Lecture 12 Oct 4, 2018, W. C. Chew

Transmission Line Theory

Transmission lines were the first electromagnetic waveguides ever invented. They were driven by the need in telegraphy. It is best to introduce transmission line theory from the viewpoint of circuit theory. The circuit theory explains why waveguides can be made sloppily when circuit theory is valid. For instance in the long wavelength limit, we can make twisted-pair waveguides with abandon, and they still work well. We can explain the propagation of EM signal on a twisted-pair transmission line using circuit analysis.

Keep in mind that two pieces of metal can accumulate attractive charges between them, giving rise to capacitive coupling. Moreover, a piece of wire carrying a current generates a magnetic field, and hence, yielding stored energy in the magnetic field. This stored energy is the source of the inductive effect. Hence, we can model two pieces of metal with a lumped element model as shown in Fig. 1.

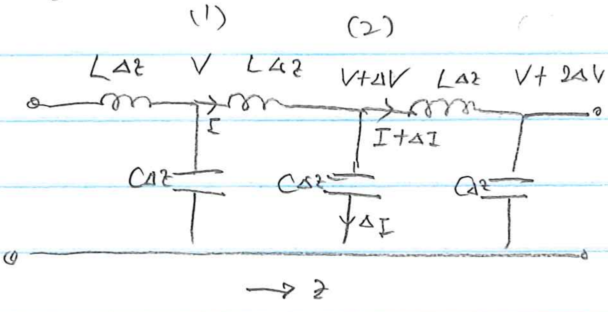

Fig. 1

Then, writing the circuit equation between nodes 1 and 2, we have

$$V-(V+\Delta V) = L\Delta z \frac{\partial I}{\partial t} \quad (1)$$

or

$$\Delta V = -L \Delta z \frac{\partial I}{\partial t} \quad (2)$$

where L is the inductance per unit length. Then, writing the current relation at node 2, one gets

$$-\Delta I = C\Delta z \frac{\partial}{\partial t}(V+\Delta V) \approx C\Delta z \frac{\partial V}{\partial t} \quad (3)$$

where C is the capacitance per unit length. In the limit when $\Delta z \to 0$, one gets

$$\frac{\partial V(z,t)}{\partial z} = -L \frac{\partial I(z,t)}{\partial t} \quad (4)$$

$$\frac{\partial I(z,t)}{\partial z} = -C \frac{\partial V(z,t)}{\partial t} \quad (5)$$

The above are the telegrapher's equations. The two coupled first-order system, and they can be converted into second-order system easily. Therefore,

$$\frac{\partial^2 V}{\partial z^2} - LC\frac{\partial^2 V}{\partial t^2} = 0 \quad (7)$$

$$\frac{\partial^2 I}{\partial z^2} - LC\frac{\partial^2 I}{\partial t^2} = 0 \quad (8)$$

The above are wave equations where the velocity of the wave is given by

$$v = \frac{1}{\sqrt{LC}} \quad (9)$$

Furthermore, if we assume that

$$V(z,t) = f(z - vt) \tag{10}$$

substituting into (4) yields

$$-L\frac{\partial I}{\partial t} = f'(z - vt) \tag{11}$$

or that

$$I = \frac{1}{Lv} f(z - vt) = \sqrt{\frac{C}{L}}\, f(z - vt) \tag{12}$$

Consequently,

$$\frac{V(z,t)}{I(z,t)} = \sqrt{\frac{L}{C}} = Z_0 \tag{13}$$

where Z_0 is the characteristic impedance.*

<u>Lossy Transmission Line</u>

The above analysis can be easily generalized to the lossy case. In this case, it is best to perform the analysis in the frequency domain using phasor technique, since impedances and complex numbers will be involved.

To include loss, we use the lumped-element model as shown

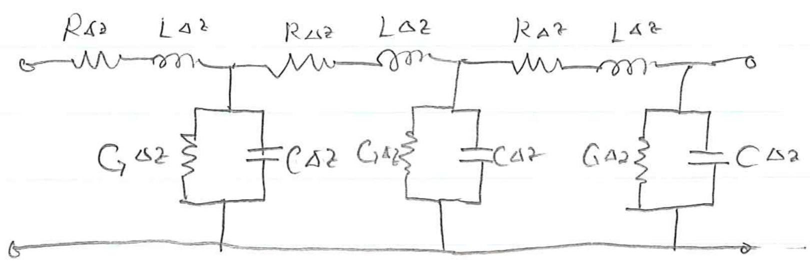

Fig. 2.

Using phasor technique, eq (4) can be written as

$$\frac{d}{dz}V(z,\omega) = -j\omega L I(z,\omega) \tag{14}$$

$$\frac{d}{dz}I(z,\omega) = -j\omega C V(z,\omega) \tag{15}$$

To include loss, we generalize the above to

$$\frac{d}{dz} V = -Z I \quad (16)$$

$$\frac{d}{dz} I = -Y V \quad (17)$$

where the series impedance is

$$Z = j\omega L + R \quad (18).$$

and the shunt admittance is

$$Y = j\omega C + G \quad (19)$$

Then,

$$\frac{d^2 V}{dz^2} - ZYV = 0 \quad (20)$$

$$\frac{d^2 I}{dz^2} - ZYI = 0 \quad (21)$$

or

$$\frac{d^2 V}{dz^2} - \gamma^2 V = 0 \quad (22)$$

$$\frac{d^2 I}{dz^2} - \gamma^2 I = 0 \quad (23)$$

The above are 2nd order 1D Helmholtz wave equations where the general solution is

$$V(z) = V_+ e^{-\gamma z} + V_- e^{\gamma z} \quad (24)$$

where

$$\gamma = \sqrt{ZY} = \sqrt{(j\omega L + R)(j\omega C + G)} = \alpha + j\beta \quad (25)$$

or thus

$$V(z) = V_+ e^{-\alpha z - j\beta z} + V_- e^{\alpha z + j\beta z} \quad (26)$$

Letting $V_\pm = |V_\pm| e^{j\phi_\pm}$, the above can be converted back to the time domain as

$$V(z,t) = \mathrm{Re}\{V(z,\omega) e^{j\omega t}\}$$
$$= |V_+| e^{-\alpha z} \cos(\omega t - \beta z + \phi_+) + |V_-| e^{\alpha z} \cos(\omega t + \beta z + \phi_-) \quad (27)$$

The first term corresponds to a decaying wave moving to the right while the second term is also a decaying wave moving to the left.

When there is no loss, or $R = G = 0$, then

$$V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z} \qquad (28)$$

Notice that for the lossy case, the characteristic impedance is

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{j\omega L + R}{j\omega C + G}} \qquad (29)$$

In the absence of loss, the above becomes

$$Z_0 = \sqrt{\frac{L}{C}} . \qquad (30)$$

The current assumes a similar expression like (24), namely

$$I(z) = I_+ e^{-\gamma z} + I_- e^{\gamma z} \qquad (31)$$

and in general

$$\frac{V_+}{I_+} = \frac{V_-}{I_-} = Z_0 = \sqrt{\frac{Z}{Y}} \qquad (32)$$

For the general case, again $\gamma = \alpha + j\beta = \sqrt{ZY} = \sqrt{(j\omega L + R)(j\omega C + G)}$.

When $R = G = 0$, $\alpha = 0$, and $\gamma = j\beta = \sqrt{(j\omega L)(j\omega C)} = j\omega\sqrt{LC}$.

In comparison with the 1 D wave equation, where

$$E_x(z) = E_{0+} e^{-jk_0 z} + E_{0-} e^{jk_0 z} \qquad (33)$$

We see a lot of similarity between (28), (31), and (33).