ECE 604, Lecture 9

September 25, 2018

1 Introduction

In this lecture, we will cover the following topics:

- Energy and Power

- Emergence of Wave Phenomenon, Triumph of Maxwell's Equations

Additional Reading:

- Sections 3.11, 3.12, 3.13, Ramo et al.

2 Energy and Power

Consider the first two of Maxwell's equations where fictitious magnetic current is included and that the medium is isotropic. We need to consider only the first two equations since in electrodynamics, invoking charge conservation, the third and the fourth equations are derivable from the first two. They are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{M}_i = -\mu \frac{\partial \mathbf{H}}{\partial t} - \mathbf{M} \tag{2.1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_i + \sigma \mathbf{E} \tag{2.2}$$

where $\mathbf{M}_i$ and $\mathbf{J}_i$ are impressed current sources, while $\mathbf{J} = \sigma \mathbf{E}$ is the induced current source. Here, $\mathbf{J} = \sigma \mathbf{E}$ is similar to ohm's law. We can show from (2.1) and (2.2) that

$$\mathbf{H} \cdot \nabla \times \mathbf{E} = -\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} - \mathbf{H} \cdot \mathbf{M}_i \tag{2.3}$$

$$\mathbf{E} \cdot \nabla \times \mathbf{H} = \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \mathbf{E} \cdot \mathbf{J}_i + \sigma \mathbf{E} \cdot \mathbf{E} \tag{2.4}$$

Using the identity, which is the same as the product rule for derivatives, we have

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \tag{2.5}$$

Therefore, from (2.3), (2.4), and (2.5) we have

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\left(\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} + \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{M}_i + \mathbf{E} \cdot \mathbf{J}_i \right) \tag{2.6}$$

The physical meaning of the above is more lucid if we first consider $\sigma = 0$, and $\mathbf{M}_i = \mathbf{J}_i = 0$, or the absence of conductive loss and the impressed current sources. Then the above becomes

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\left(\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} + \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \right) \tag{2.7}$$

Rewriting each term on the right-hand side of the above, we have

$$\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2} \mu \frac{\partial}{\partial t} \mathbf{H} \cdot \mathbf{H} = \frac{\partial}{\partial t} \left(\frac{1}{2} \mu |\mathbf{H}|^2 \right) = \frac{\partial}{\partial t} W_m \tag{2.8}$$

$$\varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2} \varepsilon \frac{\partial}{\partial t} \mathbf{E} \cdot \mathbf{E} = \frac{\partial}{\partial t} \left(\frac{1}{2} \varepsilon |\mathbf{E}|^2 \right) = \frac{\partial}{\partial t} W_e \tag{2.9}$$

Then (2.7) becomes

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t} (W_m + W_e) \tag{2.10}$$

where $W_m = \frac{1}{2}\mu|\mathbf{H}|^2$ and $W_e = \frac{1}{2}\varepsilon|\mathbf{E}|^2$. Equation (2.10) is reminiscent of the current continuity equation, namely,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \varrho}{\partial t} \tag{2.11}$$

In the above, current density is due to charge density flow.

Hence, $\mathbf{E} \times \mathbf{H}$ has the meaning of power density, and W_m and W_e are the energy density stored in the magnetic field and electric field, respectively. In fact, one can show that $\mathbf{E} \times \mathbf{H}$ has the unit of V m^{-1} times A m^{-1} which is W m^{-2}, where V is volt, A is ampere, and W is watt which is joule s^{-1}. Hence, it has the unit of power density.

Similarly, $W_m = \frac{1}{2}\mu|\mathbf{H}|^2$ has the unit of H m^{-1} times A^2 m^{-2} = J m^{-3}, where H is henry, A is ampere, and J is joule. Therefore, it has the unit of energy density. We can also ascertain the unit of $\frac{1}{2}\mu|\mathbf{H}|^2$ easily by noticing that the energy stored in an inductor is $\frac{1}{2}LI^2$ which is in terms of joules, and is due to henry times A^2.

Also $W_e = \frac{1}{2}\varepsilon|\mathbf{E}|^2$ has the unit of F m^{-1} times V^2 m^{-2} = J m^{-3} where F is farad, V is voltage, and J is joule, which is energy density again. We can ascertain the unit of $\frac{1}{2}\varepsilon|\mathbf{E}|^2$ easily by noticing that the energy stored in a capacitor is $\frac{1}{2}CV^2$ which has the unit of joules, and is due to farad times V^2.

The vector quantity

$$\mathbf{S}_p = \mathbf{E} \times \mathbf{H} \tag{2.12}$$

is called the Poynting's vector, and (2.10) becomes

$$\nabla \cdot \mathbf{S}_p = -\frac{\partial}{\partial t}W_t \tag{2.13}$$

where $W_t = W_e + W_m$ is the total energy density stored. The above is similar to the current continuity equation mentioned above. Analogous to that current density is charge density flow, power density is energy density flow.

Now, if we let $\sigma \neq 0$, then the term to be included is then $\sigma \mathbf{E} \cdot \mathbf{E} = \sigma|\mathbf{E}|^2$ which has the unit of S m^{-1} times V^2 m^{-2}, or W m^{-3} where S is siemens. We gather this unit by noticing that $\frac{1}{2}\frac{V^2}{R}$ is the power dissipated in a resistor of R ohms with a unit of watts. The reciprocal unit of ohms, which used to be mhos is now siemens. With $\sigma \neq 0$, (2.13) becomes

$$\nabla \cdot \mathbf{S}_p = -\frac{\partial}{\partial t}W_t - \sigma|\mathbf{E}|^2 = -\frac{\partial}{\partial t}W_e - P_d \tag{2.14}$$

Here, $\nabla \cdot \mathbf{S}_p$ has physical meaning of power density oozing out from a point, and $P_d = \sigma|\mathbf{E}|^2$ has the physical meaning of power density dissipated (siphoned) at a point by the conductive loss in the medium which is proportional to $\sigma|\mathbf{E}|^2$.

Now if we set $\mathbf{J}_i$ and $\mathbf{M}_i$ to be nonzero, (2.14) is augmented by the last two terms in (2.6), or

$$\nabla \cdot \mathbf{S}_p = -\frac{\partial}{\partial t}W_t - P_d - \mathbf{H} \cdot \mathbf{M}_i - \mathbf{E} \cdot \mathbf{J}_i \tag{2.15}$$

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The last two terms can be interpreted as the power density supplied by the impressed currents $\mathbf{M}_i$ and $\mathbf{J}_i$. Hence, (2.15) becomes

$$\nabla \cdot \mathbf{S}_p = -\frac{\partial}{\partial t} W_t - P_d + P_s \qquad (2.16)$$

where

$$P_s = -\mathbf{H} \cdot \mathbf{M}_i - \mathbf{E} \cdot \mathbf{J}_i \qquad (2.17)$$

where P_s is the power supplied by the impressed current sources. The last term is positive if $\mathbf{E}$ and $\mathbf{J}_i$ have opposite signs. This reminds us of a battery where the positive charges move from a region of lower potential to a region of higher potential. The positive charges move from one end of a battery to the other end of the battery. Hence, they are doing an "uphill climb" due to chemical processes within the battery.
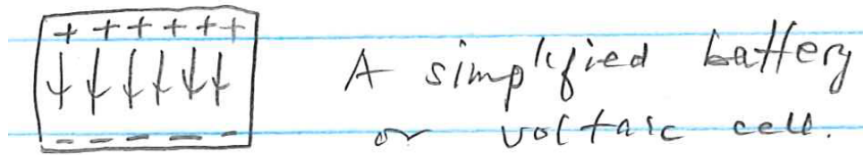

Figure 1:

In the above, one can easily work out that P_s has the unit of W m^{-3} which is power supplied density. One can also choose to rewrite (2.16) in integral form by integrating it over a volume V and invoking the divergence theorem yielding

$$\int_S d\mathbf{S} \cdot \mathbf{S}_p = -\frac{d}{dt} \int_V W_t dV - \int_V P_d dV + \int_V P_s dV \qquad (2.18)$$

The left-hand side is

$$\int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{H}) \qquad (2.19)$$

which represents the power flowing out of the surface S.

3 Emergence of Wave Phenomenon, Triumph of Maxwell's Equations

One of the major triumphs of Maxwell's equations is the prediction of the wave phenomenon. This was experimentally verified by Heinrich Hertz in 1888, some

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23 years after the completion of Maxwell's theory. Then it was realized that electromagnetic wave propagates at a tremendous velocity which is the velocity of light. This was also the defining moment which revealed that the field of electricity and magnetism and the field of optics were both described by Maxwell's equations or electromagnetic theory.

To see this, we consider the first two Maxwell's equations in vacuum or a source-free medium.[1] They are

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{3.1}$$

$$\nabla \times \mathbf{H} = -\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{3.2}$$

Taking the curl of (3.1), we have

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \frac{\partial}{\partial t} \nabla \times \mathbf{H} \tag{3.3}$$

It is understood that in the above, the double curl operator implies $\nabla \times (\nabla \times \mathbf{E})$. Substituting (3.2) into (3.3), we have

$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E} \tag{3.4}$$

Furthermore, using the identity that

$$\nabla \times \nabla \times \mathbf{E} = \nabla \nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} \tag{3.5}$$

and that $\nabla \cdot \mathbf{E} = 0$ in a source-free medium, we have

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E} = 0 \tag{3.6}$$

To see the simplest form of wave emerging in the above, we can let $\mathbf{E} = \hat{x} E_x(z, t)$ so that $\nabla \cdot \mathbf{E} = 0$ satisfying the source-free condition. Then (3.6) becomes

$$\frac{\partial^2}{\partial z^2} E_x(z, t) - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} E_x(z, t) = 0 \tag{3.7}$$

Eq. (3.7) is known mathematically as the wave equation. It can also be written as

$$\frac{\partial^2}{\partial z^2} f(z, t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} f(z, t) = 0 \tag{3.8}$$

where $c_0^2 = (\mu_0 \varepsilon_0)^{-1}$. Eq. (3.8) can also be factorized as

$$\left(\frac{\partial}{\partial z} - \frac{1}{c_0} \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial z} + \frac{1}{c_0} \frac{\partial}{\partial t} \right) f(z, t) = 0 \tag{3.9}$$

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[1]Since the third and the fourth Maxwell's equations are derivable from the first two.

or

$$\left(\frac{\partial}{\partial z} + \frac{1}{c_0}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} - \frac{1}{c_0}\frac{\partial}{\partial t}\right)f(z,t) = 0 \tag{3.10}$$

The above implies that we have

$$\left(\frac{\partial}{\partial z} + \frac{1}{c_0}\frac{\partial}{\partial t}\right)f_+(z,t) = 0 \tag{3.11}$$

or

$$\left(\frac{\partial}{\partial z} - \frac{1}{c_0}\frac{\partial}{\partial t}\right)f_-(z,t) = 0 \tag{3.12}$$

Equation (3.11) and (3.11) are known as the one-way wave equations or advective equations. From the above factorization, it is seen that the solutions of these one-way wave equations are also the solutions of the original wave equation given by (3.8). Their general solutions are then

$$f_+(z,t) = F_+(z - c_0 t) \tag{3.13}$$
$$f_-(z,t) = F_-(z + c_0 t) \tag{3.14}$$

Eq. (3.13) constitutes a right-traveling wave function of any shape while (3.14) constitutes a left-traveling wave function of any shape. Since Eqs. (3.13) and (3.14) are also solutions to (3.8), we can write the general solution to the wave equation as

$$f(z,t) = F_+(z - c_0 t) + F_-(z + c_0 t) \tag{3.15}$$

This is a wonderful result since F_+ and F_- are arbitrary functions of any shape (see Figure 2); they can be used to encode information for communication!
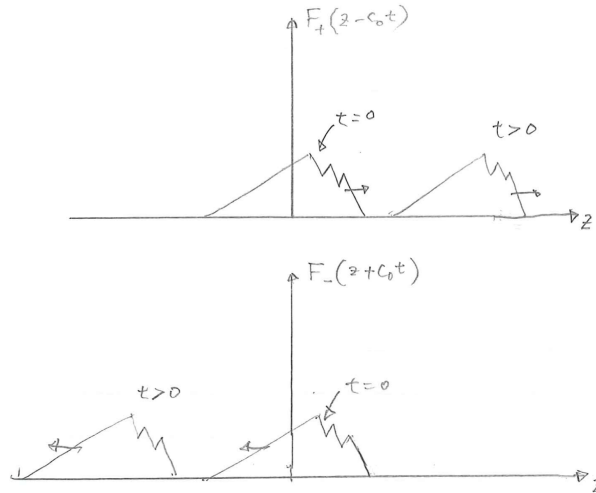

Figure 2:

Furthermore, one can calculate the velocity of this wave to be

$$c_0 = 299,792,458 m/s \simeq 3 \times 10^8 m/s \qquad (3.16)$$

where $c_0 = \sqrt{1/\mu_0 \varepsilon_0}$. Since there is only one independent constant in the wave equation, the value of μ_0 is defined to be $4\pi \times 10^{-7}$ henry m^{-1}, while the value of ε_0 has been measured to be about 8.854×10^{-12} farad m^{-1}. Now it has been decided that the velocity of light is defined to be the integer given in (3.16). A meter is defined to be the distance traveled by light in $1/(299792458)$ seconds. Hence, the more accurate that unit of time or second can be calibrated, the more accurate can we calibrate the unit of length or meter.