ECE 604, Lecture 8

September 20, 2018

1 Introduction

In this lecture, we will cover the following topics:

- Maxwell's Equations
- Fictitious Magnetic Currents
- Boundary Conditions
- Inductance

Additional Reading:

- Sections 2.2, 2.3, 2.4, 2.5, 2.13, 2.14, 2.16, 2.17, Ramo et al.

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2 Maxwell's Equations

Maxwell's equations were completed in 1865, and they have a tremendous impact in our modern world. We first write down their integral forms:[1]

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S}, \quad \text{Faraday's Law} \tag{2.1}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \frac{d}{dt} \iint_S \mathbf{D} \cdot d\mathbf{S} + I, \quad \text{Ampere's Law} \tag{2.2}$$

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = Q, \quad \text{Coulomb's Law} \tag{2.3}$$

$$\oiint_S \mathbf{B} \cdot d\mathbf{S} = 0,, \quad \text{Gauss's Law} \tag{2.4}$$

The basic quantities with their units are

$$\mathbf{E} : \text{V/m}, \qquad \mathbf{H} : \text{A/m} \tag{2.5}$$

$$\mathbf{D} : \text{C/m}^2, \qquad \mathbf{B} : \text{W/m}^2 \tag{2.6}$$

$$I : \text{A}, \qquad Q : \text{C} \tag{2.7}$$

where V is voltage, A is ampere, C is coulomb, and W is weber in the above. We can convert the above integral forms into partial differential equation form by using Stokes' theorem and Gauss' theorem as we did before. For example, using Stokes' theorem, we can write

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} \tag{2.8}$$

Therefore, Faraday's law (2.1) becomes

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S} = \iint_S \left(-\frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S} \tag{2.9}$$

where we have exchanged the order of total time derivative and the integral, assuming that the surface integral does not change with time. After the exchange, the total time derivative becomes a partial time derivative, since $\mathbf{B}$ is a function of both space and time. In the limit when the area $S \to 0$ above becomes a point-wise relationship. In other words,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.10}$$

which is Faraday's law in its full glory. This equation was experimentally derived in 1831, during the height of the age of telegraphy.

We can apply the same treatment to Ampere's law to get

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{2.11}$$

[1] The textbook uses partial time derivatives, but total time derivative is more appropriate.

The above is sometimes called the generalized Ampere's law as the original Ampere's law does not have the second term. The term $\frac{\partial \mathbf{D}}{\partial t}$ was the contribution of James Clerk Maxwell in 1865, known as the displacement current.

Similarly, we can apply Gauss' divergence theorem to get

$$\nabla \cdot \mathbf{D} = \varrho \tag{2.12}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.13}$$

Equations (2.10) to (2.13) constitute the four fundamental equations of electromagnetic theory, now known as Maxwell's equations. Maxwell, in addition to adding the extra term $\frac{\partial \mathbf{D}}{\partial t}$, the displacement current, to generalized Ampere's law, was the first to write these equations down lucidly and mathematically. Also the present form of what textbooks call Maxwell's equations, (2.10) to (2.13), were actually written down by his admirer, Oliver Heaviside.[2] Maxwell himself first wrote down electromagnetic theory using vector potential $\mathbf{A}$ and scalar potential Φ. We will learn more above vector and scalar potential formulation of electromagnetic theory later.

3 Fictitious Magnetic Currents

Even though magnetic charges or monopoles do not exist, magnetic dipoles do. For instance, a magnet can be regarded as a magnetic dipole. Also, it is believed that electrons have spins, and these spins make electrons behave like tiny magnetic dipoles in the presence of a magnetic field.

Also if we form electric current into a loop, it produces a magnetic field that looks like the electric field of an electric dipole. Hence, a magnetic dipole can be made using a small electric current loop (see Figure 1).

[2] Quote by Heaviside: "I remember my first look at the great treatise of Maxwell's when I was a young man. I saw that it was great, greater and greatest, with prodigious possibilities in its power. I was determined to master the book and set to work. I was very ignorant. I had no knowledge of mathematical analysis (having learned only school algebra and trigonometry which I had largely forgotten) and thus my work was laid out for me. It took me several years before I could understand as much as I possibly could. Then I set Maxwell aside and followed my own course. And I progressed much more quickly. It will be understood that I preach the gospel according to my interpretation of Maxwell." See also Yaghjian, PIER, vol. 149, pp. 217-249, 2015.

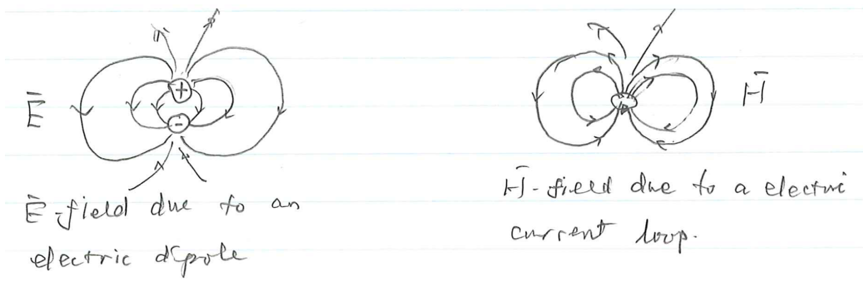

Figure 1:

Because of these similarities, it is common to introduce fictitious magnetic charges and magnetic currents into Maxwell's equations. One can think that these magnetic charges always occur in pair and together. Thus, they do not contradict the absence of magnetic monopole.

Hence, Maxwell's equation can be alternatively written as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{M} \tag{3.1}$$

$$\nabla \times \mathbf{H} = -\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{3.2}$$

$$\nabla \cdot \mathbf{D} = \varrho \tag{3.3}$$

$$\nabla \cdot \mathbf{B} = \varrho_m \tag{3.4}$$

where $\mathbf{M}$ is the magnetic current density, while ϱ_m is the magnetic charge density. By so doing, Maxwell's equation also become more symmetrical mathematically.

When one takes the divergence of (3.2), one gets

$$\nabla \cdot (\nabla \times \mathbf{H}) = -\frac{\partial}{\partial t} \nabla \cdot \mathbf{D} + \nabla \cdot \mathbf{J} \tag{3.5}$$

Since $\nabla \cdot (\nabla \times \mathbf{H}) = 0$, and that $\nabla \cdot \mathbf{D} = \varrho$, the above is the same as

$$0 = \frac{\partial}{\partial t} \varrho + \nabla \cdot \mathbf{J} \tag{3.6}$$

which is the current continuity equation. In other words, one says that Maxwell's equations are consistent with charge conservation since the current continuity equation is a statement of charge conservation.

Alternatively, we can say that Gauss' law, $\nabla \cdot \mathbf{D} = \varrho$, is derivable from Ampere's law, $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$ if we invoke charge conservation and current continuity. In this view point, one can say that the third and the fourth Maxwell's equation are derivable from the first two, unless $\frac{\partial}{\partial t} = 0$, or for electrostatics.

By converting the current continuity equation (3.5) into integral form, we have

$$\iiint_V \nabla \cdot \mathbf{J} dV = -\iiint_V \frac{\partial \varrho}{\partial t} dV \tag{3.7}$$

Using Gauss' divergence theorem, we can rewrite the above as

$$\oiint_S d\mathbf{S} \cdot \mathbf{J} = -\frac{d}{dt} Q \tag{3.8}$$

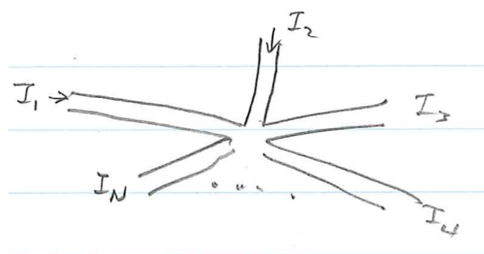

Figure 2:

When we apply the above to many wires going into a junction, and assuming that there is no charge accumulation at the junction such that $\frac{dQ}{dt} \approx 0$, then we have

$$\oiint_S d\mathbf{S} \cdot \mathbf{J} \approx 0 \tag{3.9}$$

Applying the above to the wire-junction problem shown in Figure 2, where the integration of the current density over the closed surface S will give rise to the sum of currents flowing out of this surface, we arrive at

$$\sum_{i=1}^{N} I_i = 0 \tag{3.10}$$

which is Kirchhoff's current law.

Also, if we take Faraday's law, assuming that $\mathbf{M} = 0$, and integrating it over a closed loop, we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{3.11}$$

When the quantities are slowly varying with time such that $\frac{d}{dt} \approx 0$, or $\int_S \mathbf{B} \cdot d\mathbf{S}$ is small, we have

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \tag{3.12}$$

which is the electrostatic form of Faraday's law.

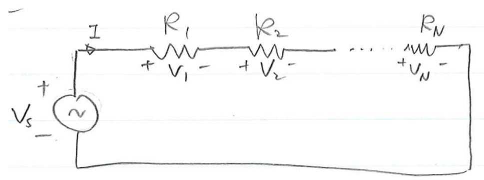

Figure 3:

If we integrate this Faraday's law around a closed loop as shown in Figure 3, and noting that $\int_a^b \mathbf{E} \cdot d\mathbf{l} = -V_{ba}$, the voltage difference from point a to point b, we get that

$$-V_s + V_1 + \cdots + V_n = -V_s + \sum_{i=1}^{N} V_i = 0 \tag{3.13}$$

or

$$V_s = \sum_{i=1}^{N} V_i \tag{3.14}$$

The above is just Kirchhoff voltage law.

4 Boundary Conditions

Given our experience in deriving boundary conditions before, from (3.1), assuming a magnetic current sheet $\mathbf{M}_s$ at an interface, we have from Faraday's law that

$$\hat{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = -\mathbf{M}_s \tag{4.1}$$

For $\mathbf{M}_s = 0$, we retrieve the previously derived boundary condition that

$$\hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{4.2}$$

or that tangential $\mathbf{E}$ is continuous. From Ampere's law, we have that

$$\hat{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = -\mathbf{J}_s \tag{4.3}$$

or that the jump discontinuity in the tangential $\mathbf{H}$ field is due to the presence of a surface current sheet $\mathbf{J}_s$. For electric flux $\mathbf{D}$, we have

$$\hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \varrho_s \tag{4.4}$$

or that the jump discontinuity in the normal flux $\mathbf{D}$ is due to the presence of a surface electric charge density ϱ_s. For magnetic flux $\mathbf{B}$, we have

$$\hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = \varrho_{ms} \tag{4.5}$$

or that the jump discontinuity in the normal flux $\mathbf{B}$ is due to the presence of a surface magnetic charge density ϱ_{ms}. For $\varrho_{ms} = 0$, then $\hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0$, a previously derived boundary condition.

We have seen that for electrostatics, a conductive medium alone can expel or screen the electric field away from a conductive region by re-orientation of the electric charges. However, for electrodynamics, a conductive medium alone is not sufficient to expel the electric field completely from the conductor. Imaging a time-varying electric dipole placed next to a conductor as shown in Figure 4. When the dipole's electric field varies, the charges on the surface of the conductor will vary as well, as charges on the dipole will attract charges of opposite polarity on the surface of the conductor. Hence, there has to be a current flowing in the conductor to maintain this constant re-orientation of the charges on the conductor. Hence, the time-varying electric field cannot be zero inside the conductor.

To expel the electric field completely from a time-varying field excitation, one needs a perfect electric conductor with $\sigma \to \infty$. The mere presence of non-zero electric field inside the perfect conductor gives rise to an infinite current, which is not possible nor physical. Hence, the electric field has to be expelled from a perfect conductor or a superconductor.

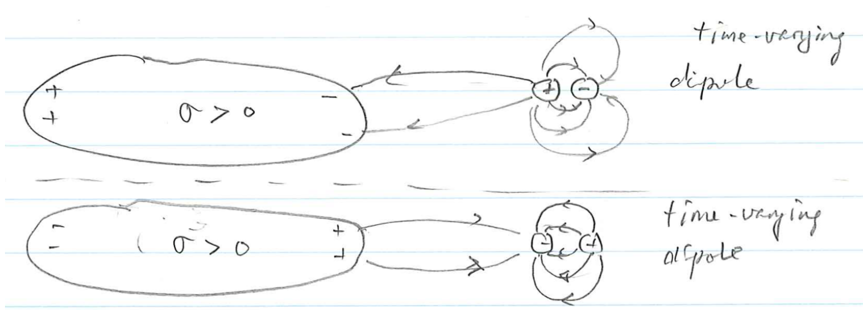

Figure 4:

We have seen that in magnetostatics, one needs a perfect conductor or super-conductor to completely expel the magnetic field. This is again necessary for a time-varying magnetic field.

In summary, on a PEC surface, $\hat{n} \times \mathbf{E} = 0$ due to the absence of electric field inside the conductor. The continuity of the tangential component of the electric field ensures that tangential electric field is zero on a PEC surface.

For the magnetic field, the normal component of magnetic field is continuous, since there are no magnetic monopole charges on the surface of the PEC. But the magnetic field inside the PEC is zero. Therefore, $\hat{n} \cdot \mathbf{B} = 0$ on a PEC surface, since magnetic monopole charges are absent (see Figure 5).

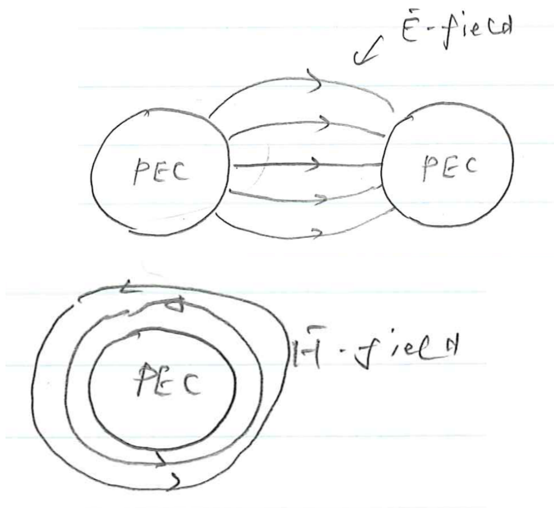

Figure 5:

5 Inductance

Inductance is a concept that follows from time-varying Faraday's law which is

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{5.1}$$

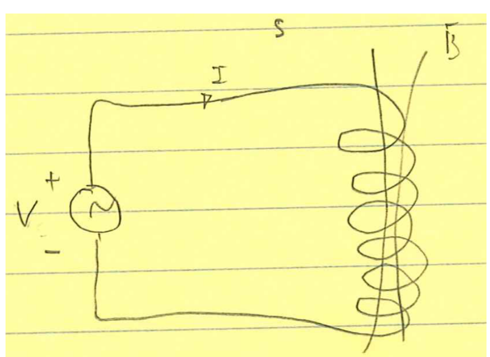

Figure 6:

If we take C to be a closed contour integral along the conducting wire, then $\mathbf{E} = 0$ inside the wire, and

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -V \tag{5.2}$$

The flux linkage due to the $\mathbf{B}$ field, is

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} \tag{5.3}$$

where S is a rather convoluted surface enclosed by C. Then (5.1) becomes

$$V = \frac{d}{dt}\Psi \tag{5.4}$$

But Ψ is linearly proportional to I: The stronger the current, the stronger is the $\mathbf{B}$ field generated by I and the stronger would be the flux linkage according to (5.3). Hence $\Psi = LI$, and the constant of proportionality is L, the inductance. Thus

$$V = L\frac{dI}{dt} \tag{5.5}$$

The above is just the I-V relation of an inductor.

6 Finding L, the inductance for a Solenoid

For a solenoid, $H_z = \frac{NI}{l}$. This formula is derived in the textbook, but can be easily derived with the following assumption. Imagine a solenoid as shown in Figure 7. Assume that the $\mathbf{H}$ field, which is $\hat{z}H_z$, is completely trapped inside the solenoid so that there is no field outside the solenoid. Then taking a contour integral of Ampere's law around the contour C, the only contribution is from the non-zero field inside the solenoid pointing in the z direction where z is the axis of the solenoid. Thus

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = lH_z = nlI \tag{6.1}$$

where nlI is the total current passing through the surface enclosed by C, and n is the number of turns per unit length. From the above, one gets $H_z = nI$.

One calculates the flux linkage by performing the integral $\int_S \mathbf{B} \cdot d\mathbf{S}$. Each loop contributes a flux linkage of $\Psi = \mu H_z A$ and its time variation produces a voltage or a voltage source. One can think of these voltage sources produced by the loops add in series and thus, the N turns will contribute a total flux of $\mu H_z AN$. Therefore,

$$\int_S \mathbf{B} \cdot d\mathbf{S} = \mu H_z AN = \mu nNAI \tag{6.2}$$

Using that $n \approx N/l$, ignoring fringing field effect, one gets

$$L = \frac{\mu N^2 A}{l} \tag{6.3}$$

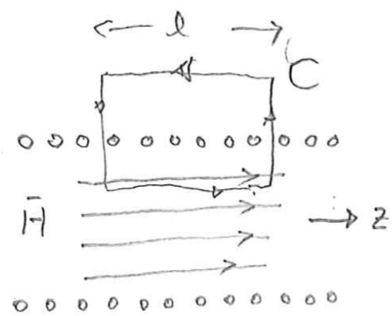

Figure 7: