ECE 604, Lecture 7

September 18, 2018

1 Introduction

In this lecture, we will cover the following topics:

- Scalar Magnetic Potential Φ_m
- Boundary Conditions
- Magnetic Energy Density
- Energy Stored in an Inductor

Additional Reading:

- Sections 2.5, 2.13–2.17, Ramo et al.

Printed on September 25, 2018 at 15 : 55: W.C. Chew and D. Jiao.

2 Scalar Magnetic Potential Φ_m

In a source-free region, Ampere's law becomes

$$\nabla \times \mathbf{H} = 0 \tag{2.1}$$

Hence, we can also use the identity that $\nabla \times \nabla \Phi_m = 0$ to suggest that

$$\mathbf{H} = -\nabla \Phi_m \tag{2.2}$$

where Φ_m is analogous to the electric scalar potential. There we have let $\mathbf{E} = -\nabla \Phi$ so that $\nabla \times \mathbf{E} = 0$. Thus Φ_m here is called the magnetic scalar potential.

From Gauss's law for the magnetic field such that

$$\nabla \cdot \mathbf{B} = 0 \tag{2.3}$$

and that $\mathbf{B} = \mu \mathbf{H}$ from the constitutive relation, then we arrive at the generalized Laplace's equation

$$\nabla \cdot \mu \nabla \Phi_m = 0 \tag{2.4}$$

For a homogenous medium where μ is independent of position, then $\nabla \cdot \mu \nabla \Phi_m = \mu \nabla \cdot \nabla \Phi_m = \mu \nabla^2 \Phi_m$. Thus,

$$\nabla^2 \Phi_m = 0 \tag{2.5}$$

which is the simple Laplace's equation

Ampere's law when a current source $\mathbf{J}$ is present is given by

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{2.6}$$

Next, we investigate the kind of boundary condition induced by such an equation.

3 Boundary Condition for H

To derive the boundary condition induced by the partial differential equation (2.6), we can use the conventional method described by most textbooks by integrating (2.6) over a small loop straddling the interface between two media μ_1 and μ_2. This is illustrated by Prof. Dan Jiao's lecture notes.

Alternatively, we can use the unconventional methods by projecting the partial differential equation onto a local orthogonal coordinate system at the interface as shown in Figure 1

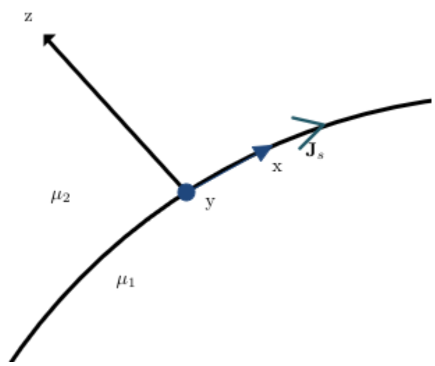

Figure 1: My student helper has not drawn the local coordinates to be that orthogonal.

To be general, we also include the presence of a current sheet at the interface. Rewriting (2.6) in a local coordinate system, assuming that $\mathbf{J} = \hat{x} J_{sx} \delta(z)$, then

$$\nabla \times \mathbf{H} = \hat{x} \left(\frac{\partial}{\partial y} H_z - \frac{\partial}{\partial z} H_y \right) = \hat{x} J_{sx} \delta(z) \tag{3.1}$$

From the above, a current sheet, or a surface current density becomes a delta function singularity when expressed as a volume current density; hence, we have the form of the right-hand side of the above equation. From the above, the only term that can produce a $\delta(z)$ singularity on the left-hand side is the $-\frac{\partial}{\partial z} H_y$ term. Therefore, we conclude that

$$-\frac{\partial}{\partial z} H_y = J_{sx} \delta(z) \tag{3.2}$$

In other words, H_y has to have a jump discontinuity at the interface where the current sheet resides. Or that

$$H_y(z = 0^+) - H_y(z = 0^-) = -J_{sx} \tag{3.3}$$

The above implies that

$$H_{2y} - H_{1y} = -J_{sx} \tag{3.4}$$

But H_y is just the tangential component of the $\mathbf{H}$ field. Now if we repeat the same exercise with $\mathbf{J} = \hat{y} J_{sy} \delta(z)$, at the interface, we have

$$H_{2x} - H_{1x} = J_{sy} \tag{3.5}$$

Now, (3.4) and (3.5) can be rewritten using a cross product as

$$\hat{z} \times (\hat{y} H_{2y} - \hat{y} H_{1y}) = \hat{x} J_{sx} \tag{3.6}$$

$$\hat{z} \times (\hat{x} H_{2x} - \hat{x} H_{1x}) = \hat{y} J_{sy} \tag{3.7}$$

The above two equations can be combined as one to give

$$\hat{z} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s \tag{3.8}$$

Taking $\hat{z} = \hat{n}$ in general, we have

$$\hat{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s \tag{3.9}$$

4 Boundary Condition for B

The **B** field satisfies the partial differential equation given by (2.3). It induces a different boundary condition for **B**.

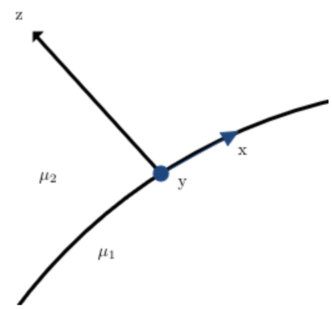

Figure 2: Not quite orthogonal

Again, writing Gauss's law using a local coordinates shown in Figure 2, we have

$$\frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z = 0 \tag{4.1}$$

Since there are no medium discontinuities in the x and y directions, there cannot be jump discontinuity for B_x and B_y. The only possibility for discontinuity is on B_z since it goes through a medium interface. Focussing on B_z alone, the normal component, it implies that

$$\frac{\partial}{\partial z}B_z = \text{finite} \tag{4.2}$$

since there cannot be infinities coming from the other terms in the above equation. Integrating the above across the interface, over an infinitesimal length, we have

$$B_z(z = 0^+) - B_z(z = 0^-) = 0 \tag{4.3}$$

or that

$$\hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \tag{4.4}$$

5 Perfect Conductor and Conductor

We have seen that for a finite conductor, as long as $\sigma \neq 0$, the charges will re-orient themselves until the electric field is expelled from the conductor; otherwise, the current will keep flowing. But there are no magnetic charges nor magnetic conductors in this world. So this physical phenomenon does not happen for magnetic field: in other words, magnetic field cannot be expelled from an electric conductor. However, a magnetic field is expelled from a perfect conductor or a superconductor. You can only fully understand this physical phenomenon if we study Maxwell's equations in their full glory or in their time-varying form.

In a perfect conductor where $\sigma \to \infty$, it is unstable for the magnetic field $\mathbf{B}$ to be nonzero. As time varying magnetic field gives rise to an electric field by the time-varying form of Faraday's law, a small time variation of the $\mathbf{B}$ field will give rise to infinite current flow in a perfect conductor. Therefore to avoid this ludicrous situation, and to be stable, $\mathbf{B} = 0$ in a perfect conductor or a superconductor.

So if medium 1 is a perfect electric conductor, then $\mathbf{B}_1 = \mathbf{H}_1 = 0$. The boundary conditions (3.9) and (4.4) become

$$\hat{n} \times \mathbf{H}_2 = \mathbf{J}_s \tag{5.1}$$

$$\hat{n} \cdot \mathbf{B}_2 = 0 \tag{5.2}$$

The $\mathbf{B}$ field is expelled from the perfect conductor, and there is no normal component of the $\mathbf{B}$ field as there cannot be magnetic charges, as shown in Figure 3.

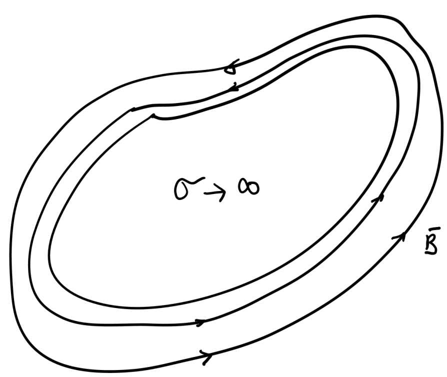

Figure 3:

6 Boundary Conditions for Φ_m

As mentioned before, from Gauss' law that $\nabla \cdot \mathbf{B} = 0$, and Ampere's law for a source-free region that $\nabla \times \mathbf{H} = 0$, we conclude that

$$\mathbf{H} = -\nabla \Phi_m \tag{6.1}$$

$$\nabla \cdot \mu \nabla \Phi_m = 0 \tag{6.2}$$

Here, (6.2) is the generalized Laplace's equation. From it, we can deduce the boundary condition that

$$\hat{n} \cdot \mu_1 \nabla \Phi_m = \hat{n} \cdot \mu_2 \nabla \Phi_{m2} \tag{6.3}$$

across an interface. The above is similar to

$$\mu_1 \frac{\partial}{\partial n} \Phi_{m1} = \mu_2 \frac{\partial}{\partial n} \Phi_{m2} \tag{6.4}$$

where $\hat{n} \cdot \nabla = \frac{\partial}{\partial n}$ is the normal derivative. Also, since $\nabla \Phi_m$ cannot be infinite, because $\nabla \cdot \mu \nabla \Phi_m = 0$, Φ_m must be a continuous function. Consequently, we have

$$\Phi_{m1} = \Phi_{m2} \tag{6.5}$$

at a medium interface.

7 Magnetic Energy Density

Unlike the electric energy stored where we can do a Gedanken experiment of moving an electric charge against an electric field as work done and use that to determine the energy stored in the electric field, no such Gedanken experiment exists for magnetic field. At this point, we have to derive the magnetic energy stored by analogy. Since the energy stored in the electric field is given by

$$U_E = \frac{1}{2}\int_V \mathbf{E}\cdot\mathbf{D}dV \tag{7.1}$$

by the same token, the energy stored in the magnetic field is

$$U_H = \frac{1}{2}\int_V \mathbf{H}\cdot\mathbf{B}dV \tag{7.2}$$

Only when we study Maxwell's equation in its full form can we prove the physical meaning of the above.

For an isotropic medium when $\mathbf{B} = \mu\mathbf{H}$, then the above becomes

$$U_H = \frac{1}{2}\int_V \mu\mathbf{H}\cdot\mathbf{H}dV = \frac{1}{2}\int_V \mu|\mathbf{H}^2|dV \tag{7.3}$$

The above is consistent with the energy stored in an inductor which is

$$W_H = \frac{1}{2}LI^2 \tag{7.4}$$

8 Energy Stored in an Inductor

The energy stored in an inductor can be derived by considering the instantaneous power in an inductor which is given by

$$p = vi = Li\frac{di}{dt} \tag{8.1}$$

where we have used the fact that across an inductor, $v = L\frac{di}{dt}$. Integrating the above, we get

$$W_H = \int_0^t pdt = \int_0^I Li'di' = \frac{1}{2}LI^2 \tag{8.2}$$

Notice that (8.2) is proved by considering a time-varying system. Hence, (7.2) and (7.3) can be properly proved by considering a time-varying system, which will be described by the full form of Maxwell's equations.