ECE 604, Lecture 5

September 04, 2018

1 Introduction

In this lecture, we will cover the following topics:

- Green's Function Derivation from Coulomb's Law
- Green's Function Derivation from Partial Differential Equation
- Uniqueness Theorem

Additional Reading:

- Section 1.17, Ramo et al, Section 5.2, Kong

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2 Green's Function—From Coulomb's Law

A Green's function is a point source response to a partial differential equation (PDE). We can take the Poisson's equation as an illustration,

$$\nabla^2 \Phi(\mathbf{r}) = -\frac{\varrho((r)}{\varepsilon} \tag{2.1}$$

To find the Green's function, we let the right-hand side be a point source, and seek the solution of

$$\nabla^2 g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \tag{2.2}$$

Equations (2.1) and (2.2) are very similar except for the right-hand side. The solution to (2.2) is the solution due to a point source, and hence, is called the Green's function $g(\mathbf{r}, \mathbf{r}')$. We know the solution to (2.1) when the right-hand side is a point charge located at $r = \mathbf{r}'$, namely,

$$\varrho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}') \tag{2.3}$$

We know what $\Phi(\mathbf{r})$ is like for a point source: in this case from Coulomb's law, that is

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon|\mathbf{r} - \mathbf{r}'|} \tag{2.4}$$

By comparing (2.1) (2.2) (2.3) and (2.4), we conclude that the Green's function is

$$g(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} = g(\mathbf{r} - \mathbf{r}') \tag{2.5}$$

Since $g(\mathbf{r}, \mathbf{r}')$ only depends on $|\mathbf{r} - \mathbf{r}'|$, we can rewrite it as $g(\mathbf{r} - \mathbf{r}')$. Eq. (2.5) is known as the Green's function of the partial differential equation given by (2.1). Mathematicians also call the Green's function the fundamental solution, as it is of fundamental importance to the partial differential equation.

We can write a source $\varrho(\mathbf{r})$ as a linear superposition of point sources mathematically as

$$\varrho(\mathbf{r}) = \int_V dV' \varrho(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') \tag{2.6}$$

The right-hand side is a linear superposition of point sources. Can we also write $\Phi(\mathbf{r})$ then as linear superposition of point-source response? Or

$$\Phi(\mathbf{r}) = \int_V dV' \frac{\varrho(\mathbf{r})}{4\pi\varepsilon|\mathbf{r} - \mathbf{r}'|} \tag{2.7}$$

In other words,

$$\Phi(\mathbf{r}) = \frac{1}{\varepsilon} \int_V dV' \varrho(\mathbf{r}) g(\mathbf{r} - \mathbf{r}') \tag{2.8}$$

To confirm that (2.8) is a solution if (2.1), we can back substitute it into the left-hand side of (2.1), and exchange the order of differentiation and integration. That is

$$\nabla^2 \Phi(\mathbf{r}) = \frac{1}{\varepsilon} \int_V dV' \varrho(\mathbf{r}) \nabla^2 g(\mathbf{r} - \mathbf{r}') \tag{2.9}$$

Using (2.2) for the definition of $g(\mathbf{r} - \mathbf{r}')$, we in fact see that the right-hand side of (2.9) is in fact the right-hand side of (2.1), confirming that (2.8) is the solution of (2.1) for a general source $\varrho(\mathbf{r})$. Notice that (2.9) is a convolutional integral. In other words, the general solution to the Poisson's equation as given by (2.8) is obtained by convolving a general source with the point-source response $g(\mathbf{r} - \mathbf{r}')$, or the Green's function.

3 Green's Function from Partial Differential Equation

Instead of relying on Coulomb's law to determine the Green's function, it can be also derived mathematically from the relevant partial differential equation. To this end, we put the point source at the orgin such that (2.2) becomes

$$\nabla^2 g(\mathbf{r}) = -\delta(\mathbf{r}) \tag{3.1}$$

where we have let $\mathbf{r}' = 0$, calling $g(\mathbf{r})$ the solution to (3.1). Due to symmetry, $g(\mathbf{r})$ has to be spherically symmetric; hence $g(\mathbf{r}) = g(r)$. In this case, the Laplacian operator ∇^2 only depends on r but not on θ and ϕ in spherical coordinates. From lookup table, it is

$$\nabla^2 = \nabla \cdot \nabla = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \tag{3.2}$$

Away from $r \neq 0$, $\nabla^2 g(r) = 0$, and possible solution for $g(r)$ is

$$g(r) = \frac{C}{r} + D \tag{3.3}$$

However, we are seeking solutions such that $g(r) \to 0$, as $r \to \infty$; therefore $D = 0$, and

$$g(r) = \frac{C}{r} \tag{3.4}$$

But

$$\nabla \cdot \nabla \frac{C}{r} = -\delta(x)\delta(y)\delta(z) \tag{3.5}$$

Integrating the above about a small sphere of radius a around the origin, one gets, after invoking Gauss's divergence theorem that

$$\int_{\Delta S} dS \hat{n} \cdot \nabla \frac{C}{r} = -1 \tag{3.6}$$

where ΔS is the surface of a small sphere if radius a, moreover, $\hat{n} \cdot \nabla \frac{C}{r} = \frac{\partial}{\partial r}\frac{C}{r} = -\frac{C}{r^2}$. For a small sphere of radius a, the left-hand side of the above evaluates to a simple function, and the above equation becomes

$$-4\pi a^2 (\frac{C}{a^2}) = -1 \tag{3.7}$$

or that

$$C = \frac{1}{4\pi} \tag{3.8}$$

Hence,

$$g(r) = \frac{1}{4\pi r} \tag{3.9}$$

In general,

$$g(\mathbf{r} - \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{3.10}$$

only depends on $\mathbf{r} - \mathbf{r}'|$.

4 Uniqueness Theorem

This theorem says that if the boundary condition for solving a partial differential equation is stipulated, the solution is unique. Such a problem is also called a boundary value problem (BVP). For example, we wish to solve for $\Phi(\mathbf{r})$ of Poisson's equation with a source term $\varrho(\mathbf{r})$ in a certain volume bounded by a surface S (see Figure 1).

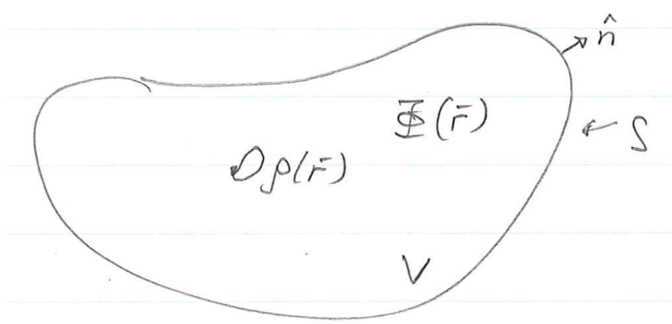

Figure 1:

The relevant boundary value problem is to solve the generalized Poisson's equation,

$$\nabla \cdot \varepsilon \nabla \Phi = -\varrho(\mathbf{r}) \tag{4.1}$$

with stipulated boundary condition. The question to ask is if there could be two different solutions for $\Phi(\mathbf{r})$, namely $\Phi_1(\mathbf{r})$ and $\Phi_2(\mathbf{r})$ that satisfy the above equation and the same boundary condition. In other words,

$$\nabla \cdot \varepsilon \nabla \Phi_1(\mathbf{r}) = -\varrho(\mathbf{r}) \tag{4.2}$$

$$\nabla \cdot \varepsilon \nabla \Phi_2(\mathbf{r}) = -\varrho(\mathbf{r}) \tag{4.3}$$

If this is really the case, then taking the difference of the above two equations, we have

$$\nabla \cdot \varepsilon \nabla \delta \Phi(\mathbf{r}) = 0 \tag{4.4}$$

where

$$\delta \Phi(\mathbf{r}) = \Phi_1(\mathbf{r}) - \Phi_2(\mathbf{r}) \tag{4.5}$$

The above does not imply that $\delta\Phi = 0$ as yet. Hence, more work is needed![1]

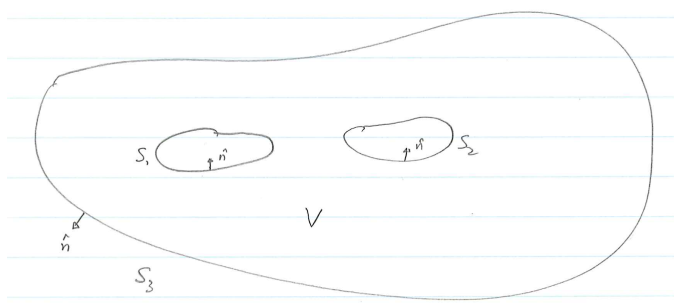

Figure 2:

In this proof, we shall allow for the possibility of a not simply connected surface as shown in Figure 2. The total surface $S = S_1 \cup S_2 \cup S_3$,[2] and V is enclosed by S. To prove the theorem, we multiply (4.4) by $\delta\Phi(\mathbf{r})$ and integrate over V, or

$$\int_V dV \delta\Phi \nabla \cdot \varepsilon \nabla \delta \Phi = 0 \tag{4.6}$$

We invoke Gauss's divergence theorem[3] that allows us to convert the volume integral into a surface integral.

To this end, one notices that, by the product rule of derivatives, we have

$$\nabla \cdot (\delta\Phi \varepsilon \nabla \delta\Phi) = \nabla \delta\Phi \cdot \varepsilon \nabla \delta \Phi + \delta\Phi \nabla \cdot \varepsilon \nabla \delta\Phi \tag{4.7}$$

Integrating the above equation, both the left-hand side and right-hand side over a volume V, Gauss's divergence theorem allows one to convert the left-hand side into a surface integral, namely

$$\int_S dS \hat{n} \cdot (\delta\Phi \varepsilon \nabla \delta\Phi) = \int_V dV \varepsilon (\nabla \delta\Phi)^2 + \int_V dV \delta\Phi \underbrace{\nabla \cdot \varepsilon \nabla \delta\Phi}_{=0} \tag{4.8}$$

[1] We are in a very similar situation in linear algebra where $\overline{\mathbf{A}} \cdot \mathbf{x} = 0$ does not imply that $\mathbf{x} = 0$.

[2] The cup symbol means "union". It is the mathematically formal way to say "plus".

[3] The Gauss' divergence theorem and Stokes' theorem have all been related to Green's theorem, but Green never wrote them in his original work. They are somewhat related to integration by parts.

The last term on the right-hand side is zero by virtue of (4.4) or (4.6). Consequently,

$$\int_S dS\hat{n} \cdot (\delta\Phi\varepsilon\nabla\delta\Phi) = \int_V dV\varepsilon(\nabla\delta\Phi)^2 \tag{4.9}$$

Therefore, if

(i) $\delta\Phi = 0$, or $\hat{n} \cdot \nabla\delta\Phi = 0$ on S, or

(ii) $\delta\Phi = 0$ on part of S and $\hat{n} \cdot \nabla\delta\Phi = 0$, on the rest of S,

then, the left-hand side of the above is zero, and we have

$$\int_V dV\varepsilon(\nabla\delta\Phi)^2 = 0 \tag{4.10}$$

Since $(\nabla\delta\Phi)^2 = \nabla\delta\Phi \cdot \nabla\delta\Phi = |\nabla\delta\Phi|^2 > 0$ always if $\delta\Phi \neq 0$, the above is possible only if $\delta\Phi = 0$ (unless $\delta\Phi$ is a constant).

We further elaborate on the boundary conditions. The above is equivalent to

(i) $\Phi_1 = \Phi_2$ on S, or $\hat{n} \cdot \nabla\Phi_1 = \hat{n} \cdot \nabla\Phi_2$ on S

(ii) $\Phi_1 = \Phi_2$ on part of S, and $\hat{n} \cdot \nabla\Phi_1 = \hat{n} \cdot \nabla\Phi_2$ on the rest of S.

The specification of Φ on S is known as the Dirichlet boundary condition, while the specification of $\hat{n} \cdot \nabla\Phi = \frac{\partial}{\partial n}\Phi$ or its normal derivative on S is known as the Neumann boundary condition. With these boundary conditions and mixture thereof, there could only be a unique solution to the boundary value problem related to the generalized Poisson's equation

$$\nabla \cdot \varepsilon\nabla\Phi = -\varrho \tag{4.11}$$