ECE 604, Lecture 25

December 6, 2018

In this lecture, we will cover the following topics:

- Quantum Coherent State of Light
  - More on Connecting Electromagnetic Oscillation to the Quantum Pendulum
  - Time-Dependent Quantum State
- Derivation of the Coherent State
  - Quantum Harmonic Oscillator Revisited
  - Coherent State Revisited
  - Quantum Randomness and Quantum Observables
  - Time Evolution of the Coherent State
  - More on Creation and Annihilation Operators
  - Connecting with Electromagnetic Fields

Additional Reading:

- W.C. Chew, Quantum Mechanics Made Simple Lecture Notes.
- D.A.B. Miller, Quantum Mechanics for Scientists and Engineers.
- C.C. Gerry and P.L. Knight, Introductory Quantum Optics.
- M. Fox, Quantum Optics: An Introduction.

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1 Quantum Coherent State of Light

We have seen that a photon number state[1] of a quantum pendulum do not have a classical correspondence as the average or expectation value of the position of the pendulum is always zero for all time for this state. Therefore, we have to seek a time-dependent quantum state that has the classical equivalence of a pendulum. This is the coherent state, which is the contribution of many researchers, most notably, George Sudarshan (1931–2018) and Roy Glauber (born 1925) in 1963. Glauber was awarded the Nobel prize in 2005 when Sudershan was still alive.

1.1 More on Connecting Quantum Pendulum to Electromagnetic Oscillator

We see that the electromagnetic oscillator in a cavity is similar or homomorphic to a pendulum. We have next to elevate a classical pendulum to become a quantum pendulum. The classical Hamiltonian is

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2 = E \tag{1.1}$$

Using de Broglie's observation $p = \hbar k$ for a quantum particle, and that $E = \hbar\omega$ for photon, H is elevated to be an operator $\hat{H}$ operating on a wave function $\psi(x,t)$, where

$$p = \hbar k \rightarrow -i\hbar\frac{\partial}{\partial x}, \qquad E = \hbar\omega = i\hbar\frac{\partial}{\partial t}. \tag{1.2}$$

Then (1.1) becomes

$$\hat{H}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_0^2 x^2\right]\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t). \tag{1.3}$$

By separation of variables, letting $\psi(x,t) = \psi_n(x)e^{-i\omega_n t}$, then (1.3) becomes an eigenvalue problem

$$\hat{H}\psi_n(x) = \hbar\omega_n\psi_n(x), \tag{1.4}$$

where (1.4) can be solved in closed form as shown in the previous lecture.

Our next task is to connect the electromagnetic oscillator to this pendulum. In general, the total energy or the Hamiltonian of an electromagnetic system is

$$H = \frac{1}{2}\int\limits_V d\mathbf{r}\left[\varepsilon\mathbf{E}^2(\mathbf{r},t) + \frac{1}{\mu}\mathbf{B}^2(\mathbf{r},t)\right]. \tag{1.5}$$

We reduce the above to a one dimensional (1D) case for simplicity by letting

$$\mathbf{E} = \hat{x}E_x \tag{1.6}$$

$$\mathbf{B} = \hat{y}B_y \tag{1.7}$$

[1]In quantum theory, a "state" is synonymous with a state vector or a function.

so that the total energy is

$$H = \frac{\mathcal{A}}{2} \int_0^L dz \left[\varepsilon E_x^2(z,t) + \frac{1}{\mu} B_y^2(z,t) \right], \tag{1.8}$$

where $\mathcal{A}$ is the cross-section area. We can let $\mathcal{A} = 1$ so that the dimension of H is J·m^{-2}.

Furthermore, we will consider only one mode in the 1D cavity so that

$$E_x(z,t) = E_0(t) \sin(k_l z) \tag{1.9}$$

$$B_y(z,t) = B_0(t) \cos(k_l z). \tag{1.10}$$

Then (1.8) becomes

$$H = \frac{L}{4} \varepsilon E_0^2(t) + \frac{L}{4\mu} B_0^2(t). \tag{1.11}$$

We will do what is common practice: To express the above in terms of the vector potential $\mathbf{A}$. In this case, we can let $\mathbf{A} = \hat{x} A_x$, $\nabla \cdot \mathbf{A} = 0$ so that $\partial_x A_x = 0$, and letting $\Phi = 0$. Then $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\dot{\mathbf{A}}$, and the classical Hamiltonian from (1.5) for a Maxwellian system becomes

$$H = \frac{1}{2} \int_V d\mathbf{r} \left[\varepsilon \dot{\mathbf{A}}^2(\mathbf{r},t) + \frac{1}{\mu} \left(\nabla \times \mathbf{A}(\mathbf{r},t) \right)^2 \right]. \tag{1.12}$$

For the 1D case, the above implies that $B_y = \partial_z A_x$, and $E_x = -\partial_t A_x = -\dot{A}_x$. Hence, we let

$$A_x = A_0(t) \sin(k_l z) \tag{1.13}$$

$$B_y = k_l A_0(t) \cos(k_l z). \tag{1.14}$$

The Hamiltonian (1.11) then becomes

$$H = \frac{L}{4} \varepsilon \left(\dot{A}_0(t) \right)^2 + \frac{L}{4\mu} k_l^2 A_0^2(t). \tag{1.15}$$

The form (1.15) now has all the elements that make it resemble the pendulum Hamiltonian. We can let $\Pi(t) = \varepsilon \dot{A}_0(t)$, yielding

$$H = \frac{L}{4\varepsilon} \left(\Pi(t) \right)^2 + \frac{L}{4\mu} k_l^2 A_0^2(t). \tag{1.16}$$

Then, with the following map,

$$\sqrt{\frac{L}{2}} \Pi(t) \to p(t) \tag{1.17}$$

$$\varepsilon \to m \tag{1.18}$$

$$\frac{k_l^2}{\mu} = \omega_l^2 \varepsilon \to k = m\omega_0^2 \tag{1.19}$$

$$\sqrt{\frac{L}{2}} A_0(t) \to x, \tag{1.20}$$

the electromagnetic Hamiltonian becomes a pendulum Hamiltonian. The conjugate variables in Hamiltonian mechanics are then $\sqrt{\frac{L}{2}}\Pi(t)$ and $\sqrt{\frac{L}{2}}A_0(t)$.

We can use the above homomorphism to elevate the electromagnetic oscillator to become a quantum oscillator. If x becomes fuzzy in the quantum world, A_0 or A_x will become fuzzy also.

1.2 Time-Dependent Quantum State

The coherent state is a contribution of George Sudarshan (1931–2018) and Roy Glauber (born 1925) in 1963. Glauber was awarded the Nobel prize in 2005 when Sudershan was still alive.

To connect the number state with the coherent state, we need a linear superposition of the time-varying eigensolutions, or eigenstates. In other words, we just look at the linear superposition of two eigenstates. Then,

$$\psi(x,t) = c_n e^{-i\omega_n t}\psi_n(x) + c_m e^{-i\omega_m t}\psi_m(x). \tag{1.21}$$

Then the probability of finding a particle between x and $x+\Delta x$ is $|\psi(x,t)|^2\,\Delta x$, where

$$\begin{aligned}|\psi(x,t)|^2 &= |c_n|^2\,|\psi_n(x)|^2 + |c_m|^2\,|\psi_m(x)|^2 \\ &\quad + 2\Re e\left[c_n c_m^* \psi_n(x)\psi_m^*(x)e^{-i(\omega_n-\omega_m)t}\right].\end{aligned} \tag{1.22}$$

Clearly, $|\psi(x,t)|^2$ is a time-varying function due to the interference between the two time-varying eigenfunctions. A plot of the probability density function of this time varying state is shown in Figure 1.

A natural extension is to use an infinite number of photon number states to arrive at the coherent state[2]

$$\psi_\alpha(\xi,t) = \sum_{n=0}^{\infty} C_{\alpha,n} e^{-i(n+\frac{1}{2})\omega_0 t}\psi_n(\xi), \tag{1.23}$$

where $\xi = \sqrt{m\omega_0/\hbar}x$, $C_{\alpha,n} = e^{-|\alpha|^2/2}\alpha^n/\sqrt{n!}$. Since each photon number state carries n photons, this state $\psi_\alpha(\xi,t)$ defined above can be associated with an average number of photons $N = |\alpha|^2$. The total energy associated with this state or wave packet is $(N+1/2)\hbar\omega_0$.

For this coherent state, even though the average number of photons $N=1$, an infinite number of photon number states is involved. Plots of the distributions of the coefficients for different N's are shown in Figure 2.

Also, as N becomes larger, this coherent state of a quantum pendulum resembles a classical pendulum as shown in Figure 3.

[2]This is a heads-up for the coherent state, which will be derived in the next section.

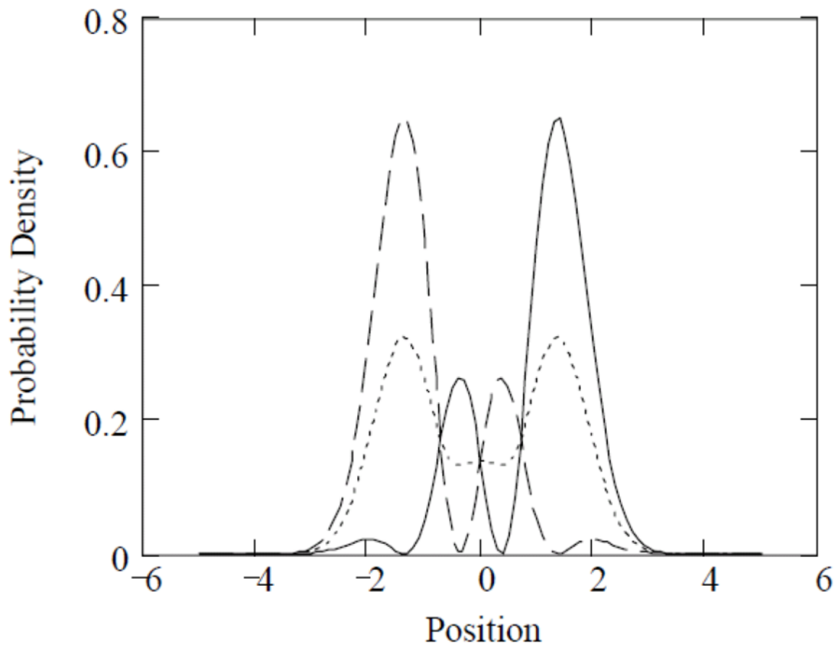

Figure 1: The time evolution of the linear superposition of the first two eigenstates of the quantum pendulum. (i) Beginning of a period (solid line). (ii) One quarter and three quarters way through the period (dotted line). (iii) Half way through the period (Courtesy of DAB Miller).

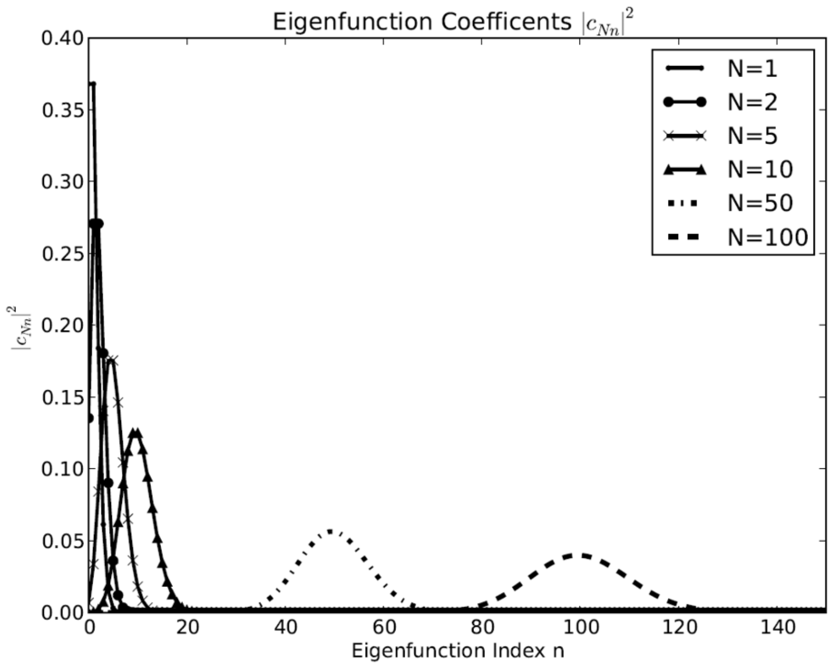

Figure 2: The plot of $|C_{Nn}|^2 = P_n$ as a function of n for different values of N, where N is the average number of photons in the coherent state.

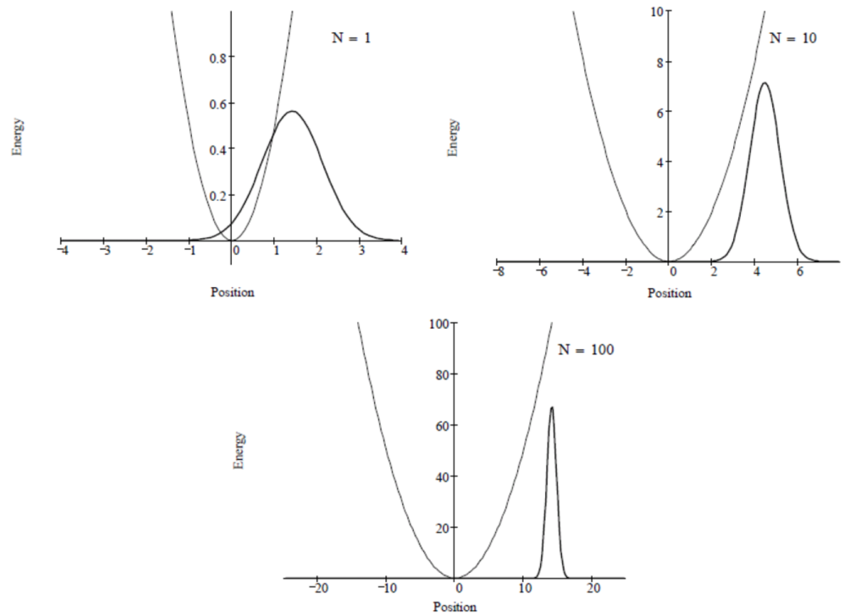

Figure 3: Coherent state for different values of N at $t = 0$ (Courtesy of DAB Miller).

2 Derivation of the Coherent States

Now that we have looked at the physical nature of the coherent state, it will be interesting to see how it is derived. The derivation of the coherent state is more math than physics. Nevertheless, the derivation is interesting. Since this knowledge is not very old, we are going to present it according to what has been found in the literature. Perhaps, some day, one of you will find a simpler derivation than what exists. There are deeper mathematical methods to derive this coherent state like Bogoliubov transform which is outside the scope of this course.

2.1 Quantum Harmonic Oscillator Revisited

To this end, we revisit the quantum harmonic oscillator or the quantum pendulum with more mathematical depth. Rewriting the eigen-equation for the photon number state for the quantum harmonic oscillator, we have

$$\hat{H}\psi_n(x) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega_0^2 x^2\right]\psi_n(x) = E_n\psi_n(x). \tag{2.1}$$

The above can be changed into a dimensionless form first by dividing $\hbar\omega_0$, and then let $\xi = \sqrt{\frac{m\omega_0}{\hbar}}x$, be a dimensionless variable. The above then becomes

$$\frac{1}{2}\left(\frac{\partial^2}{\partial\xi^2} + \xi^2\right)\psi(\xi) = \frac{E}{\hbar\omega_0}\psi(\xi) \tag{2.2}$$

which looks almost like $A^2 - B^2$, and hence motivates its factorization. To this end, it can be rewritten

$$\frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right)\frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) = \frac{1}{2}\left(\frac{d^2}{d\xi^2} + \xi^2\right) - \frac{1}{2}\left(\frac{d}{d\xi}\xi - \xi\frac{d}{d\xi}\right) \tag{2.3}$$

It can be shown easily that as operators (meaning that they will act on a function to their right),

$$\left(\frac{d}{d\xi}\xi - \xi\frac{d}{d\xi}\right) = \hat{I} \tag{2.4}$$

Therefore

$$\frac{1}{2}\left(-\frac{d^2}{d\xi^2} + \xi^2\right) = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right)\frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) + \frac{1}{2} \tag{2.5}$$

We define the operator

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right) \tag{2.6}$$

The above is the creations, or raising operator and the reason for its name is obviated later. Moreover, we define

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) \tag{2.7}$$

which represents the annihilation or lowering operator.

Therefore, Schrödinger equation for quantum harmonic oscillator can be rewritten more concisely as

$$\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\psi = \frac{E}{\hbar\omega_0}\psi \tag{2.8}$$

In mathematics, a function is analogous to a vector. So ψ is the implicit representation of a vector. The operator

$$\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

is an implicit representation of an operator, and in this case, a differential operator. So the above is analogous to the matrix eigenvalue equation $\overline{\mathbf{A}} \cdot \mathbf{x} = \lambda\mathbf{x}$.

Consequently, the Hamiltonian operator can be expressed concisely as

$$\hat{H} = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \tag{2.9}$$

Equation (2.8) above is in implicit math notation. In implicit Dirac notation, it is

$$\left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) |\psi\rangle = \frac{E}{\hbar\omega_0} |\psi\rangle \tag{2.10}$$

In the above, $\psi(\xi)$ is a function which is a vector in a functional space. It is denoted as ψ in math notation and $|\psi\rangle$ in Dirac notation. This is also known as the "ket". The conjugate transpose of a vector in Dirac notation is called a "bra" which is denoted as $\langle\psi|$. Hence, the inner product between two vectors is denoted as $\langle\psi_1|\psi_2\rangle$ in Dirac notation.[3]

If we denote a number state by $\psi_n(x)$ in explicit notation, ψ_n in math notation or $|\psi_n\rangle$ in Dirac notation, then we have

$$\left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) |\psi_n\rangle = \frac{E_n}{\hbar\omega_0} |\psi_n\rangle = \left(n + \frac{1}{2} \right) |\psi_n\rangle \tag{2.11}$$

where we have used he fact that $E_n = (n + 1/2)\hbar\omega_0$. Therefore, by comparing terms in the above, we have

$$\hat{a}^\dagger \hat{a} |\psi_n\rangle = n|\psi_n\rangle \tag{2.12}$$

and the operator $\hat{a}^\dagger \hat{a}$ is also known as the number operator because of the above. It is often denoted as

$$\hat{n} = \hat{a}^\dagger \hat{a} \tag{2.13}$$

Furthermore, one can show by direct substitution that

$$\left[\hat{a}, \hat{a}^\dagger \right] = \hat{I} \tag{2.14}$$

It can be further shown by direct substitution that

$$\hat{a}|\psi_n\rangle = \sqrt{n}|\psi_{n-1}\rangle \Leftrightarrow \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{2.15}$$

$$\hat{a}^\dagger|\psi_n\rangle = \sqrt{n+1}|\psi_{n+1}\rangle \Leftrightarrow \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{2.16}$$

hence their names on lowering and raising operator.[4]

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[3] There is a one-to-one correspondence of Dirac notation to matrix algebra notation. $\overline{\mathbf{A}}\cdot\mathbf{x} \leftrightarrow \hat{A}|x\rangle, \quad \langle x| \leftrightarrow \mathbf{x}^\dagger \quad \langle x_1|x_2\rangle \leftrightarrow \mathbf{x}_1^\dagger \cdot \mathbf{x}_2$.

[4] The above notation for a vector could appear cryptic to the uninitiated. To parse it, one can always down-convert from an abstract notation to a more explicit notation. Namely $|n\rangle \rightarrow |\psi_n\rangle \rightarrow \psi_n(\xi)$.

2.2 Coherent State Revisited

Now, endowed with the needed mathematical tools, we can study the derivation of the coherent state. To say succinctly, the coherent state is the eigenstate of the annihilation operator, namely that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \tag{2.17}$$

Here, we use α as an eigenvalue as well as an index or identifier of the state $|\alpha\rangle$.[5] Since the number state $|n\rangle$ is complete, the coherent state $|\alpha\rangle$ can be expanded in terms of the number state $|n\rangle$. Or that

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n|n\rangle \tag{2.18}$$

When the annihilation operator is applied to the above, we have

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} C_n\hat{a}|n\rangle = \sum_{n=1}^{\infty} C_n\hat{a}|n\rangle = \sum_{n=1}^{\infty} C_n\sqrt{n}|n-1\rangle \tag{2.19}$$

$$= \sum_{n=0}^{\infty} C_{n+1}\sqrt{n+1}|n\rangle \tag{2.20}$$

Equating the above with $\alpha|\alpha\rangle$, then

$$\sum_{n=0}^{\infty} C_{n+1}\sqrt{n+1}|n\rangle = \alpha\sum_{n=0}^{\infty} C_n|n\rangle \tag{2.21}$$

By the orthonormality of the number states $|n\rangle$, then we can take the inner product of the above with $\langle n|$ and making use of the orthonormal relation that $\langle n'|n\rangle = \delta_{n'n}$. Then we arrive at

$$C_{n+1} = \alpha C_n/\sqrt{n+1} \tag{2.22}$$

Or recursively

$$C_n = C_{n-1}\alpha/\sqrt{n} = C_{n-2}\alpha^2/\sqrt{n(n-1)} = \ldots = C_0\alpha^n/\sqrt{n!} \tag{2.23}$$

Consequently

$$|\alpha\rangle = C_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{2.24}$$

But due to the probabilistic interpretation of quantum mechanics, the state vector $|\alpha\rangle$ is normalized to one, or that[6]

$$\langle\alpha|\alpha\rangle = 1 \tag{2.25}$$

[5]This notation is cryptic, but one can always down-convert it as $|\alpha\rangle \to |f_\alpha\rangle \to f_\alpha(\xi)$.
[6]The expression can be written more explicitly as $\langle\alpha|\alpha\rangle = \langle f_\alpha|f_\alpha\rangle = \int_{\infty}^{\infty} d\xi f_\alpha^*(\xi)f_\alpha(\xi) = 1$.

Then

$$\langle\alpha|\alpha\rangle = C_0^* C_0 \sum_{n,n'}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{n'}}{\sqrt{n'!}} \langle n'|n\rangle \tag{2.26}$$

$$= |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |C_0|^2 e^{|\alpha|^2} = 1 \tag{2.27}$$

Therefore, $C_0 = e^{-|\alpha|^2/2}$, or that

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{2.28}$$

In the above, we have make use of $\langle n'|n\rangle = \delta_{n'n}$, or that the eigenstate are orthonormal. Also since $\hat{a}$ is not a Hermitian operator its eigenvalue α can be a complex number.

2.3 Time Evolution of the Coherent State

The previous derivation assumes that the coherent state is time independent, because all the photon number states $|n\rangle$ are time independent. We can insert time evolution in the coherent state by letting

$$|n\rangle \Rightarrow e^{-i\omega_n t}|n\rangle \tag{2.29}$$

Then the time dependent coherent state becomes[7]

$$|\alpha, t\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n e^{-i\omega_n t}}{\sqrt{n!}} |n\rangle \tag{2.30}$$

By letting $\omega_n = \omega_0 \left(n + \frac{1}{2}\right)$, the above can be written as

$$|\alpha, t\rangle = e^{-i\omega_0 t/2} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\left(\alpha e^{-i\omega_0 t}\right)^n}{\sqrt{n!}} |n\rangle \tag{2.31}$$

$$= e^{-i\omega_0 t/2} |\alpha e^{-i\omega_0 t}\rangle \tag{2.32}$$

Now we see that the last factor in (2.31) is similar to the expression for a coherent state in (2.28). Therefore, we can express the above more succinctly by replacing α in (2.28) with $\alpha e^{-i\omega_0 t}$ as

$$\hat{a}|\alpha, t\rangle = e^{-i\omega_0 t/2} \left(\alpha e^{-i\omega_0 t}\right) |\alpha e^{-i\omega_0 t}\rangle \tag{2.33}$$

The eigenvalue of the annihilation operator $\hat{a}$ is now a complex number.

[7]Note that $|\alpha, t\rangle$ is a shorthand for $f_\alpha(\xi, t)$.

2.4 More on the Creation and Annihilation Operator

Since

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\xi} + \xi\right) \tag{2.34}$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{d}{d\xi} + \xi\right) \tag{2.35}$$

one can define a normalized dimensionless momentum operator

$$\hat{P} = -i\frac{d}{d\xi} \tag{2.36}$$

Then,

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(-i\hat{P} + \hat{\xi}\right) \tag{2.37}$$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(i\hat{P} + \hat{\xi}\right) \tag{2.38}$$

where we define $\hat{\xi} = \xi\hat{I}$, where $\hat{I}$ is the identity operator. In the above, $\hat{\xi}$ is a normalized coordinate operator. We also notice that

$$\hat{\xi} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right) = \xi\hat{I} \tag{2.39}$$

$$\hat{P} = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right) = -i\frac{d}{d\xi} \tag{2.40}$$

Notice that both $\hat{\xi}$ and $\hat{P}$ are Hermitian operators in the above, and hence, their expectation values are real. With this, the average or expectation position of the pendulum, $x \sim \xi$, can be found by taking expectation with respect to the coherent state, or

$$\langle\alpha|\hat{\xi}|\alpha\rangle = \frac{1}{\sqrt{2}}\langle\alpha|\hat{a}^\dagger + \hat{a}|\alpha\rangle \tag{2.41}$$

Since by taking the complex conjugation of (2.17)[8]

$$\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^* \tag{2.42}$$

and (2.41) becomes

$$\langle\xi(t)\rangle = \langle\alpha|\hat{\xi}|\alpha\rangle = \frac{1}{\sqrt{2}}\left(\alpha^* + \alpha\right)\langle\alpha|\alpha\rangle = \sqrt{2}\Re e[\alpha] \neq 0 \tag{2.43}$$

For the time-dependent case, when we let $\alpha \to \tilde{\alpha}(t) = \alpha e^{-i\omega_0 t}$, then, letting $\alpha = |\alpha|e^{-i\psi}$,

$$\langle\xi(t)\rangle = \sqrt{2}|\alpha|\cos(\omega_0 t + \psi) \tag{2.44}$$

[8]Dirac notation is homomorphic with matrix algebra notation. $(\overline{\mathbf{a}} \cdot \mathbf{x})^\dagger = \mathbf{x}^\dagger \cdot (\overline{\mathbf{a}})^\dagger$.

By the same token,

$$\langle P(t)\rangle = \langle\alpha|\hat{P}|\alpha\rangle = \frac{i}{\sqrt{2}}\left(\alpha^* - \alpha\right)\langle\alpha|\alpha\rangle = \sqrt{2}\Im m[\alpha] \neq 0 \tag{2.45}$$

For the time-dependent case

$$\langle P(t)\rangle = -\sqrt{2}|\alpha|\sin(\omega_0 t + \psi) \tag{2.46}$$

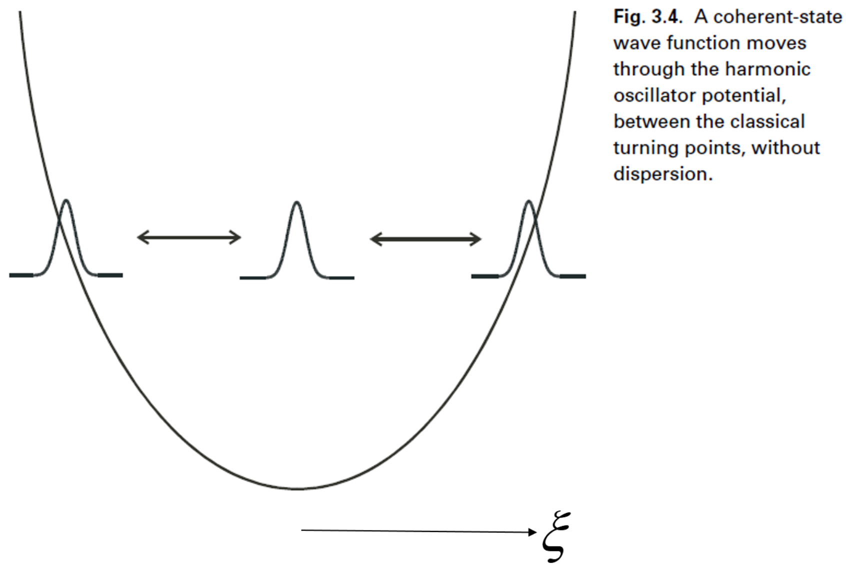

Figure 4: The time evolution of the coherent state. It follows the motion of a classical pendulum or harmonic oscillator (Courtesy of Gerry and Knight).

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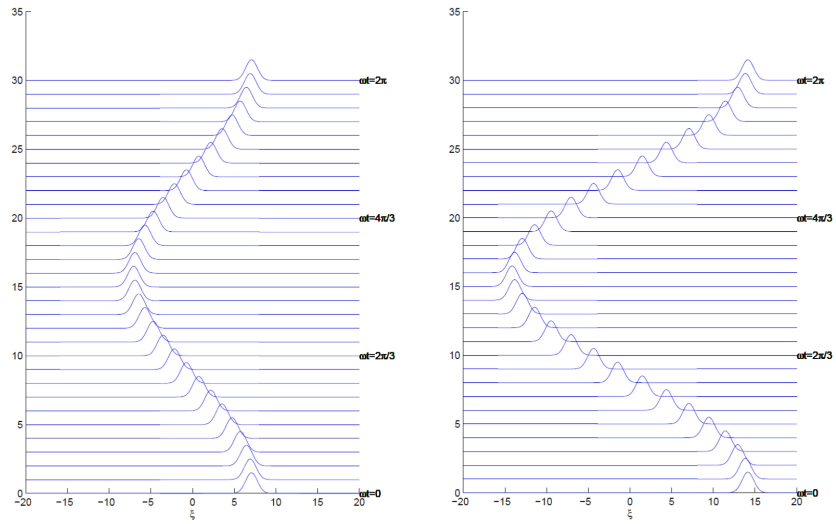

Figure 5: The time evolution of the coherent state for different α's. The left figure is for $\alpha = 5$ while the right figure is for $\alpha = 10$. Recall that $N = |\alpha|^2$.

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2.5 Quantum Randomness and Quantum Observables

We saw previously that in classical mechanics, the conjugate variables p and x are deterministic variables. But in the quantum world, they become random variables. It was quite easy to see that x is a random variable in the quantum world. But the momentum p is elevated to become a differential operator $\hat{p}$, and it is not clear that it is a random variable anymore. But we found its expectation value nevertheless in the previous lecture.

It turns out that we have to extend the concept of the average of a random variable to taking the "average" of an operator, which is the elevated form of a random variable. Now that we know Dirac notation, we can write the expectation value of the operator $\hat{p}$ with respect to a quantum state ψ as

$$\langle p \rangle = \langle \psi | \hat{p} | \psi \rangle \tag{2.47}$$

The above is the elevated way of taking the "average" of an operator. As mentioned before, Dirac notation is homomorphic to matrix algebra notation. The above is similar to $\boldsymbol{\psi}^\dagger \cdot \overline{\mathbf{P}} \cdot \boldsymbol{\psi}$. This quantity is always real if $\overline{\mathbf{P}}$ is a Hermitian matrix. Hence, in (2.47), the expectation value is always real if $\hat{p}$ is Hermitian. In fact, it can be proved the it is Hermitian in the function space that it is defined.

Operators that correspond to measurable quantities are called observables in quantum theory, and they are replaced by operators in the quantum world. We can take expectation values of these operators with respect to the quantum state involved. Therefore, these observables will have a mean and standard deviation. In the previous section, we elevated the position variable ξ to become an operator $\hat{\xi} = \xi \hat{I}$. This operator is clearly Hermitian, and hence, the expectation value of this position operator is always real. From the previous section, we see that the normalized momentum operator is always Hermitian, and hence, its expectation value is always real. The difference of these quantum observables compared to classical variables is that the quantum observables have a mean and a standard deviation just like a random variable.

2.6 Connecting with Electromagnetic Fields

In the classical case, we have connected the momentum p and position x of a classical pendulum with the field amplitudes Π and A_0 of a 1D cavity electromagnetic mode. In the quantum pendulum p and x are elevated to become operators $\hat{p}$ and $\hat{x}$ which are endowed with randomness. Hence, for the electromagnetic case, the fields are elevated to become operators $\hat{\Pi}$ and $\hat{A}_0$.

Previously, we have written Schrodinger equation in normalized coordinates, where

$$\xi = \sqrt{\frac{m\omega_0}{\hbar}} x \tag{2.48}$$

Using the map that $x \to \sqrt{\frac{L}{2}} A_0$, $m \to \varepsilon$, $\omega_0 \to \omega_l$, then

$$\xi = \sqrt{\frac{\varepsilon \omega_l L}{2\hbar}} A_0 \tag{2.49}$$

In the above, the field amplitude of the mode A_0 is mapped to the normalized coordinate ξ. Hence, in the quantum world, ξ is a random variable as is A_0. A wave function in terms of ξ is mapped to a wave function in terms of A_0.

The Schrodinger equation in terms of ξ which is related to A_0 becomes

$$\frac{1}{2}\hbar\omega_l \left(-\frac{d^2}{d\xi^2} + \xi^2 \right) \psi(\xi, t) = i\hbar \partial_t \psi(\xi, t) \tag{2.50}$$

where ω_l is now the resonant frequency of the cavity mode that is mapped to the quantum harmonic oscillator. The above can be written with $\hat{a}^\dagger$ and $\hat{a}$ and Dirac notation to arrive at

$$\hbar\omega_l \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) |\psi, t\rangle = i\hbar \partial_t |\psi, t\rangle \tag{2.51}$$

We have shown previously that

$$\hat{a}^\dagger + \hat{a} = \sqrt{2}\hat{\xi} \tag{2.52}$$

$$\hat{a}^\dagger - \hat{a} = -\sqrt{2} i \hat{P} \tag{2.53}$$

It can be shown that for the pendulum

$$\hat{p} = \sqrt{m\hbar\omega_0}\hat{P} \tag{2.54}$$

Then we use the map that $\omega_0 \to \omega_l$ and $m \to \varepsilon$, to arrive at

$$\hat{\Pi} = \sqrt{\frac{2\varepsilon\hbar\omega_l}{L}}\hat{P} = -\varepsilon\hat{E}_0 \tag{2.55}$$

or that

$$\hat{E}_0 = -\sqrt{\frac{2\hbar\omega_l}{\varepsilon L}}\hat{P} = \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon L}}\left(\hat{a}^\dagger - \hat{a}\right) \tag{2.56}$$

Now that E_0 has been elevated to be a quantum operator $\hat{E}_0$, from (1.9), we have

$$\hat{E}_x(z) = \hat{E}_0 \sin(k_l z) \tag{2.57}$$

Therefore,

$$\hat{E}_x(z) = \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon L}}\left(\hat{a}^\dagger - \hat{a}\right)\sin(k_l z) \tag{2.58}$$

Notice that in the above, $\hat{\Pi}$, $\hat{E}_0$, and $\hat{E}_x(z)$ are all Hermitian operators and they correspond to quantum observables that have randomness associated with them.

To let $\hat{E}_x$ have any meaning, it should act on a quantum state. For example,

$$|\psi_E\rangle = \hat{E}_x|\psi\rangle \tag{2.59}$$

Notice that thus far, all the operators derived are independent of time. To derive time dependence of these operators, one needs to find their expectation value with respect to time-dependent state vectors.[9]

To illustrate this, we can take expectation value of the quantum operator $\hat{E}_x(z)$ with respect to a time dependent state vector, like the time-dependent coherent state, Thus

$$\begin{aligned}\langle E_x(z,t)\rangle = \langle\alpha,t|\hat{E}_x(z)|\alpha,t\rangle &= \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon L}}\langle\alpha,t|\hat{a}^\dagger - \hat{a}|\alpha,t\rangle \\ &= \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{\varepsilon L}}\left(\tilde{\alpha}^*(t) - \tilde{\alpha}(t)\right)\langle\alpha,t|\alpha,t\rangle = -2\sqrt{\frac{\hbar\omega_l}{\varepsilon L}}\Im m(\tilde{\alpha})\end{aligned} \tag{2.60}$$

Using the time-dependent $\tilde{\alpha}(t) = \alpha e^{-i\omega_l t} = |\alpha|e^{-i(\omega_l t+\psi)}$ in the above, we have

$$\langle E_x(z,t)\rangle = 2\sqrt{\frac{\hbar\omega_l}{\varepsilon L}}|\alpha|\sin(\omega_l t + \psi) \tag{2.61}$$

where $\tilde{\alpha}(t) = \alpha e^{-i\omega_l t}$. The above, which is the average of a random field, resembles a classical field. But since it is rooted in a random variable, it has a standard deviation in addition to having a mean.

[9]This is known as the Schrodinger picture.