ECE 604, Lecture 4

August 30, 2018

1 Introduction

In this lecture, we will cover the following topics:

- Boundary Conditions–Unconventional View
- Electric Field at a Dielectric Interface
- Energy Stored–Review

Additional Reading:

- Textbook 1.14, 1.22

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2 Boundary Conditions–Unconventional View

In this lecture, we will espouse some unconventional way of arriving at boundary conditions at dielectric interface. Instead of working with pill boxes as most text books would do, we work directly with the partial differential equations involved to arrive at the boundary conditions. This method will allow one to eyeball a partial differential equation and infer directly that boundary condition that partial differential equation will induce.

2.1 Faraday's Law

From Faraday's law, it follows that

$$\nabla \times \mathbf{E} = 0 \tag{2.1}$$

From this, we have deduce earlier that

$$E_{1t} = E_{2t} \tag{2.2}$$

or tangential $\mathbf{E}$ is continuous. But we will arrive at this using a different approach below.

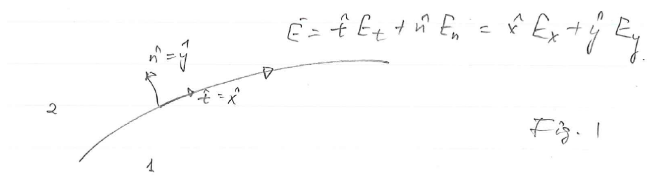

Figure 1:

Think of $\hat{t}$ and $\hat{n}$ as the local $\hat{x}$ and $\hat{y}$ coordinates, then

$$\nabla \times \mathbf{E} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right) \times (\hat{x}E_x + \hat{y}E_y) \tag{2.3}$$

$$= \hat{z}\frac{\partial}{\partial x}E_y - \hat{z}\frac{\partial}{\partial y}E_x \tag{2.4}$$

Using the distributive property of cross product, and evaluating the cross product in cartesian coordinates, the above can be evaluated easily. The cross product produces four terms, but only two of the four terms are non-zero as shown above. Since $\nabla \times \mathbf{E}$ is finite, the above implies that $\frac{\partial}{\partial y}E_y$ and $\frac{\partial}{\partial y}E_x$ have to be finite. In order words, E_x is continuous in the y direction and E_y is continuous in the x direction. Since in the local coordinate system, $E_x = E_t$, then E_t is continuous across the boundary. The above implies that

$$E_{1t} = E_{2t} \tag{2.5}$$

2.2 Gauss's Law

From Gauss's law, we have

$$\nabla \cdot \mathbf{D} = \varrho \tag{2.6}$$

where ϱ is the volume charge density.

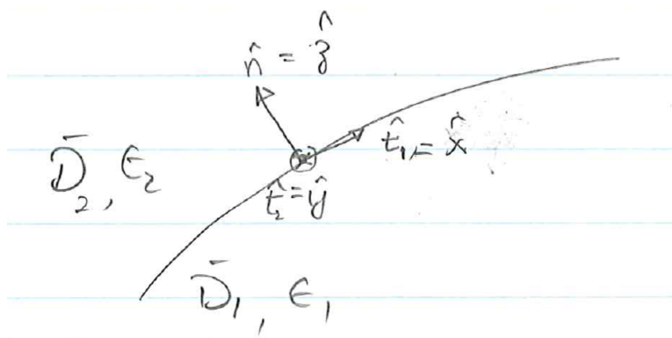

Figure 2:

Expressing the above in local coordinates, then

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x} D_x + \frac{\partial}{\partial y} D_y + \frac{\partial}{\partial z} = \varrho \tag{2.7}$$

If there is a surface layer charge at the interface, then the volume charge density must be infinitely large, and can be expressed in terms of a delta function, or $\varrho = \varrho_s \delta(z)$. By looking at the above expression, the only term that can produce a $\delta(z)$ is from $\frac{\partial}{\partial z} D_z$. In other words, D_z has a jump discontinuity at $z = 0$ the other terms do not.
Then

$$\frac{\partial}{\partial z} D_z = \varrho_s \delta(z) \tag{2.8}$$

Integrating the above from $z - \Delta$ to $z + \Delta$, we get

$$D_z(z) \mid_{z-\Delta}^{z+\Delta} = \varrho_s \tag{2.9}$$

or

$$D_z(z^+) - D_z(z^-) = \varrho_s \tag{2.10}$$

where $z^+ = \lim_{\Delta \to 0} z + \Delta$, $z^- = \lim_{\Delta \to 0} z - \Delta$. Since $D_z(z^+) = D_{2n}$, $D_z(z^-) = D_{1n}$, the above becomes

$$D_{2n} - D_{1n} = \varrho_s \tag{2.11}$$

or that

$$\hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \varrho_s \tag{2.12}$$

3 E Field at a Dielectric Interface

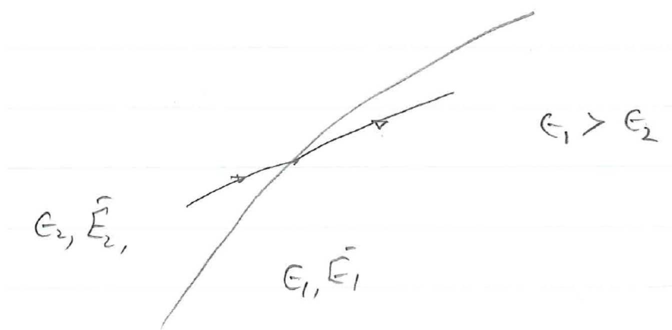

Figure 3:

Since we have, from Faraday's law and Gauss's law that

$$E_{1t} = E_{2t} \tag{3.1}$$

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} \tag{3.2}$$

if $\varepsilon_1 > \varepsilon_2$, then $E_{1n} < E_{2n}$. The electric field line will appear to be as shown in Figure 4.

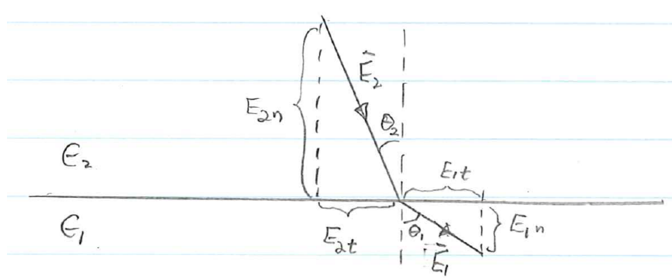

Figure 4:

When $\varepsilon_1 \to \infty$, there are two scenarios to consider:

1. When $\mathbf{E}_2$ or θ_2 is fixed as $\varepsilon_1 \to \infty$. In this case, both the right-hand side and the left-hand side of equation (3.2) are finite. Then E_{1n} has to become zero, to keep $\varepsilon_1 E_{1n}$ finite. But $E_{1t} = E_{2t}$ remain finite. Hence, $\theta_1 \to \pi/2$.

2. When $\mathbf{E}_1$ or θ_1 is fixed as $\varepsilon_1 \to \infty$. To satisfy (3.2), then it is necessary that $E_{1n} \to 0$. This is possible if $\mathbf{E}_1$ becomes zero or very small. But in this case, $E_{1t} = E_{2t}$ become very small as well. Hence, $\theta_2 \to 0$ or $\mathbf{E}_2$ becomes normal to the interface.

In either case, $\theta_1 > \theta_2$. A relationship between θ_1 and θ_2 is derived in the textbook given by

$$\varepsilon_1 \tan\theta_2 = \varepsilon_2 \tan\theta_1 \tag{3.3}$$

which can be used to verify the above scenarios.

The second scenario is similar to when region 1 becomes a PEC (perfect electric conductor). Then, $E_{1t} = E_{2t} = 0$, and E_{2n} is finite, and the electric field is normal to the interface. A difference is that interfacial charges will occur at the conductor-dielectric interface, which will shield out the field in the conductor region completely.

4 Conductive Media Case

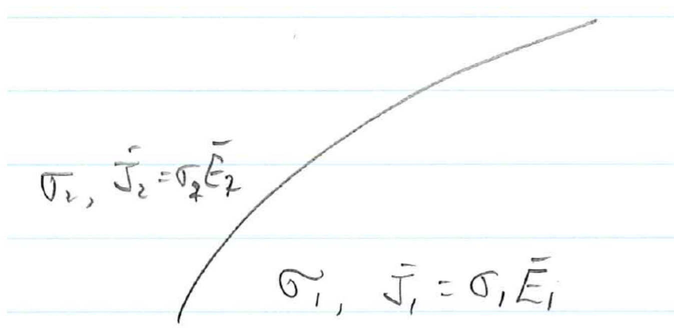

Figure 5:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \varrho}{\partial t} \tag{4.1}$$

For the static case, $\frac{\partial \varrho}{\partial t} = 0$, and we have finally

$$\nabla \cdot \mathbf{J} = 0 \tag{4.2}$$

Just like the Gauss's law case, the above implies that $\frac{\partial}{\partial n} J_n = 0$ or that J_n is continuous, or that $J_{1n} = J_{2n}$. In other words,

$$\hat{n} \cdot (\mathbf{J}_2 - \mathbf{J}_1) = 0 \tag{4.3}$$

Hence, using $\mathbf{J} = \sigma \mathbf{E}$, we have

$$\sigma_2 E_{2n} - \sigma_1 E_{1n} = 0 \tag{4.4}$$

Again, from Ampere's law,

$$E_{2t} - E_{1t} = 0 \tag{4.5}$$

But Gauss's law implies that

$$\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \varrho_s \tag{4.6}$$

Hence, surface charge accumulation is necessary, unless $\sigma_2/\sigma_1 = \varepsilon_2 = \varepsilon_1$.

5 Electric Energy

Pairwise energy stored between two charges q_1 and q_2 is

$$U_{12} = \frac{q_1 q_2}{4\pi\varepsilon R_{12}} \tag{5.1}$$

where R_{12} is the distance between the two charges. The above follows because

$$\Phi = \frac{q}{4\pi\varepsilon r} \tag{5.2}$$

The above is the energy needed to move a unit charge close to another charge of value q so that the distance between the unit charge and charge q is r apart.

Figure 6:

When we have a cluster of N charges, the total pairwise energy stored between them is

$$U_E = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{q_i q_j}{4\pi\varepsilon R_{ij}}, \quad i \neq j \tag{5.3}$$

But since

$$\Phi_i = \sum_{j=1}^{N} \frac{q_j}{4\pi\varepsilon R_{ij}},, \quad i \neq j \tag{5.4}$$

then (5.3) becomes

$$U_E = \frac{1}{2}\sum_{i=1}^{N} q_i \Phi_i \tag{5.5}$$

Replacing the above by charge density, we have

$$U_E = \frac{1}{2}\int_v \varrho(\mathbf{r})\Phi(\mathbf{r})dV \tag{5.6}$$

where ϱ

$$\varrho(\mathbf{r}) = \sum_{i=1}^{N} q_i \delta(\mathbf{r} - \mathbf{r}_i) \tag{5.7}$$

Since $\nabla \cdot \mathbf{D} = \varrho$, we have

$$U_E = \frac{1}{2}\int_v (\nabla \cdot \mathbf{D})\Phi dV \tag{5.8}$$

Using integration by parts in 3D (see Section 6), we have

$$U_E = -\frac{1}{2}\int_v \mathbf{D} \cdot \nabla\Phi dV = \frac{1}{2}\int_v \mathbf{D} \cdot \mathbf{E} dV \tag{5.9}$$

If $\mathbf{D} = \varepsilon\mathbf{E}$, then the above becomes

$$U_E = \frac{1}{2}\int_V \varepsilon\mathbf{E} \cdot \mathbf{E} dV = \frac{1}{2}\int_V \varepsilon|\mathbf{E}|^2 dV \tag{5.10}$$

5.1 Energy in a Capacitor

For a parallel plate capacitor, the field between the two plates is almost uniform. Then we can approximate the total energy stored as

$$U_E = \frac{1}{2}(\mathrm{Vol})DE = \frac{1}{2}(Ad)\frac{\varepsilon V}{d}\frac{V}{d} \tag{5.11}$$

$$= \frac{1}{2}\left(\frac{\varepsilon A}{d}\right)V^2 = \frac{1}{2}CV^2 \tag{5.12}$$

6 Integration by Parts in 3D

We have used integration by parts in 3D to rewrite the integral below:

$$U_E = \frac{1}{2}\int_v (\nabla \cdot \mathbf{D})\Phi dV = -\frac{1}{2}\int_v \mathbf{D} \cdot (\nabla\Phi) dV \tag{6.1}$$

To prove the above, we first use the identity for derivative of product in 3D, namely,

$$\nabla \cdot (\Phi \mathbf{D}) = (\nabla \Phi) \cdot \mathbf{D} + \Phi \nabla \cdot \mathbf{D} \tag{6.2}$$

Noticing the divergence on the left-hand side and integrating the above over a volume V, and invoking Gauss's divergence theorem, one gets

$$\oint_S (\Phi \mathbf{D}) \cdot d\mathbf{S} = \int_V (\nabla \Phi) \cdot \mathbf{D} dV + \int_V \Phi (\nabla \cdot \mathbf{D}) dV \tag{6.3}$$

Next, we let $V \to \infty$, and hence, $S \to \infty$. It is not clear if the left-hand side becomes vanishingly small, as the surface area S that we are integrating over becomes infinitely large.

However, if Φ is due to a cluster of point charges, then $\Phi \sim O(1/r)$ and $\mathbf{D} \sim O(1/r^2)$ when $\mathbf{r} \to \infty$, or that the integrand on the left-hand side $\mathbf{\Phi D} \sim O(1/r^3)$. But S grows as r^2 when $\mathbf{r} \to \infty$, and hence, the left-hand side indeed becomes vanishingly small. Then indeed,

$$\int_V (\nabla \Phi) \cdot \mathbf{D} dV = - \int_V \Phi (\nabla \cdot \mathbf{D}) dV \tag{6.4}$$