ECE 604, Lecture 3

August 28, 2018

1 Introduction

In this lecture, we will cover the following topics:

- Boundary Conditions
- Electric Energy
- Capacitance

Additional Reading:

- Textbook 1.14, 1.22

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2 Boundary Conditions in 1D Poisson's Equation

Boundary conditions are manifestation of the partial differential form of Maxwell's equations. Take for example a one dimensional Poisson's equation that

$$\frac{d}{dx}\varepsilon(x)\frac{d}{dx}\Phi(x) = -\varrho(x) \tag{2.1}$$

where $\varepsilon(x)$ represents material property that has the form given in Figure 1.

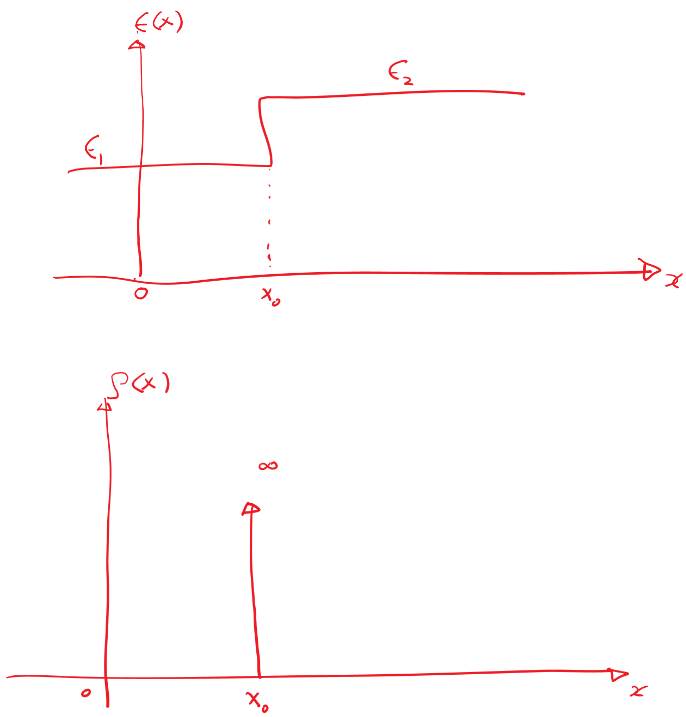

Figure 1:

In the above, $\varrho(x)$ represents a charge distribution. In this case, the charge distribution is everywhere zero except at the location of the charge sheet, where the charge density is infinite: it is represented mathematically by a delta function[1] in space.

[1]This function has been attributed to Dirac who used in pervasively, but Cauchy was aware of such a function.

To find the boundary condition of the potential Φ(x) at x_0 , we integrate (2.1) over an infinitesimal width around x_0, namely

$$\int_{x_0-\Delta}^{x_0+\Delta} dx \frac{d}{dx}\varepsilon(x)\frac{d}{dx}\Phi(x) = \int_{x_0-\Delta}^{x_0+\Delta} dx \varrho(x) \tag{2.2}$$

or

$$\varepsilon(x)\frac{d}{dx}\Phi(x)\Big|_{x_0-\Delta}^{x_0+\Delta} \cong -\varrho_s \tag{2.3}$$

$$\lim_{\Delta\to 0} \varepsilon(x^+)\frac{d}{dx}\Phi(x^+) - \varepsilon(x^-)\frac{d}{dx}\Phi(x^-) \cong -\varrho_s, \tag{2.4}$$

Since $\mathbf{E} = \nabla\Phi$,

$$E_x(x) = -\frac{d}{dx}\Phi(x), \tag{2.5}$$

The above implies that

$$\varepsilon(x^+)E_x(x^+) - \varepsilon(x^-)E_x(x^-) = \varrho_s \tag{2.6}$$

or

$$D_x(x^+) - D_x(x^-) = \varrho_s \tag{2.7}$$

where

$$D_x(x) = \varepsilon(x)E_x(x) \tag{2.8}$$

The lesson learned from above is that boundary condition is obtained by integrating the differential form of an equation over an infinitesimal small segment.

3 Boundary Conditions from Differential Equations

We have already defined the partial differential form of Faraday's Law for statics, or when $\frac{\partial}{\partial t} = 0$, and Gauss's law. They are

$$\nabla \times \mathbf{E} = 0 \tag{3.1}$$

$$\nabla \cdot \mathbf{D} = \varrho \tag{3.2}$$

4 Boundary Condition from Faraday's Law

We will start with Faraday's law, and integrating it over a small cross-section straddling two media interface.

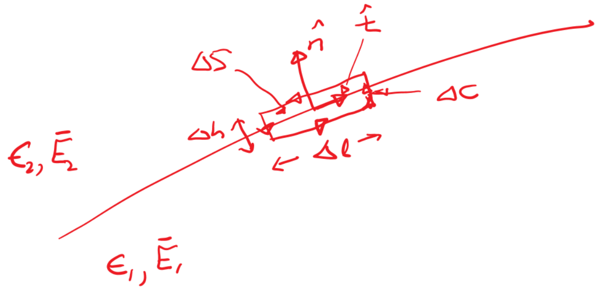

Figure 2:

Converting the surface integral into a line integral using Stokes's theorem, one gets

$$\int_{\Delta S} (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \oint_{\Delta C} \mathbf{E} \cdot d\mathbf{l} \tag{4.1}$$

When $\Delta h \approx 0$, $\Delta l \approx 0$, we can approximate the line integral as (see Figure 2)

$$\oint_{\Delta C} \mathbf{E} \cdot d\mathbf{l} \approx \mathbf{E}_1 \cdot \hat{t}\Delta l + \mathbf{E}_1 \cdot \hat{n}\frac{\Delta h}{2} + \mathbf{E}_2 \cdot \hat{n}\frac{\Delta h}{2} \tag{4.2}$$

$$-\mathbf{E}_2 \cdot \hat{t}\Delta l - \mathbf{E}_2 \cdot \hat{n}\frac{\Delta h}{2} - \mathbf{E}_1 \cdot \hat{n}\frac{\Delta h}{2} = 0 \tag{4.3}$$

Letting $\Delta h \to 0$, then

$$\mathbf{E}_1 \cdot \hat{t}\Delta l - \mathbf{E}_2 \cdot \hat{t}\Delta l = 0 \tag{4.4}$$

or

$$E_{1t} = E_{2t} \tag{4.5}$$

The above implies that the tangential componential of $\mathbf{E}$ is always continuous due to Faraday's law.

5 Boundary Condition from Gauss's Law

Gauss's law tells us that

$$\nabla \cdot \mathbf{D}(\mathbf{r}) \equiv \varrho(\mathbf{r}) \tag{5.1}$$

in partial differential form

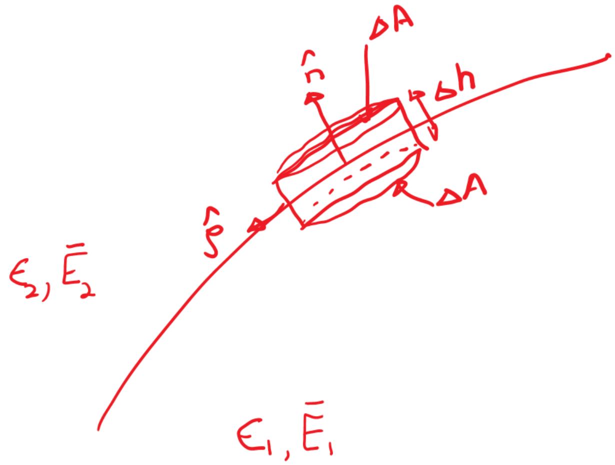

Figure 3:

In the above, $\varrho(\mathbf{r})$ is the volume charge density. However, if there is a surface charge density residing at the media interface, the volume charge density is infinitely large at the interface. Integrating (5.1) over ΔV representing a small pill box straddling the dielectric interface (see Figure 3), we have

$$\iiint_{\Delta V} \nabla \cdot \mathbf{D} dV = \iiint_{\Delta V} \varrho dV \tag{5.2}$$

By using Gauss's divergence theorem, the left-hand side becomes

$$\oiint_{\Delta S} \mathbf{D} \cdot d\mathbf{S} = \oiint_{\Delta S} \mathbf{D} \cdot \hat{n} dS$$
$$= \mathbf{D}_2 \cdot \hat{n}\Delta A - \mathbf{D}_1 \cdot \hat{n}\Delta A + \frac{1}{2}\mathbf{D}_2 \cdot \hat{\rho} 2\pi a \Delta h + \frac{1}{2}\mathbf{D}_1 \cdot \hat{\rho} 2\pi a \Delta h \tag{5.3}$$

But

$$\iiint_{\Delta V} \varrho dV = Q = \varrho_s \Delta A \tag{5.4}$$

Since

$$\oiint_{\Delta S} \mathbf{D} \cdot d\mathbf{S} = \iiint_{\Delta V} \varrho dV = Q = \varrho_s \Delta A \tag{5.5}$$

we have, after letting $\Delta h \to 0$ and let ΔA remain small,

$$\mathbf{D}_2 \cdot \hat{n}\Delta A - \mathbf{D}_1 \cdot \hat{n}\Delta A = \varrho_s \Delta A \tag{5.6}$$

or that

$$\hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \varrho_s \tag{5.7}$$

The way to argue that the field in a conductor has to be zero differs from that for the perfect conductor. The proverbial arguments given is that:

"If medium 1 is a perfect conductor, then $\sigma \to \infty$ but $\mathbf{J}_1 = \sigma \mathbf{E}_1$. An infinitesimal small $\mathbf{E}_1$ will give rise to an infinite current $\mathbf{J}_1$. To avoid this ludicrous situation, thus $\mathbf{E}_1 = 0$. This implies that $\mathbf{D}_1 = 0$ as well."

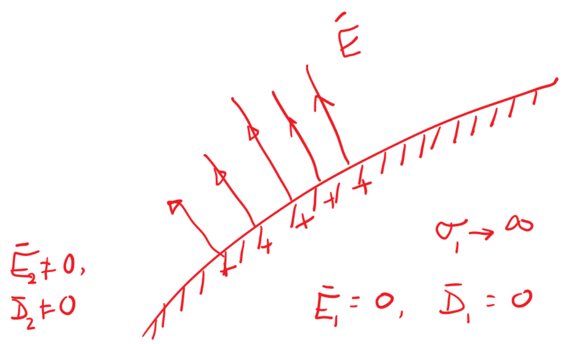

Figure 4:

Since tangential $\mathbf{E}$ is continuous, from Faraday's law, it is still true that

$$E_{2t} = E_{1t} = 0 \tag{5.8}$$

But since

$$\hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \varrho_s \tag{5.9}$$

and that $\mathbf{D}_1 = 0$, then

$$\hat{n} \cdot \mathbf{D}_2 = \varrho_s \tag{5.10}$$

In other words, normal $\mathbf{D}_2 \neq 0$, tangential $\mathbf{E}_2 = 0$. The sketch of the electric field in the vicinity of a perfect conducting surface is shown in Figure 4.

The above argument is true for electrodynamic problems. However, one does not need the above argument regarding the shielding of the static electric field from a conducting region. In the situation of the two conducting objects example below, as long as the electric fields are non-zero in the objects, currents will keep flowing. They flow until the charges in the two objects orient themselves so that electric current cannot flow anymore. This happens when the charges produce internal fields that cancel each other giving rise to zero field inside the two objects. Then the boundary condition that the fields have to be normal to the conducting object surface is still true for elecrostatics. A sketch of the electric field between two conducting spheres is show in Figure 5.

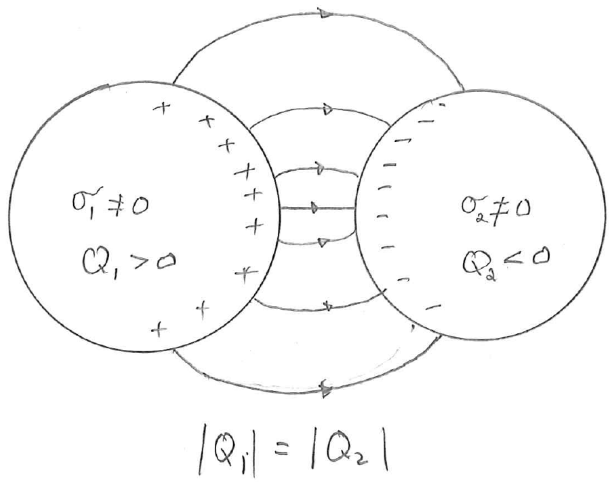

Figure 5:

6 What About a Conductor Media Interface?

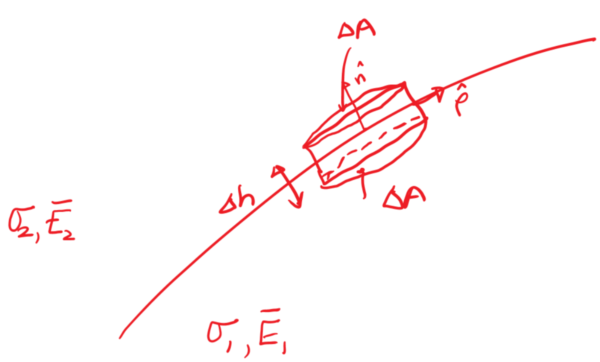

Figure 6:

The current continuity equation is

$$\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t}\varrho \tag{6.1}$$

It is a statement of charge conservation. If $\frac{\partial}{\partial t} = 0$, then

$$\nabla \cdot \mathbf{J} = 0 \tag{6.2}$$

The above is KCL (Kirchhoff current law) in disguise. The boundary condition can be obtained by integrating (6.2) over a small pill-box, or

$$\iiint_{\Delta V} dV \nabla \cdot \mathbf{J} = 0 \tag{6.3}$$

or

$$\oiint_{\Delta S} dS \hat{n} \cdot \mathbf{J} = 0 \tag{6.4}$$

Consequently, one obtains that (see Figure 6)

$$\mathbf{J}_2 \cdot \hat{n}\Delta A + \mathbf{J}_1 \cdot (-\hat{n})\Delta A + \frac{1}{2}\mathbf{J}_2 \cdot \hat{\varrho} 2\pi a \Delta h + \frac{1}{2}\mathbf{J}_1 \cdot \hat{\varrho} 2\pi a \Delta h = 0 \tag{6.5}$$

Letting $\Delta h \to 0$, then

$$\hat{n} \cdot (\mathbf{J}_2 - \mathbf{J}_1) = 0 \tag{6.6}$$

In other words, $J_{2n} = J_{1n}$. Since $\mathbf{J} = \sigma \mathbf{E}$, one gets

$$\sigma_2 E_{2n} - \sigma_1 E_{1n} = 0 \tag{6.7}$$

But Ampere's law still implies that

$$E_{2t} - E_{1t} = 0 \tag{6.8}$$

And Gauss's law implies that

$$\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \varrho_s \tag{6.9}$$

In order for (6.7) to be consistent with (6.9), $\varrho_s \neq 0$. Hence, at a conductor interface, surface charge has to accumulate.[2]

7 Boundary Conditions from Poisson's Equation

From Poisson's equation

$$\nabla \cdot \varepsilon \Phi = -\varrho \tag{7.1}$$

[2] $\varrho_s = 0$, however, if $\varepsilon_2/\varepsilon_1$ is equal to σ_2/σ_1.

one can integrate the above over a small pill box to obtain

$$\varepsilon_2 \hat{n} \cdot \nabla \Phi_2 - \varepsilon_1 \hat{n} \cdot \nabla \Phi_1 = -\varrho_s \tag{7.2}$$

where ϱ_s is the surface charge density of a surface charge at the interface.

In the above, if we assume that ϱ at most has a delta function singularity, then $\nabla\Phi$ cannot be singular. Therefore, by integrating $\nabla\Phi$ along a line segment across the interface, then

$$\Phi_1 = \Phi_2 \tag{7.3}$$

8 Electric Energy

For a point charge, it is known from Coulomb's law that

$$\mathbf{E} = \frac{q}{4\pi\varepsilon r^2}\hat{\mathbf{r}}, \qquad \Phi = \frac{q}{4\pi\varepsilon r} \tag{8.1}$$

Since $\mathbf{E} = -\nabla\Phi$, by integrating this equation along a line connecting two points P_1 and P_2, it can be shown that[3]

$$\Phi_{P_2} - \Phi_{P_1} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l} \tag{8.2}$$

Furthermore, by noting that $\Phi \to 0$ when r $\to \infty$, one arrives at that

$$\Phi = \int_r^\infty \mathbf{E} \cdot d\mathbf{l} \tag{8.3}$$

Since $\mathbf{E}$ is the force on a unit charge, the right-hand side of the above is the work done on moving a unit charge from infinity to the point r. Thus, the potential Φ expressed in (8.1) can be interpreted as work needed to bring a unit charge next to another charge with value q.

The energy needed to bring two charges q_1 and q_2 next to each other with distance R_{12} is then

$$u_{12} = \frac{q_1 q_2}{4\pi\varepsilon R_{12}} \tag{8.4}$$

The energy contained in N charges, when they are brought to close proximity of each other is by summing over their pairwise energy. Therefore,

$$U_E = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{q_1 q_2}{4\pi\varepsilon R_{ij}}, \qquad i \neq j \tag{8.5}$$

The factor $\frac{1}{2}$ is needed because the double summation above double counts.

[3]The result is independent of the shape of the line because by Faraday's law, $\int_C \mathbf{E} \cdot d\mathbf{l} - 0$ for any C. Such a field where $\nabla \times \mathbf{E} = 0$ is known as a conservative field.

By using the fact that the potential at the location of charge i due to the other charges $j = 1, \ldots, N$ is

$$\Phi_i = \sum_{j=1}^{N} \frac{q_i}{4\pi\varepsilon R_{ij}}, \qquad j \neq i \tag{8.6}$$

implies that

$$U_E = \frac{1}{2} \sum_{i=1}^{N} q_i \Phi_i \tag{8.7}$$

When their point charges are replaced by a continuum, or by charge density, then

$$U_E = \frac{1}{2} \int_v \varrho \Phi dV \tag{8.8}$$

Using Gauss's law that $\nabla \cdot \mathbf{D} = \varrho$,

$$U_E = \frac{1}{2} \int_v (\nabla \cdot \mathbf{D}) \Phi dV \tag{8.9}$$

The above ∇ operator can be made to operate on Φ by performing integration by parts in 3D. Finally,

$$U_E = -\frac{1}{2} \int_v \mathbf{D} \cdot \nabla \Phi dV = \frac{1}{2} \int_V \mathbf{D} \cdot \mathbf{E} dV \tag{8.10}$$

For an isotropic medium, $\mathbf{D} = \varepsilon \mathbf{E}$, and upon substitution, the above becomes

$$U_E = \frac{1}{2} \int_V \varepsilon \mathbf{E} \cdot \mathbf{E} = \frac{1}{2} \int_V \varepsilon |\mathbf{E}|^2 dV \tag{8.11}$$

9 Energy Stored in a Capacitor

Assume that a parallel plate capacitor is charged to a voltage V_o, and that the separation of the parallel plate is d, and that the area of the plate is A, then the electric field between the plate is $E = V_o/d$ and $D = \varepsilon V_o/d$.[4] The volume between the plate is Vol $= Ad$. Then the stored energy is approximately given by

$$U = \frac{1}{2} \text{ Vol} DE = \frac{1}{2}(Ad) \frac{\varepsilon V_o}{d} \frac{V_o}{d} = \frac{1}{2} \left(\frac{\varepsilon A}{d} \right) V_o^2 = \frac{1}{2} C V_o^2 \tag{9.1}$$

This is the proverbial formula for stored energy in a parallel plate capacitor.

[4]This field is almost uniformly distributed save near the edge of the plates.