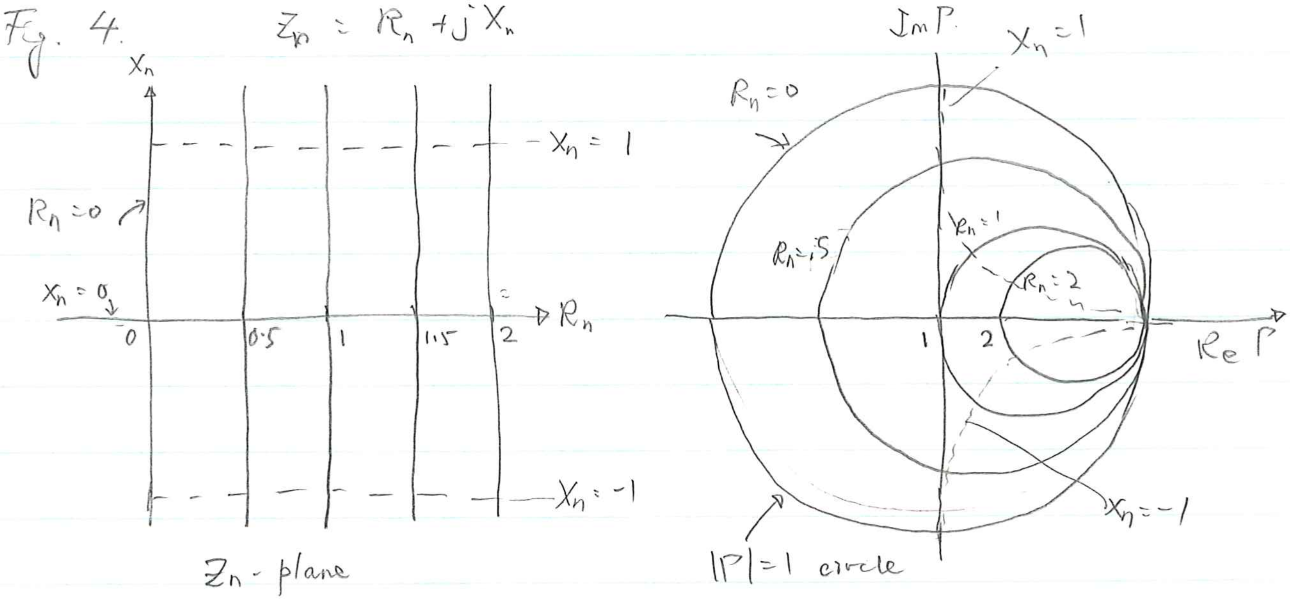
Fig 3

Hence, an ~~open-circuited~~ short terminated transmission line appears like a capacitor.

But the changing length of l can make a short or an open line look both inductive and capacitive. In general, a length of transmission line can transform a load Z_L to a range of possible values $Z(-l)$. To understand this range of values, we need to understand the Smith chart (1939). The Smith chart is essentially a graphical calculator for solving transmission line problems. Equation (11) indicates that there is a unique map between the impedance $Z(z)$ and the reflection coefficient $\Gamma(z)$. In the normalized impedance form,

$$\Gamma = \frac{Z_n - 1}{Z_n + 1}, \quad Z_n = \frac{1+\Gamma}{1-\Gamma} = \frac{1+\Gamma_L e^{-2j\beta l}}{1-\Gamma_L e^{-2j\beta l}} \qquad (17)$$

Fig. 4. $Z_n = R_n + jX_n$



Equation (17) is a bilinear transform in complex variable, a kind of conformal map that maps circles to circles.

VSWR (Voltage Standing Wave Ratio)

The general reflection coefficient for a lossless transmission line at z is given as

$$\Gamma(z) = \frac{V_-(z)}{V_+(z)} = \Gamma_L e^{2j\beta z} \qquad (18)$$

The standing wave on a terminated transmission line is given as

$$V(z) = V_0 e^{-j\beta z} + V_0 e^{j\beta z}\Gamma_L$$

$$= V_0 e^{-j\beta z}\left(1 + \Gamma_L e^{2j\beta z}\right)$$

$$= V_0 e^{-j\beta z}\left(1 + \Gamma(z)\right) \qquad (19)$$

Hence, $|V(z)|$ is not a constant of z, but

$$|V(z)| = |V_0|\,|1 + \Gamma(z)| \qquad (20)$$

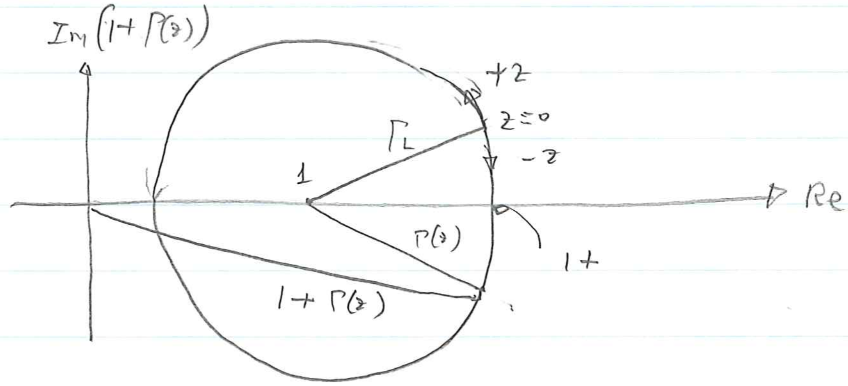


Fig 5

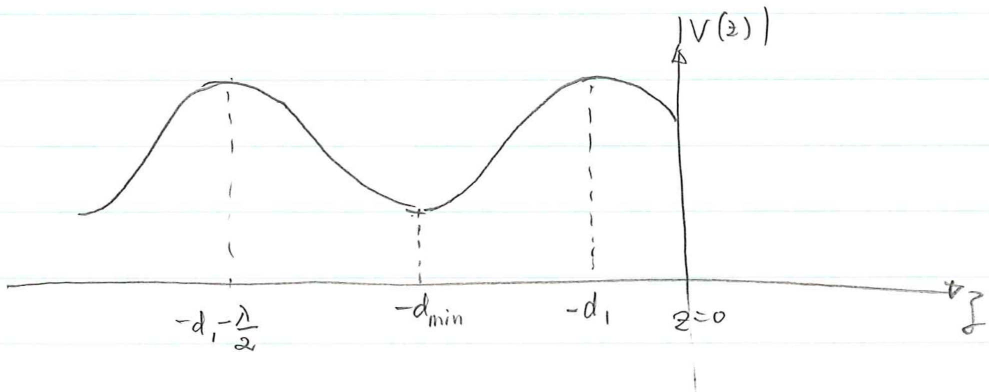


Fig 6

Using the triangular inequality,

$$|V_0|\left(1-|\Gamma(z)|\right) \le |V(z)| \le |V_0|\left(1+|\Gamma(z)|\right) \qquad (21)$$

But $|\Gamma(z)| = |\Gamma_L|$; hence

$$|V_0|\left(1-|\Gamma_L|\right) \le |V(z)| \le |V_0|\left(1+|\Gamma_L|\right) \qquad (22)$$

The voltage standing wave ratio, VSWR is

$$VSWR = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \qquad (23)$$

Conversely,

$$|\Gamma_L| = \frac{VSWR-1}{VSWR+1} \qquad (24)$$

Hence, the knowledge of VSWR yields the knowledge of $|\Gamma_L|$. The location of Γ_L in Fig. 5 is determined by the phase of Γ_L. Hence, the value of d_1 in Fig 6 is determined by the phase of Γ_L as well. Thus, by measuring the voltage standing wave pattern, one deduces the amplitude and phase of Γ_L.