ECE 604

Lecture 10, Sept 27, 2018

Time-Harmonic Fields – Linear Systems

The analysis of Maxwell's equations can be greatly simplified by assuming the fields to be time harmonic, or sinusoidal (cosinusoidal). Electrical engineers use a method called phasor technique to simplify time-harmonic solutions. This is also a poor-man's Fourier transform. That is one begets the benefits of Fourier transform technique without knowledge of Fourier transform.

To learn phasor technique, one makes use of Euler's formula that (1707-1783)

$$e^{j\alpha} = \cos\alpha + j\sin\alpha \qquad (1)$$

where $j=\sqrt{-1}$. Lo and behold, in other disciplines, $\sqrt{-1}$ is denoted by "i", but "i" is too close to the symbol for current. So the preferred symbol for $\sqrt{-1}$, an imaginary number is j: a quirkiness of convention; just as positive charges do not carry current in a wire.

From Euler's formula, one gets

$$\cos\alpha = \mathrm{Re}\left(e^{j\alpha}\right) \qquad (2)$$

Hence, all time harmonic quantity can be written as

$$V(x,y,z,t) = V'(x,y,z)\cos(\omega t+\alpha) \qquad (3)$$

$$= V'(\bar{r})\,\mathrm{Re}\left(e^{j(\omega t+\alpha)}\right) \qquad (4)$$

$$= \mathrm{Re}\left(V'(\bar{r})e^{j\alpha}e^{j\omega t}\right) \qquad (5)$$

$$= \mathrm{Re}\left(\underset{\sim}{V}(\bar{r})\,e^{j\omega t}\right) \qquad (6)$$

Now $\underset{\sim}{V}(\bar{r})$ is a complex number is called the phasor representation of $V(\bar{r},t)$, a time-harmonic quantity.

Consequently, any component of a field can be expressed as

$$E_x(x,y,z,t) = E_x(\bar{r},t) = Re[\underset{\sim}{E_x}(\bar{r})e^{j\omega t}] \quad (7)$$

The above can be repeated for y and z components.

Compactly, one can write

$$\bar{E}(\bar{r},t) = Re[\underset{\sim}{\bar{E}}(\bar{r},t)e^{j\omega t}] \quad (8)$$

$$\bar{H}(\bar{r},t) = Re[\underset{\sim}{\bar{H}}(\bar{r},t)e^{j\omega t}] \quad (9)$$

Such phasor representation of a time-harmonic field simplifies Maxwell's equations. For instance, if one writes

$$\bar{B}(\bar{r},t) = Re[\underset{\sim}{\bar{B}}(\bar{r})e^{j\omega t}] \quad (10)$$

then

$$\frac{\partial}{\partial t}\bar{B}(\bar{r},t) = \frac{\partial}{\partial t}Re[\underset{\sim}{\bar{B}}(\bar{r})e^{j\omega t}] \quad (11)$$

$$= Re\left[\frac{\partial}{\partial t}\underset{\sim}{\bar{B}}(\bar{r})e^{j\omega t}\right] \quad (12)$$

$$= Re[\underset{\sim}{\bar{B}}(\bar{r})\,j\omega\, e^{j\omega t}] \quad (13)$$

Therefore, given Faraday's law, that

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} - \bar{M} \quad (14)$$

assuming that all quantities are time harmonic, then

$$\bar{E}(\bar{r},t) = Re[\underset{\sim}{\bar{E}}(\bar{r})e^{j\omega t}], \quad (15)$$

$$\bar{M}(\bar{r},t) = Re[\underset{\sim}{\bar{M}}(\bar{r})e^{j\omega t}] \quad (16)$$

Using (13), (15), and (16) into (14), one gets

$$\nabla\times\bar{E}(\bar{r},t) = Re[\nabla\times\underset{\sim}{\bar{E}}(\bar{r})e^{j\omega t}], \quad (17)$$

and that

$$Re[\nabla\times\underset{\sim}{\bar{E}}(\bar{r})e^{j\omega t}] = -Re[\underset{\sim}{\bar{B}}(\bar{r})\,j\omega\, e^{j\omega t}]$$
$$- Re[\underset{\sim}{\bar{M}}(\bar{r})e^{j\omega t}] \quad (18)$$

Since if

$$\mathrm{Re}\left[Ae^{j\omega t}\right] = \mathrm{Re}\left[Be^{j\omega t}\right], \forall t,$$

then $A = B$; it must be true that

$$\nabla \times \underset{\sim}{\bar{E}}(\bar{r}) = -j\omega \underset{\sim}{\bar{B}}(\bar{r}) - \underset{\sim}{\bar{M}}(\bar{r}) \qquad (19)$$

Hence, finding the phasor representation of an equation is clear: whenever we have $\frac{\partial}{\partial t}$, we replace it by $j\omega$. Applying this methodically to the other equations, we have

$$\nabla \times \underset{\sim}{\bar{H}}(\bar{r}) = j\omega \underset{\sim}{\bar{D}}(\bar{r}) + \underset{\sim}{\bar{J}}(\bar{r}) \qquad (20)$$

$$\nabla \cdot \underset{\sim}{\bar{D}}(\bar{r}) = \underset{\sim}{\rho_e}(\bar{r}) \qquad (21)$$

$$\nabla \cdot \underset{\sim}{\bar{B}}(\bar{r}) = \underset{\sim}{\rho_m}(\bar{r}). \qquad (22)$$

Fourier Transform Derivation

In the phasor representation M. E. has no time derivatives; hence the equations are simplified. We can also arrive at the above simplified equations using Fourier transform technique. To this end, we use Faraday's law as an example. By setting

$$\bar{E}(\bar{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{\sim}{\bar{E}}(\bar{r},\omega)\, e^{j\omega t}\, d\omega \qquad (23)$$

$$\bar{B}(\bar{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{\sim}{\bar{B}}(\bar{r},\omega)\, e^{j\omega t}\, d\omega \qquad (24)$$

$$\bar{M}(\bar{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underset{\sim}{\bar{M}}(\bar{r},\omega)\, e^{j\omega t}\, d\omega \qquad (25)$$

Substituting the above into Faraday's law given by (14)

$\omega \equiv$ \omega

We get

$$\nabla \times \int_{-\infty}^{\infty} d\omega\, e^{j\omega t} \underset{\sim}{\bar{E}}(\bar{r},\omega) = -\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\omega\, e^{j\omega t} \underset{\sim}{\bar{B}}(\bar{r},\omega)$$

$$- \int_{-\infty}^{\infty} d\omega\, e^{j\omega t} \underset{\sim}{\bar{M}}(\bar{r},\omega) \qquad (26)$$

Using the fact that

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\omega\, e^{j\omega t} \underset{\sim}{\bar{B}}(\bar{r},\omega) = \int_{-\infty}^{\infty} d\omega\, e^{j\omega t}\, j\omega\, \underset{\sim}{\bar{B}}(\bar{r},\omega) \qquad (27)$$

and that

$$\nabla \times \int_{-\infty}^{\infty} d\omega\, e^{j\omega t} \underset{\sim}{\bar{E}}(\bar{r},\omega) = \int_{-\infty}^{\infty} d\omega\, e^{j\omega t}\, \nabla \times \underset{\sim}{\bar{E}}(\bar{r},\omega) \qquad (28)$$

Using the ~~fact~~ that

$$\int_{-\infty}^{\infty} d\omega\, e^{j\omega t} A(\omega) = \int_{-\infty}^{\infty} d\omega\, e^{j\omega t} B(\omega), \quad \forall t \qquad (29)$$

implies that $A(\omega) = B(\omega)$, and using (27) and (28) in (26), and the property (29), one gets

$$\nabla \times \underset{\sim}{\bar{E}}(\bar{r},\omega) = -j\omega\, \underset{\sim}{\bar{B}}(\bar{r},\omega) - \underset{\sim}{\bar{M}}(\bar{r},\omega) \qquad (30)$$

These equations look exactly like the phasor equations we have derived previously, save that the field $\underset{\sim}{\bar{E}}(\bar{r},\omega)$, $\underset{\sim}{\bar{B}}(\bar{r},\omega)$, and $\underset{\sim}{\bar{M}}(\bar{r},\omega)$ are now the Fourier transforms of the field $\bar{E}(\bar{r},t)$, $\bar{B}(\bar{r},t)$ and $\bar{M}(\bar{r},t)$. Moreover, the Fourier transform variables can be complex.

Repeating the exercise above for the other Maxwell's equations, we obtain them that look similar to those of their phasor representations. Hence, Maxwell's equations can be simplified either by using phasor technique or Fourier technique.

Complex Power

Consider now that in the phasor representations, $\underset{\sim}{\bar{E}}(\bar{r})$ and $\underset{\sim}{\bar{H}}(\bar{r})$ are complex vectors, they have different physical meaning. Consider the Poynting vector

$$\bar{S}(r,t) = \bar{E}(r,t) \times \bar{H}(r,t) \tag{1}$$

where all the quantities are real valued. Now, we can use phasor technique to analyze the above. Assuming time-harmonic fields, the above can be rewritten as

$$\begin{aligned}\bar{S}(r,t) &= \mathrm{Re}\left[\underset{\sim}{\bar{E}}(\bar{r})e^{j\omega t}\right] \times \mathrm{Re}\left[\underset{\sim}{\bar{H}}(\bar{r})e^{j\omega t}\right] \\ &= \frac{1}{2}\left[\underset{\sim}{\bar{E}}e^{j\omega t} + (\underset{\sim}{\bar{E}}e^{j\omega t})^*\right] \\ &\quad \times \frac{1}{2}\left[\underset{\sim}{\bar{H}}e^{j\omega t} + (\underset{\sim}{\bar{H}}e^{j\omega t})^*\right]\end{aligned} \tag{2}$$

or more elaborately,

$$\begin{aligned}\bar{S}(r,t) &= \frac{1}{4}\underset{\sim}{\bar{E}} \times \underset{\sim}{\bar{H}} e^{2j\omega t} + \frac{1}{4}\underset{\sim}{\bar{E}} \times \underset{\sim}{\bar{H}}^* + \frac{1}{4}\underset{\sim}{\bar{E}}^* \times \underset{\sim}{\bar{H}} \\ &\quad + \frac{1}{4}\underset{\sim}{\bar{E}}^* \times \underset{\sim}{\bar{H}}^* e^{-2j\omega t}\end{aligned} \tag{3}$$

Then

$$\bar{S}(r,t) = \frac{1}{2}\mathrm{Re}\left[\underset{\sim}{\bar{E}} \times \underset{\sim}{\bar{H}}^*\right] + \frac{1}{2}\mathrm{Re}\left[\underset{\sim}{\bar{E}} \times \underset{\sim}{\bar{H}} e^{2j\omega t}\right] \tag{4}$$

If we define a time average quantity that

$$\bar{S}_{av} = \langle \bar{S}(r,t) \rangle = \lim_{T\to\infty} \int_0^T \bar{S}(r,t)\,dt \tag{5}$$

then it is quite clear that

$$\bar{S}_{av} = \frac{1}{2}\mathrm{Re}\left[\underset{\sim}{\bar{E}} \times \bar{H}^*\right] \tag{6}$$

Hence, in the phasor representation, the quantity

$$\underset{\sim}{\bar{S}} = \underset{\sim}{\bar{E}} \times \underset{\sim}{\bar{H}}^* \tag{7}$$

is termed the complex Poynting vector.

The power flow associated with it is termed complex power.

To understand what complex power is, it is fruitful if we revisit complex power in our circuit theory course.

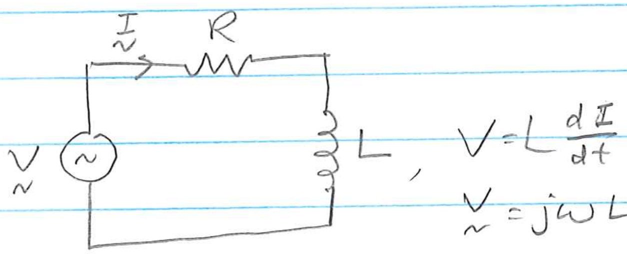

Fig. 1

The above circuit can be easily solved by using phasor technique. The impedance of the circuit is $Z = R + j\omega L$. Hence,

$$\underset{\sim}{V} = (R + j\omega L)\underset{\sim}{I} \tag{8}$$

Just as in the electromagnetic case, the complex power is taken to be

$$\underset{\sim}{P} = \underset{\sim}{V}\,\underset{\sim}{I}^*, \quad P_{inst}(t) = V(t)\,I(t) \tag{9}$$

It is quite easy to show that

$$P_{av} = \langle P_{inst}(t) \rangle = \frac{1}{2} Re\left[\underset{\sim}{P}\right] \tag{10}$$

where $P_{inst}(t)$ is the instantaneous power. It is clear that if $V(t)$ is sinusoidal, it can be written as

$$V(t) = V_0 \cos(\omega t) = Re\left[\underset{\sim}{V} e^{j\omega t}\right] \tag{11}$$

where we assume that $\underset{\sim}{V} = V_0$. Then from (8), it is clear that $V(t)$ and $I(t)$ are not in phase. Namely,

$$I(t) = I_0 \cos(\omega t + \alpha) = Re\left[\underset{\sim}{I} e^{j\omega t}\right] \tag{12}$$

where $\underset{\sim}{I} = I_0 e^{j\alpha}$. Then

$$
\begin{aligned}
P_{inst}(t) &= V_0 I_0 \cos(\omega t) \cos(\omega t + \alpha) \\
&= V_0 I_0 \cos(\omega t)\left[\cos(\omega t)\cos(\alpha) - \sin(\omega t)\sin\alpha\right] \\
&= V_0 I_0 \cos^2(\omega t)\cos\alpha - V_0 I_0 \cos(\omega t)\sin(\omega t)\sin\alpha
\end{aligned} \tag{13}
$$

Now taking the average of (13), we get

$$P_{av} = \langle P_{inst} \rangle = \frac{1}{2} V_0 I_0 \cos\alpha = \frac{1}{2} Re[\underset{\sim}{V} \underset{\sim}{I}]$$

$$= \frac{1}{2} Re[\underset{\sim}{P}] \qquad (14)$$

On the other hand,

$$P_{reactive} = \frac{1}{2} Im[\underset{\sim}{P}] = \frac{1}{2} Im[V_0 I_0 e^{j\alpha}] = \frac{1}{2} V_0 I_0 \sin\alpha \qquad (15)$$

We see that the amplitude of the time-varying term in (13) is precisely proportional to $Im[\underset{\sim}{P}]$.

The reason for the existence of imaginary part of $\underset{\sim}{P}$ is because $V(t)$ and $I(t)$ are out of phase or $\underset{\sim}{V} = V_0$, $\underset{\sim}{I} = I_0 e^{j\alpha}$. The reason why they are out of phase is because the circuit has a reactive part to it. Hence the imaginary part of complex power is also called the reactive power. In a reactive circuit, the plot of instantaneous power is shown in Fig. 2

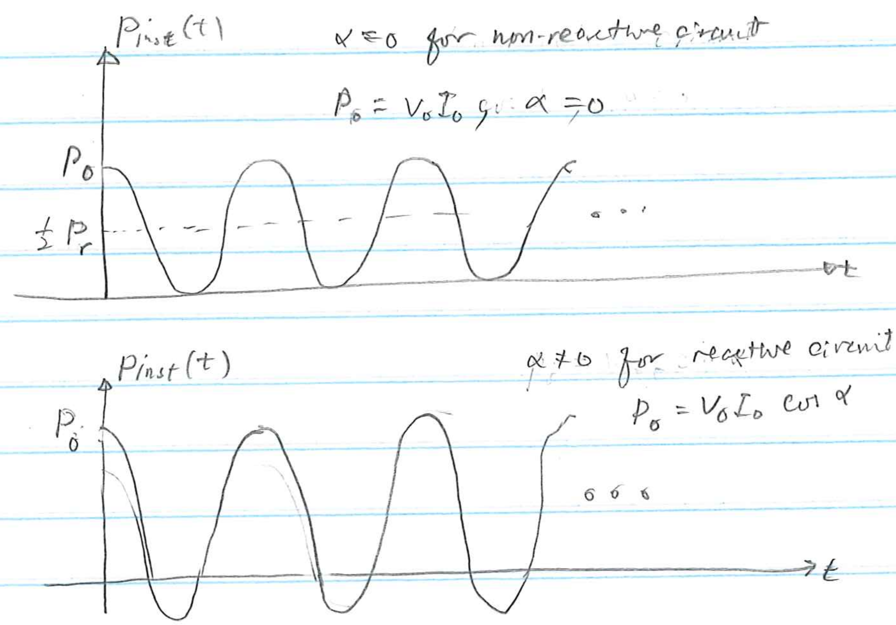

Fig. 2