## More on Complex Power

Now that we realize that complex power is quite different from instantaneous power. The real part of complex power is proportional to the time average of the instantaneous power while the imaginary part is proportional to a time varying power that averages to zero. This part is termed the reactive power which is proportional to stored energy in the system. Reactive power corresponds to time varying part of the instantaneous power that can be both positive or negative. Hence, it corresponds to power that are drained from the source as well as power returned to the source.

Because of the above observation, it is prudent to look at the conservative property of complex power for a Maxwellian system. We shall consider a system where conductive loss as well as impressed sources are present. Using phasor technique for time-harmonic fields, Maxwell's equations can be written in the frequency domain as

$$\nabla\times\bar{E} = -j\omega\mu\bar{H} - \bar{M}_i \tag{1}$$

$$\nabla\times\bar{H} = j\omega\epsilon\bar{E} + \sigma\bar{E} + \bar{J}_i \tag{2}$$

From now on, we neglect to use under-tilde and assume that these are complex vectors. To this end, we take

$$\nabla\cdot(\bar{E}\times\bar{H}^*) = \bar{H}^*\cdot\nabla\times\bar{E} - \bar{E}\cdot\nabla\times\bar{H}^* \tag{3}$$

Using (1) and (2) in (3), we have

$$\nabla\cdot(\bar{E}\times\bar{H}^*) = -\bar{H}^*\cdot(j\omega\mu\bar{H}) - \bar{H}^*\cdot\bar{M}_i - \bar{E}\cdot(-j\omega\epsilon\bar{E}^*) - \sigma\bar{E}\cdot\bar{E}^* - \bar{E}\cdot\bar{J}_i^* \tag{4}$$

The above can be further rearranged as

$$\nabla\cdot(\bar{E}\times\bar{H}^*) = -j\omega\mu\,\bar{H}\cdot\bar{H}^* + j\omega\epsilon\,\bar{E}\cdot\bar{E}^* - \sigma\bar{E}\cdot\bar{E}^* - \bar{H}^*\cdot\bar{M}_i - \bar{E}\cdot\bar{J}_i^* \qquad (5)$$

Noticing that

$$\langle V(t)\,I(t)\rangle = \frac{1}{2}\mathrm{Re}\left[\underset{\sim}{V}(\omega)\,\underset{\sim}{I}^*(\omega)\right]$$

we can show that

$$\langle W_H\rangle = \frac{1}{2}\mu\langle\bar{H}(\bar{r},t)\cdot\bar{H}(\bar{r},t)\rangle = \frac{1}{2}\mu\langle|\bar{H}(\bar{r},t)|^2\rangle = \frac{1}{4}\mu|\bar{H}(\bar{r},\omega)|^2 \qquad (6)$$

We take care to write eqn (5) as

$$\nabla\cdot(\bar{E}\times\bar{H}^*) = -j\omega\mu|\bar{H}|^2 + j\omega\epsilon|\bar{E}|^2 - \sigma|\bar{E}|^2 - \bar{H}^*\cdot\bar{M}_i - \bar{E}\cdot\bar{J}_i^* \qquad (7)$$

Had $\bar{H}(\bar{r},t)$ being sinusoidal, (6) implies that

$$\nabla\cdot(\bar{E}\times\bar{H}^*) = -j\omega\,4\left[\langle W_H\rangle - \langle W_E\rangle\right] - \sigma|\bar{E}|^2 - \bar{H}^*\cdot\bar{M}_i - \bar{E}\cdot\bar{J}_i^* \qquad (8)$$

where

$$\langle W_H\rangle = \frac{1}{4}\mu|\bar{H}(\omega)|^2 = \frac{1}{2}\mu\langle|\bar{H}(t)|^2\rangle \qquad (9)$$

$$\langle W_E\rangle = \frac{1}{4}\epsilon|\bar{E}(\omega)|^2 = \frac{1}{2}\epsilon\langle|\bar{E}(t)|^2\rangle \qquad (10)$$

If we assume that $\bar{M}_i = \bar{J}_i = 0$ to begin with, taking $\frac{1}{2}$ the real part of (7) gives

$$\nabla\cdot\frac{1}{2}\mathrm{Re}(\bar{E}\times\bar{H}^*) = -\frac{1}{2}\sigma|\bar{E}|^2 \qquad (11)$$

a statement of energy conservation. Taking $\frac{1}{2}$ the imaginary part of (8) gives us

$$\nabla\cdot\frac{1}{2}\mathrm{Im}(\bar{E}\times\bar{H}^*) = -\omega\,2\left[\langle W_H\rangle - \langle W_E\rangle\right] \qquad (12)$$

Eq. (11) implies that $\nabla\cdot\frac{1}{2}\mathrm{Re}(\bar{E}\times\bar{H}^*)$ is the time-average power exuding from a point in space while $\frac{1}{2}\sigma|\bar{E}|^2$ is the time-average power dissipating in the lossy conductor at that point in space. In (12), $\nabla\cdot\frac{1}{2}\mathrm{Im}(\bar{E}\times\bar{H}^*)$ is the reactive power leaving that point in space while $-2\omega[\langle W_H\rangle - \langle W_E\rangle]$ is the reactive power entering the point.

Then we can think of $-\frac{1}{2}\bar{H}^*\cdot\bar{M}_i$ and $-\frac{1}{2}\bar{E}\cdot\bar{J}_i^*$ as complex power supplied to the system

## Wave Phenomenon in the Frequency Domain

Given that we have seen the emergence of wave phenomenon in the time domain, it will be interesting to ask how this phenomenon presents itself for time-harmonic field or in the frequency domain. In the frequency domain, the source-free Maxwell's equations are

$$\nabla \times \bar{E}(\bar{r}) = -j\omega\mu_0 \bar{H}(\bar{r}) \quad (1)$$

$$\nabla \times \bar{H}(\bar{r}) = j\omega\epsilon_0 \bar{E}(\bar{r}) \quad (2)$$

Taking the curl of (1) and then substituting (2) into its RHS, one obtains

$$\nabla \times \nabla \times \bar{E}(\bar{r}) = -j\omega\mu_0 \nabla \times \bar{H}(\bar{r}) = \omega^2\mu_0\epsilon_0 \bar{E}(\bar{r}) \quad (3)$$

Again, using the identity that

$$\nabla \times \nabla \times \bar{E} = \nabla(\nabla \cdot \bar{E}) - \nabla \cdot \nabla \bar{E} = -\nabla^2 \bar{E} \quad (4)$$

Since $\nabla \cdot \bar{E} = 0$ in a source-free medium, (3) becomes

$$(\nabla^2 + \omega^2\mu_0\epsilon_0)\bar{E}(\bar{r}) = 0 \quad (5)$$

This is known as the Helmholtz wave equation.

For simplicity of seeing the wave phenomenon, we let $\bar{E} = \hat{x} E_x(z)$, $\nabla \cdot \bar{E} = 0$, and (5) becomes

$$\left(\frac{d^2}{dz^2} + k_0^2\right) E_x(z) = 0 \quad (6)$$

where $k_0^2 = \omega^2\mu_0\epsilon_0 = \omega^2/c_0^2$. The general solution to (6) is of the form

$$E_x(z) = E_{0+} e^{-jk_0 z} + E_{0-} e^{jk_0 z} \quad (7)$$

One can convert the above back to the time domain by using that $E_x(z,t) = \mathrm{Re}\left[E_x(z,\omega) e^{j\omega t}\right]$, or that

$$E_x(z,t) = |E_{0+}| \cos(\omega t - k_0 z + \alpha_+) + |E_{0-}| \cos(\omega t + k_0 z + \alpha_-) \tag{8}$$

where we have assumed that

$$E_{0\pm} = |E_{0\pm}| e^{j\alpha_\pm} \tag{9}$$

One can see that the first term on the RHS is a sinusoidal plane wave travelling to the right, while the second term is a sinusoidal plane wave travelling to the left, with velocity $c_0$. This can be appreciated by rewriting

$$\cos(\omega t \mp k_0 z + \alpha_\pm) = \cos\left[\frac{\omega}{c_0}(c_0 t \mp z_0) + \alpha_\pm\right] \tag{10}$$

where we have used the fact that $k_0 = \frac{\omega}{c_0}$. Moreover, for a fixed $t$, (or $t=0$) as a function of $z$, the function is sinusoidal and is proportional to $\cos(\mp k_0 z + \alpha_\pm)$. From this, we can see a spatial wavelength that when $k_0 z = 2n\pi$, $n \in Q$ where $Q$ is the set of integers, the function repeats itself. From, this, we deduce that

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{\omega}{c_0} = \frac{2\pi f}{c_0} \tag{11}$$

where $\lambda_0$ is the wavelength. From the above, one sees that

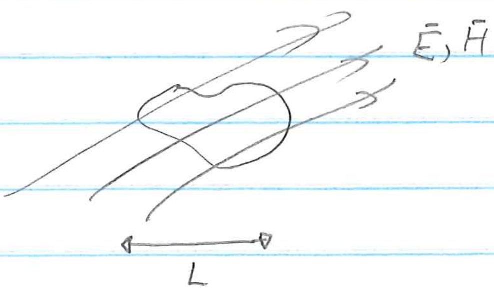
$$\lambda_0 = c_0/f \tag{12}$$

where $f$ is the frequency in hertz. One can see that because $c_0$ is a humongous number, $\lambda_0$ can be very large. You can plug in the frequency of your local AM station to see how big $\lambda_0$ is.

## When is Circuit Theory Valid ?

Historically, circuit theory was discovered first. It consists of elements like resistors, capacitors, and inductors. Static electromagnetic theory, or quasi-static electromagnetic theory was used to derived the formulas for these elements. Given that we have now seen electromagnetic theory in its full form, we like to ponder when one can use static or quasistatic theory to describe electromagnetic phenomena.

To see this lucidly, it is best to write Maxwell's equations in dimensionless units or the same units.




Say if we want to solve M.E. for the fields close to an object of size L. This object can be a small particle like the sphere in your take home exam, or it could be a capacitor, or an inductor. It is clear that the $\bar{E}$ and H fields will have to satisfy boundary conditions in the vicinity of the object. They become great contortionist in order to do so. Hence, we do not expect a constant field but that the field will vary on the length scale of L. So we renormalize our length scale by this length L by defining a new coordinate system such that

$$x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad z' = \frac{z}{L} \qquad (1)$$

By so doing, then

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y} = \frac{1}{L}\frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z} = \frac{1}{L}\frac{\partial}{\partial z'} \qquad (2)$$

Then, M. E. becomes

$$\frac{1}{L}\nabla' \times \bar{E} = -j\omega\mu_0 \bar{H} \qquad (3)$$

$$\frac{1}{L}\nabla' \times \bar{H} = j\omega\epsilon_0 \bar{E} + \bar{J} \qquad (4)$$

Here, we still have apples and oranges to compare since $\bar{E}$ and $\bar{H}$ have different units. To bring them to the same unit, we define a new $\bar{E}'$ such that

$$\eta_0 \bar{E}' = \bar{E} \qquad (5)$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0} \cong 377\,\Omega$ in vacuum. Then, (3) and (4) become

$$\frac{\eta_0}{L}\nabla' \times \bar{E}' = -j\omega\mu_0 \bar{H} \qquad (6)$$

$$\frac{1}{L}\nabla' \times \bar{H} = j\omega\epsilon_0\eta_0 \bar{E}' + \bar{J} \qquad (7)$$

In this change, the above becomes

$$\nabla' \times \bar{E}' = -j\omega\mu_0 \frac{L}{\eta_0} \bar{H} \qquad (8)$$

$$\nabla' \times \bar{H} = j\omega\epsilon_0\eta_0 L \bar{E}' + L\bar{J} \qquad (9)$$

The above can be further simplified to become

$$\nabla' \times \bar{E}' = -j\frac{\omega}{c_0} L \bar{H} \qquad (10)$$

$$\nabla' \times \bar{H} = j\frac{\omega}{c_0} L \bar{E}' + L\bar{J} \qquad (11)$$

Therefore, one can ignore the frequency dependent term if

$$\frac{\omega}{c_0} L \ll 1 \quad (12)$$

or

$$2\pi \frac{L}{\lambda_0} \ll 1 \quad (13)$$

Therefore, the above are criteria for the validity of the static or quasistatic approximation. In other words, you can solve an optics problem where ω is humongous, with a plasmonic nanoparticle using quasi-static analysis if the particle is small enough compared to wavelength of the light.

Also in circuit theory where static analysis prevails, we better check if (13) is satisfied before we use circuit theory comfortably. At 3 GHz, where the wavelength is 10 cm, when the circuit board or the computer chip is much smaller than this dimension, circuit theory can be used. But when the clock rate of the computer switches up to 10 GHz, many circuit analysis do not hold any more. Then one really has to perform electrodynamic analysis of the electromagnetic phenomena inside the circuit board, especially in the computer chassis level.