ECE 604, Lecture 6

September 6, 2018

1 Introduction

In this lecture, we will cover the following topics:

- Lorentz Force Law
- Biot-Savart Law
- Ampere's Law
- Gauss's Law for Magnetic Field
- Magnetic Vector Potential
- Vector Poisson's Equation
- Derivation of Biot-Savart Law from Ampere's Law and Gauss's Law

Additional Reading:

- Sections 2.2, 2.3, 2.4, 2.6–2.9, 2.11–2.12, Ramo et al.

Printed on September 18, 2018 at 16 : 30: W.C. Chew and D. Jiao.

2 Lorentz Force Law

The Lorentz force law is given by

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{2.1}$$

The first term is electric force from Coulomb's law while the second term is the magnetic force also called the $\mathbf{v} \times \mathbf{B}$ force. The magnetic force can also be written for an incremented current flowing in the wire of length $\mathbf{dl}$, or

$$\mathbf{dF} = I\mathbf{dl} \times \mathbf{B} \tag{2.2}$$

3 Biot-Savart Law

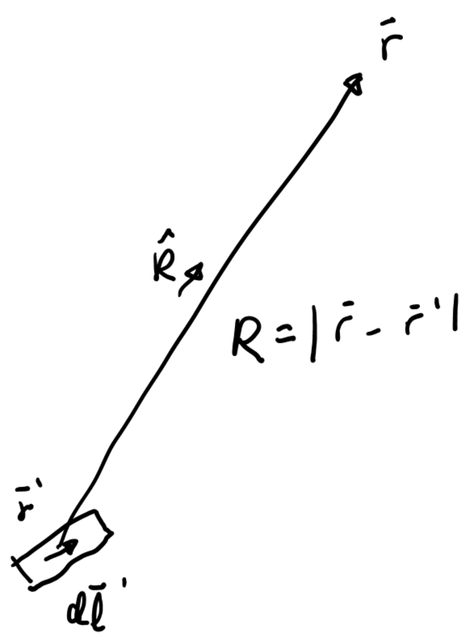

Figure 1:

Biot-Savart law states that the incremental magnetic field due to an incremental current, as shown in Figure 1, is

$$\mathbf{dH} = \frac{I\mathbf{dl} \times \mathbf{R}}{4\pi R^2} \tag{3.1}$$

where

$$R = |\mathbf{r} - \mathbf{r}'| \tag{3.2}$$

This law was first experimentally derived. But we will give a mathematical derivation of it later.

4 Ampere's Law

Ampere's law in integral form says that

$$\oint_C \mathbf{H} \cdot \mathbf{dl} = I \tag{4.1}$$

Using Stoke's theorem, one rewrites the left-hand side of the above as

$$\oint_C \mathbf{H} \cdot \mathbf{dl} = \iint_S (\nabla \times \mathbf{H}) \cdot \mathbf{dS} \tag{4.2}$$

But the right-hand side of the (4.1) can be written as

$$I = \iint_S \mathbf{J} \cdot \mathbf{dS} \tag{4.3}$$

Therefore

$$\iint_S (\nabla \times \mathbf{H}) \cdot \mathbf{dS} = \iint_S \mathbf{J} \cdot \mathbf{dS} \tag{4.4}$$

When $S \to 0$, the above implies that

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{4.5}$$

5 Gauss's Law—Magnetic

Gauss's law for magnetic field says that

$$\oiint_S \mathbf{B} \cdot \mathbf{dS} = 0 \tag{5.1}$$

But from Gauss's divergence theorem,

$$\oiint \mathbf{B} \cdot \mathbf{dS} = \iiint_V \nabla \cdot \mathbf{B} dV \tag{5.2}$$

Therefore

$$\iiint_V \nabla \cdot \mathbf{B} dV = 0 \tag{5.3}$$

When $V \to 0$, we have

$$\nabla \cdot \mathbf{B} = 0$$

which is the partial differential equation for Gauss' law.

6 Constitutive Relation

The constitutive relation between magnetic flux **B** and magnetic field **H** is given as

$$\mathbf{B} = \mu \mathbf{H}, \qquad \mu = \text{permeability H/m} \tag{6.1}$$

In free space,

$$\mu = \mu_0 = 4\pi \times 10^{-7} \text{ H/m} \tag{6.2}$$

In other materials, the permeability can be written as

$$\mu = \mu_0 \mu_r \tag{6.3}$$

Similarly, the permittivity for electric field can be written as

$$\varepsilon = \varepsilon_0 \varepsilon_r \tag{6.4}$$

7 Magnetic Vector Potential A

Since from Gauss's law

$$\nabla \cdot \mathbf{B} = 0 \tag{7.1}$$

we can let

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{7.2}$$

because

$$\nabla \cdot \nabla \times \mathbf{A} = 0 \tag{7.3}$$

This is similar to

$$\nabla \times \nabla \Phi = 0 \tag{7.4}$$

In this manner, Gauss's law is automatically satisfied.

8 Derivation of the Vector Poisson's Equation

From

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{8.1}$$

we have

$$\nabla \times \left(\frac{\mathbf{B}}{\mu} \right) = \mathbf{J} \tag{8.2}$$

Then using (7.2)

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J} \tag{8.3}$$

In a homogeneous medium, μ is a constant and hence

$$\nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{A} \tag{8.4}$$

We use the vector identity that (see handout on Some Useful Formulas)

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A} \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \end{aligned} \tag{8.5}$$

As a result, we arrive at

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} \tag{8.6}$$

However, $\mathbf{A}$ in (7.2) is not unique because one can always define

$$\mathbf{A}' = \mathbf{A} - \nabla\Psi \tag{8.7}$$

Then

$$\nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} - \nabla\Psi) = \nabla \times \mathbf{A} = \mathbf{B} \tag{8.8}$$

where we have made use of that $\nabla \times \nabla\Psi = 0$. Hence, the $\nabla\times$ of both $\mathbf{A}$ and $\mathbf{A}'$ produce the same $\mathbf{B}$.

To find $\mathbf{A}$ properly, we have to define or set the divergence of $\mathbf{A}$ or provide a gauge condition. One way is to set the divergence of $\mathbf{A}$ is to let

$$\nabla \cdot \mathbf{A} = 0 \tag{8.9}$$

This gauge condition is also known as Coulomb's gauge. Then

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} - \nabla^2\Psi \neq \nabla \cdot \mathbf{A} \tag{8.10}$$

The last non-equal sign follows if $\nabla^2\Psi \neq 0$. If we stipulate that $\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} = 0$, then $-\nabla^2\Psi = 0$. This does not necessary imply that $\Psi = 0$, but if we impose that condition that $\Psi \to 0$ when $\mathbf{r} \to \infty$, then $\Psi = 0$ everywhere. By so doing, $\mathbf{A}$ and $\mathbf{A}'$ are equal to each other, and we obtain

$$\nabla^2\mathbf{A} = -\mu\mathbf{J} \tag{8.11}$$

In cartesian coordinates, the above can be viewed as three scalar Poisson's equations. Each of the Poisson's equation can be solved using the Green's function method. Consequently, in free space

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r})}{R} dV' \tag{8.12}$$

where

$$R = |\mathbf{r} - \mathbf{r}'| \tag{8.13}$$

9 Derivation of Biot-Savart Law

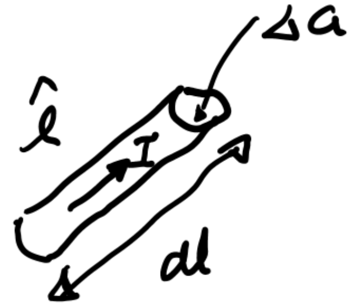

Figure 2:

From Gauss' law and Ampere's law, we have derived that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r})}{R} dV' \tag{9.1}$$

When the current element is small, and is carried by a wire of cross sectional area Δa as shown in Figure 2, we can approximate the integrand as

$$\mathbf{J}(\mathbf{r}')dV' \approx \mathbf{J}(\mathbf{r}')\Delta V' = \underbrace{(\Delta a)\Delta l}_{\Delta V}\, \underbrace{\hat{l}I/\Delta a}_{\mathbf{J}(\mathbf{r}')} \tag{9.2}$$

In the above, $\Delta V = (\Delta a)\Delta l$ and $\hat{l}I/\Delta a = \mathbf{J}(\mathbf{r}')$. Here, $\hat{l}$ is a unit vector pointing in the direction of the current flow. Hence, we can let

$$\mathbf{J}(\mathbf{r}')dV' \approx I\Delta\mathbf{l} \tag{9.3}$$

where $\Delta\mathbf{l} = \Delta l\hat{l}$. Therefore, the incremental vector potential due to an incremental current is

$$\mathbf{dA}(\mathbf{r}) \approx \frac{\mu}{4\pi}\left(\frac{\mathbf{J}(\mathbf{r}')\Delta V'}{R}\right) = \frac{\mu}{4\pi}\frac{I\Delta\mathbf{l}'}{R} \tag{9.4}$$

Since $\mathbf{B} = \nabla \times \mathbf{A}$, we have

$$\mathbf{dB} = \nabla \times \mathbf{dA}(\mathbf{r}) \cong \frac{\mu I}{4\pi}\nabla \times \frac{\Delta\mathbf{l}'}{R} = \frac{-\mu I}{4\pi}\Delta\mathbf{l}' \times \nabla\frac{1}{R} \tag{9.5}$$

where we have made use of the fact that $\nabla \times \mathbf{a} f(\mathbf{r}) = -\mathbf{a} \times \nabla f(\mathbf{r})$ when $\mathbf{a}$ is a constant vector (see one of the HW problems). The above can be simplified further by making use of the fact that

$$\nabla \frac{1}{R} = -\frac{1}{R^2} \hat{R} \tag{9.6}$$

where $\hat{R}$ is a unit vector pointing in the $\mathbf{r}-\mathbf{r}'$ direction. We have also made use of the fact that $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$. Consequently, assuming that the incremental length becomes very small, or $\Delta\mathbf{l} \rightarrow \mathbf{dl}$, we have, after using (9.6) in (9.5), that

$$\mathbf{dB} = \frac{\mu I}{4\pi} \mathbf{dl}' \times \frac{1}{R^2} \hat{R} \tag{9.7}$$

$$= \frac{\mu I \mathbf{dl}' \times \hat{R}}{4\pi R^2} \tag{9.8}$$

Since $\mathbf{B} = \mu \mathbf{H}$, we have

$$\mathbf{dH} = \frac{I \mathbf{dl}' \times \hat{R}}{4\pi R^2} \tag{9.9}$$

or

$$\mathbf{H}(\mathbf{r}) = \int \frac{I(\mathbf{r}') \mathbf{dl}' \times \hat{R}}{4\pi R^2} \tag{9.10}$$

which is Biot-Savart Law